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Asymptotics of a $_3F_2$ polynomial associated with the Catalan-Larcombe-French sequence

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is considered by using integral representations of this polynomial. This $_3F_2$ polynomial is associated with the Catalan-Larcombe-French sequence. Several other representations are mentioned, with references to the literature, and another asymptotic method is described by using a generating function of the sequence. The results are similar to those obtained by Clark (2004) who used a binomial sum for obtaining an asymptotic expansion.

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Asymptotics of a ${}_3F_2$ polynomial associated with the Catalan-Larcombe-French sequence

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Abstract

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is considered by using integral representations of this polynomial. This ${}_3F_2$ polynomial is associated with the Catalan-Larcombe-French sequence. Several other representations are mentioned, with references to the literature, and another asymptotic method is described by using a generating function of the sequence. The results are similar to those obtained by Clark (2004) who used a binomial sum for obtaining an asymptotic expansion.

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1 The problem

Find the large n asymptotics of

$$ f(n) = {}_3F_2\left(\begin{matrix} -n, \frac12, \frac12 \\ \frac12 - n, \frac12 - n \end{matrix}; -1\right) \tag{1.1} $$

Peter Larcombe conjectured that $\lim_{n\to\infty} f(n) = 2$ and Tom Koornwinder gave a proof, based on dominated convergence. See for details of the proof [6], where also a different representation of $f(n)$ is considered in the form

$$ f(n) = 2^n {}_3F_2\left(\begin{matrix} -n, -\frac12 n, \frac12 - \frac12 n \\ \frac12 - n, \frac12 - n \end{matrix}; 1\right). \tag{1.2} $$

The equivalence of these two forms follows from a quadratic transformation of the ${}_3F_2$−functions as given in [2, Ex. 4(iv), p.97], that is,

$$
\begin{aligned}
&{}_3F_2\left(\begin{matrix} a, b, c \\ 1+a-b, 1+c-c \end{matrix}\,;\, z\right) = \\
&\qquad (1-z)^{-a}{}_3F_2\left(\begin{matrix} \frac12 a, \frac12+\frac12 a, 1+a-b-c \\ 1+a-b, 1+a-c \end{matrix}\,;\, \frac{-4z}{(1-z)^2}\right).
\end{aligned}
\tag{1.3}
$$

with $a=-n$, $b=c=\frac12$, and $z=-1$. Another form is given by (see [7, Eq. (A2)])

$$
f(n)=\frac{n!}{2^n(\frac12)_n}{}_3F_2\left(\begin{matrix} -n,-n,\frac12 \\ 1,\frac12-n \end{matrix}\,;\,-1\right). \tag{1.4}
$$

In [4] an asymptotic expansion of $\frac12 f(n)$ has been derived. The asymptotic analysis is based on the representation

$$
P_n=\frac{1}{n!}\sum_{p+q=n}\binom{2n}{p}\binom{2q}{q}\frac{(2p)!\,(2q)!}{p!\,q!}. \tag{1.5}
$$

By using the relation

$$
(2n)! = 2^{2n} n! (\tfrac12)_n, \quad n=0,1,2,\ldots, \tag{1.6}
$$

it is straightforward to verify that (1.5) can be written as

$$
P_n=\frac{2^{4n}}{n!}\sum_{p=0}^{n}\frac{(\frac12)_p(\frac12)_p(\frac12)_{n-p}(\frac12)_{n-p}}{p!\,(n-p)!}. \tag{1.7}
$$

By using

$$
(a)_{n-k}=(-1)^k\frac{(a)_n}{(1-a-n)_k}, \tag{1.8}
$$

it follows that

$$
P_n=\frac{2^{4n}(\frac12)_n(\frac12)_n}{n!\,n!}\sum_{p=0}^{n}(-1)^p\frac{(-n)_p(\frac12)_p(\frac12)_p}{p!\,(\frac12-n)_p(\frac12-n)_p}, \tag{1.9}
$$

that is,

$$
P_n=\frac{2^{4n}(\frac12)_n(\frac12)_n}{n!\,n!}{}_3F_2\left(\begin{matrix} -n,\frac12,\frac12 \\ \frac12-n,\frac12-n \end{matrix}\,;\,-1\right), \tag{1.10}
$$

which gives the relation with $f(n)$ by using (1.1):

$$
P_n=\frac{2^{4n}(\frac12)_n(\frac12)_n}{n!\,n!}\,f(n)=\binom{2n}{n}^2 f(n). \tag{1.11}
$$

The numbers P_n are for $n = 0, 1, 2, , \ldots$ known as the elements of the sequence (A053175) $\{1, 8, 80, 896, 10816, \ldots\}$, called the *Catalan-Larcombe-French* sequence, which is originally discussed by Catalan [3]. See the *On-Line Encyclopedia of Integer Sequences* http://www.research.att.com/ njas/sequences/ .

In this paper we derive a complete asymptotic expansion of the numbers P_n by using integral representations of the corresponding ${}_3F_2-$functions. Our results are the same as those obtained by Clark [4], who used the binomial sum in (1.5) without reference to the ${}_3F_2-$functions.

2 Transformations

We derive an integral representation of the ${}_3F_2-$function of (1.1) by using several transformations for special functions. We start with the beta integral

$$B(x, y) = \frac{\Gamma(x)\Gamma(y)}{\Gamma(x + y)} = \int_0^1 t^{x-1}(1 - t)^{y-1}\, dt \tag{2.12}$$

and use it in the form

$$\frac{\left(\frac{1}{2}\right)_k}{\left(\frac{1}{2} - n\right)_k} = \frac{(-1)^k n!}{\sqrt{\pi}\Gamma(n + \frac{1}{2})} \int_0^1 t^{k-\frac{1}{2}}(1 - t)^{n-k-\frac{1}{2}}\, dt, \quad k = 0, 1, \ldots, n. \tag{2.13}$$

We substitute this in the representation of the ${}_3F_2-$function in (1.1)

$${}_3F_2\left(\begin{matrix}-n, \frac{1}{2}, \frac{1}{2} \\ \frac{1}{2} - n, \frac{1}{2} - n\end{matrix}; -1\right) = \sum_{k=0}^{n}(-1)^k \frac{(-n)_k(\frac{1}{2})_k(\frac{1}{2})_k}{k!\,(\frac{1}{2} - n)_k(\frac{1}{2} - n)_k}. \tag{2.14}$$

This gives after performing the $k-$summation

$$f(n) = \frac{n!}{\pi\,(\frac{1}{2})_n} \int_0^1 t^{-\frac{1}{2}}(1 - t)^{n-\frac{1}{2}}\,{}_2F_1\left(\begin{matrix}-n,\ \frac{1}{2} \\ \frac{1}{2} - n\end{matrix}; \frac{t}{1 - t}\right) dt. \tag{2.15}$$

We substitute $t = \sin^2(\theta/2)$ and obtain

$$f(n) = \frac{n!}{\pi\,(\frac{1}{2})_n} \int_0^\pi \cos^{2n}(\theta/2)\,{}_2F_1\left(\begin{matrix}-n,\ \frac{1}{2} \\ \frac{1}{2} - n\end{matrix}; \tan^2(\theta/2)\right) d\theta. \tag{2.16}$$

We apply a quadratic transformation (see [1, Eq. 15.3.26]) to obtain

$$f(n) = \frac{n!}{\pi\,(\frac{1}{2})_n} \int_0^\pi {}_2F_1\left(\begin{matrix}-\frac{1}{2}n,\ \frac{1}{2} - \frac{1}{2}n \\ \frac{1}{2} - n\end{matrix}; \sin^2\theta\right) d\theta, \tag{2.17}$$

and use the representation of the Legendre polynomial

$$P_n(x) = \frac{(2n)!}{2^n n!\, n!} x^n\,{}_2F_1\left(\begin{matrix}-\frac{1}{2}n,\ \frac{1}{2} - \frac{1}{2}n \\ \frac{1}{2} - n\end{matrix}; x^{-2}\right). \tag{2.18}$$

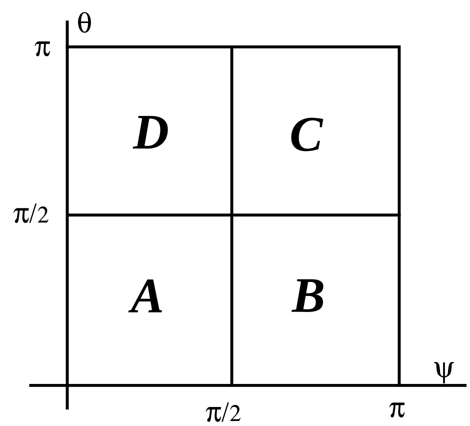

Figure 1: The domain of integration of the integral in (2.21) and subdomains A, B, C and D.

This follows from [1, Eq. (22.3.8)] and gives

$$f(n) = \frac{2^{-n} n!\, n!}{\pi\, (\frac12)_n\, (\frac12)_n} \int_0^\pi \sin^n\theta\; P_n\left(\frac{1}{\sin\theta}\right)\, d\theta. \tag{2.19}$$

Next, consider (see [8, p. 204])

$$P_n(z) = \frac{1}{\pi}\int_0^\pi \left(z + \sqrt{z^2-1}\cos\psi\right)^n\, d\psi, \quad n = 0, 1, 2, \ldots, \tag{2.20}$$

which gives the double integral

$$f(n) = \frac{n!\, n!}{\pi^2\, (\frac12)_n\, (\frac12)_n} \int_0^\pi \int_0^\pi \left(\frac{1+\cos\theta\cos\psi}{2}\right)^n\, d\theta\, d\psi. \tag{2.21}$$

3 Asymptotic analysis

The landscape of the integrand in (2.21) shows peaks at the boundary points $(0,0)$ and (π,π), where it assumes the value 1. Along the interior lines $\theta = \frac12\pi$ and $\psi = \frac12\pi$ the integrand has the value 2^{-n}. Inside the squares A and C, see Figure 1, the value of the integrand is between 2^{-n} and 1, in the squares B and D it is between 0 and 2^{-n}. In addition, the contributions from A and C are the same, and also those from B and D are the same.

From an asymptotic point of view it follows that the integral over the full square equals twice the integral over A, with an error that is of order $\mathcal{O}(2^{-n})$, while the total integral is of order $\mathcal{O}(1)$, as n is large. Hence, we concentrate on the integral over A, and write for large values of n

$$f(n)=2\frac{2^{-n}n!\,n!}{\pi^2\,(\frac12)_n\,(\frac12)_n}\left[\int_0^{\frac12\pi}\int_0^{\frac12\pi}(1+\cos\theta\cos\psi)^n\,d\theta\,d\psi+E_n\right], \tag{3.22}$$

where $E_n=\mathcal{O}(2^{-n})$. Next, we neglect E_n and put $u=\sin(\theta/2), v=\sin(\psi/2)$, and obtain

$$f(n)\sim\frac{8\,n!\,n!}{\pi^2\,(\frac12)_n\,(\frac12)_n}\int_0^{\frac12\sqrt2}\int_0^{\frac12\sqrt2}\left(1-u^2-v^2+2u^2v^2\right)^n\,\frac{du}{\sqrt{1-u^2}}\,\frac{dv}{\sqrt{1-v^2}}. \tag{3.23}$$

For the integrals in (3.22) and (3.23) asymptotic expansions can be obtained by using Laplace's method for double integrals; see [10, § VIII.10]). In our case a simpler approach is based on neglecting a part of square A by introducing polar coordinates

$$u=r\cos t,\quad v=r\sin t,\quad 0\le r\le\tfrac12\sqrt2,\quad 0\le t\le\tfrac12\pi. \tag{3.24}$$

This gives (again we make an error in the integral that is of order $\mathcal{O}(2^{-n})$)

$$f(n)\sim\frac{8\,n!\,n!}{\pi^2\,(\frac12)_n\,(\frac12)_n}\int_0^{\pi/2}\int_0^{\frac12\sqrt2}\frac{(1-r^2+2r^4\cos^2t\sin^2t)^n r\,dr\,dt}{\sqrt{(1-r^2\cos^2t)(1-r^2\sin^2t)}}. \tag{3.25}$$

We change r^2 into r, and obtain

$$f(n)\sim\frac{4\,n!\,n!}{\pi^2\,(\frac12)_n\,(\frac12)_n}\int_0^{\pi/2}\int_0^{\frac12}\frac{(1-r+2r^2\cos^2t\sin^2t)^n\,dr\,dt}{\sqrt{(1-r\cos^2t)(1-r\sin^2t)}}. \tag{3.26}$$

First the standard method for obtaining asymptotic expansions of a Laplace-type integral can be used (for the $r-$integral). The second step is done by integrating the coefficients of this expansion with respect to t.

For the $r-$integral we transform the variable of integration by putting

$$w=-\ln\left(1-r+2r^2\cos^2t\sin^2t\right). \tag{3.27}$$

This mapping is one-to-one for $r\in[0,\frac12]$, uniformly with respect to $t\in[0,\frac12\pi]$, with corresponding $w-$interval $[0,w_0]$, where $w_0=w(\frac12)$.

We obtain

$$f(n)\sim\frac{4\,n!\,n!}{\pi^2\,(\frac12)_n\,(\frac12)_n}\int_0^{\pi/2}\int_0^{w_0}e^{-nw}F(w,t)\,dw\,dt, \tag{3.28}$$

where

$$F(w,t)=\frac{1}{\sqrt{(1-r\cos^2t)(1-r\sin^2t)}}\,\frac{dr}{dw}. \tag{3.29}$$

4 Asymptotic expansion

We obtain the asymptotic expansion of $w-$integral in (3.28) by using Watson's lemma (see [10, § I.5]).

The function $F(w,t)$ is analytic in a neighborhood of the origin of the $w-$plane. We expand

$$F(w,t) = \sum_{k=0}^{\infty} c_k(t) w^k \tag{4.30}$$

and substitute this expansion in (3.28). Interchanging the order of summation and integration, and replacing the interval of the $w-$integrals by $[0,\infty)$ (a standard procedure in asymptotics) we obtain

$$f(n) \sim \frac{4\,n!\,n!}{n\pi^2\,(\frac{1}{2})_n\,(\frac{1}{2})_n} \sum_{k=0}^{\infty} C_k \frac{k!}{n^k}, \quad n\to\infty, \tag{4.31}$$

where

$$C_k = \int_0^{\pi/2} c_k(t)\,dt, \quad k=0,1,2,\ldots. \tag{4.32}$$

The coefficients $c_k(t)$ can be obtained by the following method. First we need the inverse of the transformation defined in (3.27). That is, we need coefficients b_k in the expansion

$$r(w) = \sum_{k=0}^{\infty} b_k(t) w^k. \tag{4.33}$$

We can find $r(w)$ from (3.27) as a solution of a quadratic equation, with the condition $r(w) \sim w$ as $w \to 0$, that is, $b_0(t) = 1$. However, we can also differentiate (3.27) with respect to r and substitute the expansion (4.33), and solve for the coefficients $b_k(t)$. When we have these coefficients we can expand $F(w,t)$ of (3.29) and find $c_k(t)$.

The first few coefficients $c_k(t)$ are

$$\begin{aligned}
c_0(t) &= 1,\\
c_1(t) &= \tfrac{1}{2}(-1+8s^2-8s^4),\\
c_2(t) &= \tfrac{1}{8}(1-28s^2+220s^4-384s^6+192s^8),\\
c_3(t) &= \tfrac{1}{48}(-1+92s^2-1628s^4+10752s^6-24576s^8+23040s^{10}-7680s^{12}),\\
c_4(t) &= \tfrac{1}{384}(1-280s^2+10024s^4-130848s^6+773904s^8-2054400s^{10}+\\
&\quad 2691840s^{12}-1720320s^{14}+430080s^{16}),\\
c_5(t) &= \tfrac{1}{3840}(-1+848s^2-55328s^4+1259040s^6-13396560s^8+\\
&\quad 73983360s^{10}-215329920s^{12}+349224960s^{14}-\\
&\quad 319549440s^{16}+154828800s^{18}-30965760s^{20})
\end{aligned} \tag{4.34}$$

where $s = \sin^2 t$. For the corresponding C_k we have

$$C_0 = \tfrac{1}{2}\pi, \quad C_1 = 0, \quad C_2 = \tfrac{1}{8}\pi, \quad C_3 = \tfrac{1}{8}\pi, \quad C_4 = \tfrac{55}{384}\pi, \quad C_5 = \tfrac{11}{64}\pi. \quad (4.35)$$

As a next step we can replace in (4.31) the ratios $n!/(\frac{1}{2})_n$ by the asymptotic expansion

$$\frac{n!}{(\frac{1}{2})_n} = \sqrt{\pi}\frac{\Gamma(n+1)}{\Gamma(n+\frac{1}{2})} \sim \sqrt{\pi n}\sum_{k=0}^{\infty}\frac{\gamma_k}{n^k}, \tag{4.36}$$

where

$$\gamma_0 = 1, \quad \gamma_1 = \tfrac{1}{8}, \quad \gamma_2 = \tfrac{1}{128}, \quad \gamma_3 = -\tfrac{5}{1024}, \quad \gamma_4 = -\tfrac{21}{32768}, \quad \gamma_5 = \tfrac{399}{262144}. \tag{4.37}$$

This finally gives

$$f(n) \sim 2\left(1 + \frac{1}{4n} + \frac{17}{32n^2} + \frac{207}{128n^3} + \frac{14875}{2048n^4} + \frac{352375}{8192n^5} + \ldots\right). \tag{4.38}$$

5 An alternative method

The numbers P_n were proposed as "Catalan" numbers by an associate of Catalan. They appear as coefficients in the series expansion of an elliptic integral of the first kind

$$K(k) = \int_0^{\frac{1}{2}\pi} \frac{1}{\sqrt{1-k^2\sin^2 t}}\,dt, \tag{5.39}$$

which is transformed and written as a power series in k (through an intermediate variable); this gives a generating function for the sequence $\{P_n\}$. For details we refer to [5].

In [9] a generating function for the numbers P_n is given in terms of the square of a modified Bessel function, and we use this approach to obtain an asymptotic expansion of $f(n)$. See also [7] for details on this generating function.

We consider numbers F_n defined as coefficients in the generating function

$$\left[e^{w/2}\, I_0(w/2)\right]^2 = \sum_{n=0}^{\infty} F_n w^n. \tag{5.40}$$

By considering the relation of the Bessel function with the confluent hypergeometric functions (see [1, Eq. 13.6.3]),

$$e^z I_\nu(z) = \frac{(\frac{1}{2}z)^\nu}{\Gamma(\nu+1)}\,{}_1F_1\left(\begin{matrix}\nu+\frac{1}{2}\\ 2\nu+1\end{matrix}; 2z\right), \tag{5.41}$$

we can write (5.40) in the form (see also [1, Eq. 13.1.27]),

$$\left[{}_1F_1\left(\begin{matrix}\frac{1}{2}\\ 1\end{matrix}; w\right)\right]^2 = e^{2w}\left[{}_1F_1\left(\begin{matrix}\frac{1}{2}\\ 1\end{matrix}; -w\right)\right]^2 = \sum_{n=0}^{\infty} F_n w^n. \tag{5.42}$$

This gives the representation for F_n:

$$F_n = \sum_{k=0}^{n} \frac{(\frac{1}{2})_k}{k!\,k!}\,\frac{(\frac{1}{2})_{n-k}}{(n-k)!\,(n-k)!}. \tag{5.43}$$

By using (1.8) it follows that

$$F_n = \frac{(\frac{1}{2})_n}{n!\,n!}\sum_{k=0}^{n}(-1)^k\frac{(-n)_k(-n)_k(\frac{1}{2})_k}{(\frac{1}{2}-n)_k k!\,k!}, \tag{5.44}$$

or

$$F_n = \frac{(\frac{1}{2})_n}{n!\,n!}\,{}_3F_2\left(\begin{matrix}-n, -n, \frac{1}{2}\\ 1, \frac{1}{2}-n\end{matrix}\,;\,-1\right). \tag{5.45}$$

It follows from (1.4) that

$$f(n) = \frac{n!\,n!\,n!}{2^n\,(\frac{1}{2})_n\,(\frac{1}{2})_n}\;F_n. \tag{5.46}$$

From (5.40) we obtain

$$F_n = \frac{1}{2\pi i}\int_{\mathcal{C}} \frac{\left[e^{w/2}\,I_0(w/2)\right]^2}{w^{n+1}}\,dw = \frac{1}{2\pi i}\int_{\mathcal{C}}\frac{e^{2w}}{w^{n+1}}\,h(w)\,dw, \tag{5.47}$$

where

$$h(w) = \left[e^{-w/2}\,I_0(w/2)\right]^2 = \left[{}_1F_1\left(\begin{matrix}\frac{1}{2}\\ 1\end{matrix}\,;\,-w\right)\right]^2, \tag{5.48}$$

and the contour $\mathcal{C}$ is a circle around the origin, or any contour that can be obtained from this circle by using Cauchy's theorem. The main contribution comes from the saddle point of $\frac{e^{2w}}{w^{n+1}}$, that is from $w = w_0 = n/2$.

In the standard saddle point method (see [10, § II.4]) a quadratic transformation is used to bring the main part of the integrand in the form of a Gaussian. We can obtain the same expansion by just expanding the function $h(w)$ (which is slowly varying for $w > 0$) at the saddle point.

First we expand (see [1, Eq. 13.4.9])

$${}_1F_1\left(\begin{matrix}\frac{1}{2}\\ 1\end{matrix}\,;\,-w\right) = \sum_{k=0}^{\infty} a_k(w-w_0)^k,\quad a_k = \frac{(-1)^k(\frac{1}{2})_k}{k!\,k!}\,{}_1F_1\left(\begin{matrix}\frac{1}{2}+k\\ 1+k\end{matrix}\,;\,-w_0\right) \tag{5.49}$$

and next

$$h(w) = \sum_{k=0}^{\infty} A_k(w-w_0)^k. \tag{5.50}$$

We substitute this expansion in the second integral in (5.47) and obtain the convergent expansion

$$F_n = \sum_{k=0}^{\infty} A_k\Phi_k,\quad \Phi_k = \frac{1}{2\pi i}\int_{\mathcal{C}}\frac{e^{2w}}{w^{n+1}}\,(w-w_0)^k\,dw. \tag{5.51}$$

The functions Φ_k can be evaluated by using the recursion formula (which easily follows from integrating by parts)

$$\Phi_k = -\tfrac{1}{2}(k-1)(\Phi_{k-1} + w_0\Phi_{k-2}), \quad \Phi_0 = \frac{2^n}{n!}, \quad \Phi_1 = 0. \tag{5.52}$$

An asymptotic expansion can be obtained by using a well-known expansion for a_k defined in (5.49). We have (as follows from [1, Eq. 13.5.1])

$${}_1F_1\left({a \atop c}; -x\right) \sim x^{-a}\frac{\Gamma(c)}{\Gamma(c-a)}\sum_{m=0}^{\infty}\frac{(a)_m(1+a-c)_m}{m!\,x^m}, \quad x \to +\infty, \tag{5.53}$$

from which we can obtain expansions for a_k and A_k for large values of $w_0 = n/2$. By using these expansions in (5.51) we obtain an expansion for F_n, and finally for $f(n$ by using (5.46). This expansion is the same as the one in (4.38).

Acknowledgment

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