Centrum voor Wiskunde en Informatica

REPORTRAPPORT

MAS

Modelling, Analysis and Simulation

Modelling, Analysis and Simulation

Universal trellises

P.J. Collins

REPORT MAS-E0609 FEBRUARY 2006

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ISSN 1386-3703

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ABSTRACT
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2000 Mathematics Subject Classification: 57M25; 37C29; 37C25
Keywords and Phrases: universal template, homoclinic tangle, knot, trellis
Note: This work was carried out during a three-month stay with the Department of Mathematics at Kyoto University, Japan. The author would like to thank Hiroshi Kokubu for the invitation to visit, and Eiko Kin for many fruitful discussions which were invaluable in the writing of this paper. The author was partially supported by the 21st century COE program "Formation of an International Center of Excellence in the Frontier of Mathematics and Fostering of Researchers in Future Generations".

Universal Trellises

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9 February 2006

Abstract

A flow in three-dimensions is *universal* if the periodic orbits contains all knots and links. Universal flows were shown to exist by Ghrist, and can be constructed by means of *templates*. Likewise, a planar diffeomorphism is universal if it has a suspension flow which is a universal flow. In this paper we prove the existence of a homoclinic trellis type for which any representative diffeomorphism is universal. This trellis type is remarkable in that it has zero entropy, and only two homoclinic intersection points.

Mathematics Subject Classification: Primary 57M25; Secondary 37C29, 37C25.
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1 Introduction

A flow ϕ in the three-sphere S^3 is said to be *universal* if for any link, there is a finite set of periodic orbits of ϕ realising the link type. The existence of universal flows was proved by Ghrist [Ghr97] using concepts of template theory [BW83]. Universality indicates that the orbit structure of the flow is extremely complicated; what is remarkable is that simple templates can generate universal flows.

Given a homeomorphism f of the two-disc D^2, we can construct a *suspension flow* on the solid torus $D^2 \times S^1$, and embed the solid torus in S^3 in an unknotted way. We say f is universal if the embedding of some suspension flow is universal; note that there is an ambiguity in constructing the suspension flow determined by the twisting of the boundary. In [Kin00] it was shown that the that the Smale horseshoe map is not universal, but its third iterate is. We are interested in finding weaker conditions under which we can show a homeomorphism is universal.

It is well known that if p is a saddle fixed point, and the unstable manifold $W^U(p)$ intersects the stable manifold $W^S(p)$ transversely, then the dynamics is chaotic. The Smale horseshoe example shows that the existence of a single transverse homoclinic orbit does not necessarily imply universality. In this paper we show that two homoclinic orbits on different branches, as depicted in Figure 9, is sufficient to imply universality. More precisely, we prove the following theorem.

Main Theorem. *Let f be a diffeomorphism with a periodic saddle point p with positive eigenvalues. Suppose that both branches of $W^U(p)$ intersect one of the branches of $W^S(p)$. Then f is universal.*

What is particularly remarkable about this result is that the conditions on the diffeomorphism do not give a strictly positive lower bound for the topological entropy. In other words, even relatively low dynamical complexity can result in high complexity of the periodic orbit structure. This phenomenon is possible since the knots and links can live in high iterates of the diffeomorphism.

The techniques of this paper can also be used to show that the existence of a single periodic or homoclinic orbit of the correct braid type implies universality.

The main techniques used in this paper are those of template theory and trellis theory [Col04]. A trellis is a finite piece of homoclinic or heteroclinic tangle, and provides a way of specifying the "braid type" of a homoclinic orbit. We relate trellises and templates via surface-embedded graph maps, which carry combinatorial information about the dynamics.

2 Preliminaries

In this section, we define the main objects of study, namely templates, trellises and surface-embedded graph maps. Templates and trellises are have been much studied in the dynamical systems literature, and we only give a brief discussion to introduce the fundamental concepts, referring the reader to the literature for more information. Embedded graph maps enable one to relate templates and trellises, and are related to the thick tree maps of [FM93] and the foliated surfaces of [BH95]. Since the concept is slightly different, we give a full definition.

2.1 Isotopies and suspensions

Let D be the disc, I the unit interval $[0,1]$, and S^1 the circle $\mathbb{R}/\mathbb{Z}$. By glueing the ends of the cylinder $D \times I$ by the relation $(x,0) \sim (x,1)$, we obtain the solid torus $D \times S^1$. We fix, once and for all, an unknotted, untwisted embedding of $D \times S^1$ in S^3, which fixes an immersion $D \times I \longrightarrow D \times S^1 \hookrightarrow S^3$ from the cylinder into S^3.

Let $f : D \longrightarrow D$ a homeomorphism, and ϕ an isotopy from id to f, and P a finite collection of periodic orbits of f. Then P of f gives rise to a braid $\mathcal{B}(P,\phi)$ lying in $D \times I$ given by $\mathbf{B}(P,\phi) = \bigcup_{t \in I} f_t(P) \times \{t\}$. By taking the immersion of $D \times I$ into S^3, the braid $\mathcal{B}(P,\phi)$ gives rise to a knot or link $\mathcal{L}(P,\phi)$ in S^3. We call this link the *natural suspension of P under ϕ*.

The link type so obtained depends not only on P, but also on the isotopy chosen. However, if $\Phi_{s,t}$ is a continuously varying family of homeomorphisms such that $\Phi_{s,0} = \mathrm{id}$ and $\Phi_{s,1} = f$ for all s, then the braids specified by the isotopies Φ_s are themselves isotopic. Further, if H is full-twist on $D \times I$, then $\mathcal{B}(P,\phi')$ is isotopic to $H^n(\mathcal{B}(P,\phi))$ for some n depending only on the isotopies ϕ and ϕ'.

We say that f *induces all link types* or is *universal* if there exists an isotopy ϕ from id to f such that every link is obtained from ϕ by the natural suspension. Note that in this definition, we first fix the isotopy and then look for all links.

We are interested in finding conditions under which a homeomorphism induces all link types.

2.2 Templates

Birman and Williams [BW83] introduced templates (also known as knot holders) to study link types of closed orbits of dynamically complex 3-dimensional flows. Recent results on template theory can be found in the monograph [GHS97].

Definition 2.1. A *template* or *knot holder* is a compact branched 2-manifold with boundary and with smooth expansive semiflow built from a finite number of *branchline* charts, as depicted in Fig. 1.

A branchline chart with a single exit strip is a *joining* chart, and a branchline chart with a single entry strip is a *splitting* chart.

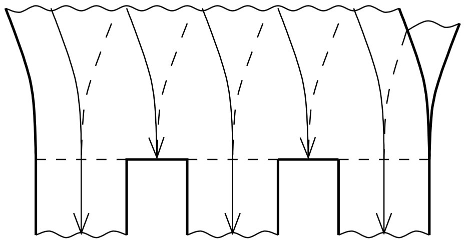

Figure 1: Local geometry of a branchline chart with two entry sheets joining and three exit sheets splitting off.

A subset $\mathcal{W}'$ of a template $\mathcal{W}$ is a *subtemplate* of $\mathcal{W}$ if $\mathcal{W}'$ with semiflow induced from $\mathcal{W}$ is itself a template.

The template theorem of Birman and Williams [BW83] shows us that if a 3-dimensional flow has a non-trivial hyperbolic chain recurrent set, then the set of links of closed orbits of the flow in this hyperbolic set is captured by a template.

One particularly useful way of constructing a template is via a *braided template* or *braid holder*. A braid holder is a subset of $D \times I$ given by branchline charts in $D \times (0, 1)$, and a fixed set of intervals in $D \times \{0\}$ and $D \times \{1\}$, such that the semiflow is monotone from top to bottom. Just as the periodic orbits of the template semiflow define knots and links, the periodic orbits of the local semiflow on a braid holder define braids. On immersing $D \times I$ into S^3 we obtain a link.

We say that a template $\mathcal{W}$ embedded in S^3 is *universal* if for each link L in S^3, there exists a finite union of periodic orbits P_L of the semiflow on $\mathcal{W}$ with the same link type as L.

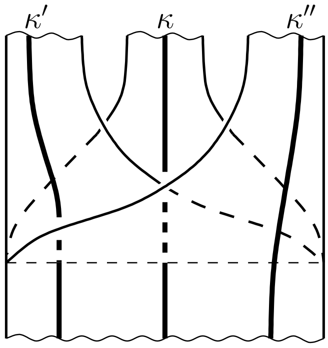

Figure 2: Pieces of the periodic orbits κ', κ and κ'' of Theorem 2.2.

Given unknotted periodic orbit κ on a template $\mathcal{T} \subset S^3$, the *twist*, $\tau(\kappa)$, is defined to be the twist number of the normal bundle to $\mathcal{T}$ along κ. The following result of Ghrist and Kin [GK04] gives sufficient conditions for universality.

Theorem 2.2. *Let $\mathcal{T}$ be a template in S^3. Suppose that there exist three disjoint periodic orbits κ, κ', and κ'' on $\mathcal{T}$ such that*

1. *They are separable unlinked unknots.*

2. *$\tau(\kappa) = 0$, $\tau(\kappa') > 0$, and $\tau(\kappa'') < 0$.*

3. *These three unknots intersect some branchline of $\mathcal{T}$ as in Figure 2 with the specified adjacencies and strip crossings.*

Then $\mathcal{T}$ is universal

2.3 Trellis theory

Let f be a diffeomorphism of a surface M, and P a finite invariant set of hyperbolic saddle points of f. A pair $T = (T^U, T^S)$ is a *trellis* for f if

1. T^U is a subset of $W^U(P)$ and T^S is a subset of $W^S(P)$.
2. T^U and T^S are both a disjoint union of finitely many embedded compact intervals with non-empty interiors,
3. $f(T^U) \supset T^U$ and $f(T^S) \subset T^S$.

The set $T^U \cap T^S$ of intersection points is denoted T^V, and the set of periodic points of T^V is denoted T^P.

Trellises T_0 and T_1 are equivalent if they are homeomorphic. If f_0 and f_1 are diffeomorphisms with trellises T_0 and T_1, then the pairs $(T_0; f_0)$ and $(T_1; f_1)$ are equivalent if there is a homeomorphism h with $h(T_0) = T_1$ such that f_0 is isotopic to $h^{-1} \circ f_1 \circ h$ relative to T_0. The equivalence classes $[T; f]$ are *trellis types*.

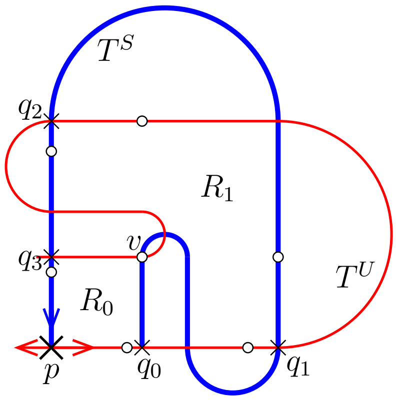

Figure 3: A trellis type for the orientation-preserving Hénon map.

Example 2.3. A trellis type for the orientation-preserving Hénon map is shown in Figure 3. The points q_0, q_1, q_2 and q_3 lie on the same homoclinic orbit. The orbit of v is indicated with white dots. It can be shown [Col04] that chaotic dynamics must be present in the regions R_0 and R_1.

A trellis type provides a way of representing homoclinic orbits. The *braid type* of a homoclinic orbit H of a diffeomorphism f is the conjugacy class of the isotopy class of f relative to H, denoted $[H; f]$. If H is a homoclinic orbit of f containing an intersection point of some trellis T, then the braid type $[H; f]$ is determined by $[T; f]$.

Given a trellis T for a diffeomorphism f of the disc D, we can cut along the unstable set T^U to obtain a topological pair $\mathcal{C}T = (\mathcal{C}^U D, \mathcal{C}^U T^S)$. Since $f(T^U) \supset T^U$, the map f induces a map $\mathcal{C}f$ on $\mathcal{C}^U D$ such that $\mathcal{C}f(\mathcal{C}^U T^S) \subset \mathcal{C}^U T^S$.

The *graph representative* of a trellis type $[T; f]$ is a map g of a topological pair (G, W) where

- G is a one-dimensional CW-complex embedded as a deformation retract of $(D \setminus T^U, T^S \setminus T^U)$, the complement of T^U.

- W is the set of intersections of G with T^S, and contains exactly one point in each segment of T^S.
- g is a map of (G, W) which is a deformation-retract of f on $(M \setminus T^U, T^S \setminus T^V)$.
- g maps the control edge z crossing segment S to the control edge $g(z)$ crossing $f(S)$.
- g is *efficient*, which means it is locally injective except at control edges, and has no invariant forests which do not contain a control edge.

The *essential graph representative* is obtained from the graph representative by restricting to $\bigcap_{n=0}^{\infty} g^n(G)$. The *topological graph representative* is obtained from the essential graph representative by collapsing all control edges to points.

Graph representatives are useful since they give an easily described combinatorial description of the dynamics. Recall that the *itinerary* of an orbit (x_i) is the sequence k_i such that $x_i \in R_{k_i}$, where the R_k are regions in D. Also recall that period-n points in some topological pair (X, Y) are *relative Nielsen equivalent* if they can be joined by curves $\alpha_i : (I, J) \longrightarrow (X, Y)$ such that $\alpha_i^{-1}(Y) = J$ for all i, and $f \circ \alpha_i \sim \alpha_{i+1 \bmod n}$ relative to endpoints.

One of the main results of trellis theory is the following theorem (see [Col04, Theorem 10]).

Theorem 2.4. *Let g be the graph representative of a trellis type $[T; f]$. Then for every period-n point of g, there is a period-n point of f which is relative Nielsen equivalent.*

In particular, any essential Nielsen class of g can be continued through the deformation-retract to an essential Nielsen class of f.

We can also analyse how the periodic orbit structure changes as the trellis changes. We say an isotopy $(f_t; T_t)$ is a *pruning* isotopy if intersections are destroyed, but never created, as t increases. The following result [Col04, Theorem 1] shows that the dynamics forced by a trellis type becomes simpler in an isotopy removing intersections.

Theorem 2.5. *Suppose $(f_t; T_t)$ is a pruning isotopy. Then any essential Nielsen class for $[T_1; f_1]$ can be continued to an essential Nielsen class of $[T_0; f_0]$.*

We are interested in whether a trellis forces all knots and links in the suspension.

Definition 2.6. A trellis type $[T; f]$ is *universal* if every diffeomorphism $\widetilde{f} \in [f]_T$ is universal.

2.4 Surface-embedded graph maps

The graph representative provides the basic topological structure which allows us to relate a trellis to a template. To use the graph representative to obtain templates and braids, we need to consider the embedding of the graph more carefully.

In this paper, by a *graph*, we mean a one-dimensional CW complex G embedded in a surface. Given a graph G, we can construct a *singularly foliated neighbourhood* $\widehat{G}$ by taking stable leaves transverse to each edge. At each valence-n vertex, we have an n-prong singularity in the foliation. (See [BH95] for a description of singular foliations.) Let $r : \widehat{G} \longrightarrow G$ be the retract preserving the stable leaves, ρ be a homotopy from id to r such that each ρ_t preserves stable leaves and ρ_t is injective for $t < 1$.

Definition 2.7. Let G be a graph, and $\widehat{G} \subset D$ a neighbourhood of G with a singular foliation $\mathscr{F}^S$ transverse to G. Let $r : \widehat{G} \longrightarrow G$ be the retract which preserves the stable leaves. Then an *embedded graph map* is an embedding $g : G \mapsto \widehat{G}$ such that $r \circ g$ is a graph map.

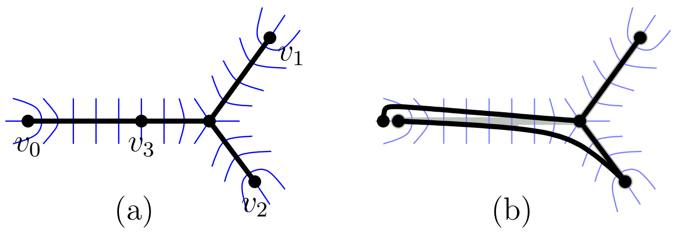

Figure 4: (a) An embedded graph G, and (b) the image of G under an embedded graph map g. The vertices are mapped $v_0 \mapsto v_0$ and $v_i \mapsto v_{i-1}$ for $i = 1, 2, 3$.

By a slight abuse of terminology, we will and say that g is an embedded graph map from G to itself, and make the actual embedding clear from a diagram, if necessary. An example of an embedded graph map is shown in Figure 4.

Not all graph maps can be embedded, but the graph representative of a trellis type (defined in Section 2.3) has a natural embedding given by the deformation-retract of the surface onto the graph.

Remark 2.8. Our embedded graph maps are similar to the *thick tree maps* of [FM93]. The results of this paper could be formulated in terms of thick trees, but this introduces complications since a map of a thick tree has finitely many extra periodic points which are not present in the original graph map.

Given an embedded graph map $g : G \longrightarrow \widehat{G}$, an isotopy γ from id to g, and a deformation retract ρ of $\widehat{G}$ onto G, define a homotopy $\rho \cdot \gamma$ from id to $r \circ g$ by

$$(\rho \cdot \gamma)_t(x) = \begin{cases} \gamma_{2t}(x) & \text{if } t \leq 1/2; \\ \rho_{2t-1}(\gamma_1(x)) & \text{if } t \geq 1/2. \end{cases} \tag{1}$$

We can use the immersion of $D \times I$ into S^3 to define a *singular braid holder* $\mathscr{B}(\rho \cdot \gamma)$ by

$$\mathscr{B}(\rho \cdot \gamma) = \bigcup_{t \in [0,1]} (\rho \cdot \gamma)_t(G) \times \{t\}.$$

Every periodic orbit P of $r \circ g$ gives rise to a braid on $\mathscr{B}(\rho \cdot \gamma)$ biven by

$$\mathcal{B}(P, \rho \cdot \gamma) = \bigcup_{t \in [0,1]} (\rho \cdot \gamma)_t(P) \times \{t\}.$$

The singularities in the braid holder arise from the vertices of G. We can construct a singular template from the singular braid holder by taking the natural suspension.

The definition of universality can be extended to embeddable graph maps.

Definition 2.9. An embedded graph map g is *universal* if there is an isotopy γ from id to g such that the immersion of the braid holder $\mathscr{B}(\rho \cdot \gamma)$ into S^3 is a template containing all knots and links.

3 Universal graph maps and trellises

In this section, we give a proof of the Main Theorem. We first consider a family of embedded graph maps, the *binary star graph maps*, and show that each is universal. We then give some results on braids forced by trellises which extend the results on Nielsen equivalence quoted in Section 2.3. Finally, we prove the main theorem by constructing a trellis type which can be pruned to give another trellis type whose topological graph representative is a binary star graph map.

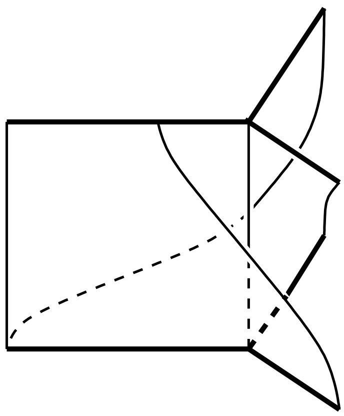

Figure 5: The braid holder given by the embedded graph map of Figure 4.

3.1 Universal graph maps

We now define the binary star graph maps, and show that these maps are universal. These maps will later be used to prove the existence of a universal template.

Definition 3.1. The (m, n) *binary star graph* has a vertex v_L of valence m and incident edges $\bar{a}_0, \bar{a}_0, \ldots, \bar{a}_{m-1}$, a vertex v_R of valence n, with incident edges $\bar{c}_0, \bar{c}_1, \ldots, \bar{c}_{n-1}$, and a vertex v_P of valence 2 with incident edges a_0 and c_0.

A *binary star graph map* g is an embedded graph map on $G_{m,n}$ such that

$$a_0 \mapsto a_0\bar{a}_1a_1, \quad a_i \mapsto a_{i+1} \text{ for } 0 < i < m-1, \qquad c_0 \mapsto c_0\bar{c}_1c_1, \quad c_i \mapsto c_{i+1} \text{ for } 0 < i < n-1,$$

and for which $g(a_{m-1}) \cap g(c_{n-1})$ is an interval b containing v_P in its interior.

A binary star graph map on $G_{5,4}$ is shown in Figure 6.

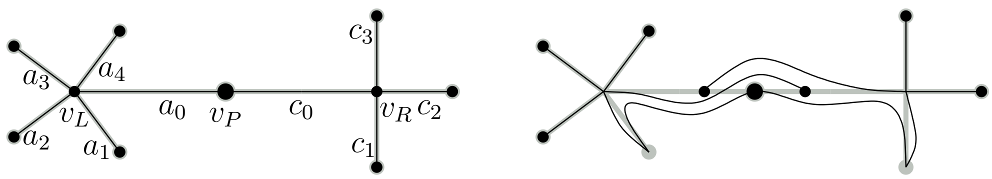

Figure 6: Binary star graph map on the $(5, 4)$ binary star.

Theorem 3.2. *Let $G_{m,n}$ be the (m, n)-binary star, and g be an embedded binary star graph map. Then g is universal.*

Proof. Let b be the intersection of $g(a_{m-1})$ and $g(c_{n-1})$. Then there exists k such that $g^k(b)$ convers a_0 and c_0. Then not only does $g^{k+m}(b)$ cover b, but there exist intervals $b_{-1}, \ldots b_{-k-m+1}$ with $b_{-i} \subset a_0$ for $i \leq k$ and $b_{-k-i} \subset a_i$ for $0 < i < m-1$ such that $g(b) \supset b_{-1}$, $g(b_{-i}) = b_{-i-1}$ for $i < k+m$, and $g(b_{-k-m+1}) = b$. Similarly, there exist intervals $b_1, \ldots b_{k+n-1}$ with $b_i \subset c_0$ for $i \leq k$ and $b_{k+i} \subset a_i$ for $0 < i < n-1$, such that $g(b) \supset b_1$, $g(b_i) = b_{i+1}$ for $i < k+n$, and $g(b_{k+n-1}) = b$. Additionally, we can ensure that b_{-k} and b_k map into a_1 and c_1 in an orientation-preserving way. By decreasing the intervals b_i, we can ensure that $g(b_0)$ covers only b_{-1}, b_0 and b_1. The intervals b_i are shown in Figure 7.

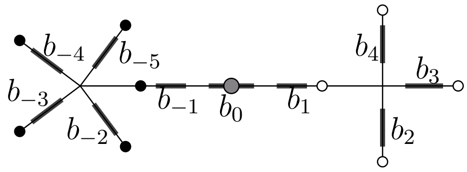

Figure 7: Universal subgraph in the (5, 4) binary star.

We now take an unknotted isotopy γ from id to g (by which we mean the suspension of the vertices v_L, v_P and v_R are unlinked under $\rho \cdot \gamma$). Under this isotopy, the restriction of $r \circ g$ to $\bigcup_{i=-k-m+1}^{k+n-1} b_i$ gives rise to a braid holder $\mathscr{B}(\rho \cdot \gamma)$ of the form shown in Figure 8, from which we can obtain a template $\mathscr{T}(\rho \cdot \gamma)$.

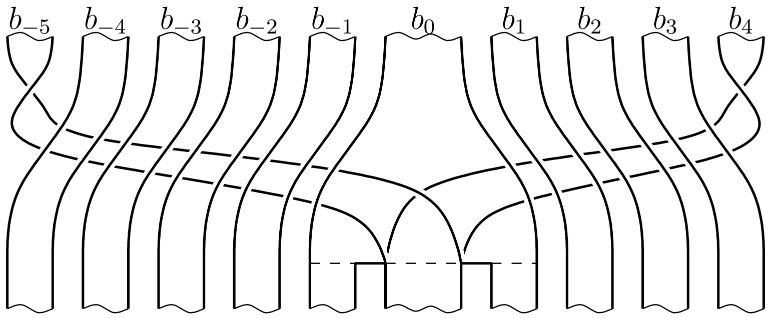

Figure 8: A braid holder for a binary star graph map.

We now show this template is universal. Let κ be the orbit on $\mathscr{T}(\rho \cdot \gamma)$ which passes through b_0 (and so is the natural suspension of the fixed-point v_P of g). Let κ' be the orbit passing successively through $b_0, b_{-1}, \ldots, b_{-k-m+1}$ and returning to b_0. Let κ'' be the orbit passing successively through $b_0, b_1, \ldots, b_{n+n-1}$ and returning to b_0. Then κ, κ' and κ'' are unlinked unknots, with twists 0, +1 and −1, respectively. The braid holder shown in Figure 8 contains a subset of the form depicted in Figure 2. Hence by Theorem 2.2, the template $\mathscr{T}(\rho \cdot \gamma)$ is universal. □

3.2 Trellises and braids

In Section 2.3, we stated results on the relative Nielsen classes and itineraries of periodic orbits occuring for a given trellis type (Theorems 2.4, 2.5). However, in this paper we are interested in the braids occuring in the suspension for a given isotopy to the identity. Fortunately, the results on relative Nielsen classes can be readily extended to results on braids.

We first say what it means for a braid to be forced by a trellis T.

Definition 3.3. Let $[T; f]$ be a trellis type, and ϕ an isotopy from id to f. We say a braid B is *forced* by T_0 in the isotopy class of ϕ if whenever $\Phi_{s,t}$ is a parameterised family of homeomorphisms such that $\Phi_{s,0} = \mathrm{id}$ and $\Phi_{s,1}$ has trellis T for all s, and $\Phi_{0,t} = \phi_t$, then the map $f' = \Phi_{1,1}$ has a periodic orbit P' with braid B' under the isotopy ϕ' given by $\phi'_t \Phi_{1,t}$ such that B' is isotopic to B.

The following theorem shows that the braids on the braid holder of the graph representative are forced by the trellis.

Theorem 3.4. *Let f be diffeomorphism with trellis T, let g be the embedded graph representative of $[T;f]$, and γ an homotopy of* id *to g such that γ_t is injective for all t. Then there exists an isotopy ϕ from* id *to f such that for any braid B on the braid holder $\mathscr{B}(\rho \cdot \gamma)$ there exists a collection of periodic orbits P of f such that $\mathcal{B}(P,\phi)$ has the same knot type in the natural suspension as B.*

Proof. The homotopy γ can be extended to an isotopy of the whole space $\mathcal{C}^U D$ in such a way that G is an attractor for γ_1. The deformation-retract ρ from $\widehat{G}$ to G can be extended to a deformation-retract from $\mathcal{C}^U D$ to G such that $\rho_t(\mathcal{C}^U T^S) \subset \mathcal{C}^U T^S$ for all t and ρ_t is injective for $t < 1$. Hence the homotopy $\rho \cdot \gamma$ extends to a homotopy δ on $\mathcal{C}^U(T^S)$ in such a way that δ_t is injective for $t < 1$ and δ_t maps $\mathcal{C}^U T^S$ into itself for $t \geq 1/2$.

Since $\mathcal{C}^U f$ is homotopic to δ_1 via a deformation-retract of $(\mathcal{C}^U D, \mathcal{C}^U T^S)$ onto G. we can construct an isotopy ν from f to γ_1 such that ν_t in jective for $t < 1$, and $\nu_t = \delta_t$ for $t \geq 1/2$, and ν_t maps $\mathcal{C}^U T^S$ into $\mathcal{C}^U T^S$.

We can therefore find a family of maps Φ such that $\Phi_{s,0} = \mathrm{id}$, $\Phi_{s,1} = \nu_s$, $\Phi_{1,t} = \delta_t$ and Φ is injective for $(s,t) \neq (1,1)$. We take the isotopy ϕ to be given by $\phi_t = \Phi_{0,t}$.

Let B be a braid on $\mathscr{B}(\gamma)$ corresponding to the collection of periodic orbits Q. Then by Theorem 2.4, Q continues to give a family of periodic orbits P_s for $\Phi_{\sigma(s),1}$, where $\sigma : I \longrightarrow I$ is a map such that $\sigma(0) = 0$ and $\sigma(1) = 1$. Further, since the points of Q lie in different relative Nielsen classes, the periods of the orbits of P_s are the same as those of Q for all s. Since $\Phi_{s,t}$ is injective for $(s,t) \neq (1,1)$ and P_1 is periodic for $\Phi_{1,1}$ we have braids B_s given by

$$B_s = \bigcup_{t \in I} \Phi_{\sigma(s),t}(P_s) \times \{t\}.$$

Since the braids B_s vary continuously, and each have n strands, the knot type of $B_0 = \mathcal{B}(P_0, \phi)$ is the same as the knot type of B. □

We now show that pruning isotopies cannot create new braids in the suspension.

Theorem 3.5. *Let $[T_1; f_1]$ is a trellis type obtained from $[T_0; f_0]$ by a pruning isotopy. Let ϕ_1 be an isotopy from* id *to f_1 Then there exists an isotopy ϕ from* id *to f_0 such that for any braid B_1 forced by T_1, there exists a braid B_0 forced by T_0 with the same link type.*

Proof. Let f_t be a pruning isotopy from f_0 to f_1. We construct $\Phi_{s,t}$ with $\Phi_{s,0} = \mathrm{id}$, $\Phi_{1,t} = (\phi_1)_t$ and $\Phi_{s,1} = f_t$ by the isotopy extension theorem. Let $(\phi_0)_t = \Phi_{0,t}$. The result follows by continuing the braid B_1 using Φ as in the proof of Theorem 3.4. □

3.3 Universal trellises

In this section, we consider trellis types on the disc which force all knots and links in the suspension flow.

For the rest of this section, let $[T;f]$ be the trellis type shown in Figure 9. The trellis type $[T;f]$ is the homoclinic trellis type such that both unstable branches of the fixed point have a single intersection with a stable branch. The main result of this section is that $[T;f]$ is universal. Since no planar trellis type with only two intersections can be universal, this is the simplest possible universal trellis type. Further, since this trellis type occurs as a subtrellis whenever both branches of T^U intersect a single branch of T^S, the universality of $[T;f]$ immediately yields our main theorem.

Remark 3.6. The trellis type $[T;f]$ is realised by the third power of the Smale horseshoe map. The fixed point p has code $\overline{011}$ for f, and the homoclinic orbits of q_L and q_R have codes $\overline{011}001\overline{011}$ and $\overline{011}111\overline{011}$, respectively.

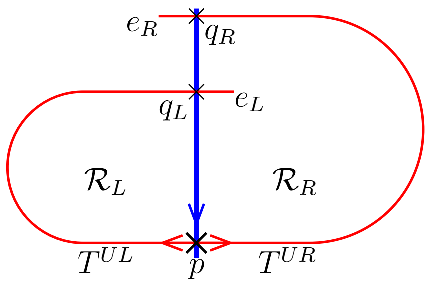

Figure 9: The universal trellis type $[T; f]$.

We use the following notation for $[T; f]$. We let $T^U = T^U[e_L, e_R]$ and $T^S = T^S[p, q_R]$, so e_L and e_R are the endpoints of the unstable branch. We assume there are transverse intersections p, q_L and q_R such that $T^U(e_L, e_R) \cap T^S[p, q_R] = \{p, q_L, q_R\}$. Further, we assume that the orientation of the intersection of T^U with T^S is positive at p, and is negative at q_L, q_R. We let H_L and H_R be the orbits of q_L and q_R, respectively. We let $\mathcal{R}_L$ be the disc bounded by $T^U[p, q_L] \cup T^S[q_L, p]$ and $\mathcal{R}_R$ the disc bounded by $T^U[p, q_R] \cup T^S[q_R, p]$. Note that the although the orbits H_L and H_R are essentially symmetric under reflection, the trellis itself is not.

In order to show that a trellis type is universal, we use the following notion of *topological universality.*

Definition 3.7. A trellis type $[T; f]$ is *topologically universal* if there exists an isotopy ϕ from id to f such that the set of braids forced by T under ϕ gives rise to all knots and links under the immersion of $D \times I$ into S^3.

Clearly, topological universality is stronger than universality. Topological universality is useful since we can use the graph representative to prove topological universality. A trellis type with zero entropy cannot be topologically universal, since it forces only finitely many periodic orbits.

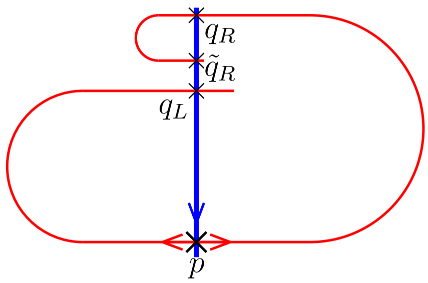

Figure 10: Extending T^U to obtain a bigon with vertices q_R and $\tilde{q}_R$.

Now consider the set $W^U(p)$. Since there is an intersection of W^U with T^S at $f(q_R)$ with negative orientation, there must be an intersection of $W^U(q_R, f(q_R))$ with $T^S(q_L, q_R)$ with positive orientation. We let $\tilde{q}_R$ be the closest such intersection to q_R along W^U, and extend T^U to obtain the trellis shown in Figure 10. We let $\mathcal{R}_B$ be the bigon bounded by $W^U[q_R, \tilde{q}_R] \cup T^S[q_R, \tilde{q}_R]$.

We now consider the construction of extra homoclinic orbits. Let x be a point of a transverse homoclinic orbit in $W^U[q_R, \tilde{q}_R]$. Then the segment of W^S through x contained in $\mathcal{R}_B$ has endpoints x and $\tilde{x}$; without loss of generality, we can take $\tilde{x} \in W^U(x, \tilde{q}_R)$.

Lemma 3.8. *Let $W^S[x, \tilde{x}]$ be a segment in the region bounded by $T^U[q_R, \tilde{q}_R] \cup T^S[q_R, \tilde{q}_R]$, where $x, \tilde{x}$ are transverse homoclinic points and $x \in T^U[q_R, \tilde{x})$.*

1. *There exists m such that $f^{-m}(W^S(x, \tilde{x}))$ intersects $T^U[q_L, e_L]$ at a point z. The arc $W^S(f^{1-m}(r), z)$ does not intersect $T^U[q_L, q_R]$, and the points $\{p, f^{-m}(x), q_L, z\}$ form the vertices of a rectangle in $\mathcal{R}_R$.*

2. *If an intersection of $f^{-m}(W^S(x, \tilde{x}))$ with $T^U(q_L, e_L)$ exists for some m, then an intersection of $f^{-m'}(W^S(x, \tilde{x}))$ with $T^U(q_L, e_L)$ satisfying the conditions in (1) exists for all $m' \geq m$.*

3. *If $T^S[y, \tilde{y}]$ is a segment such that $y \in T^U(q_R, x)$ and $\tilde{y} \in T^U(\tilde{q_R}, \tilde{x})$, then $f^{-m}(W^S(y, \tilde{y}))$ intersects $T^U(q_L, z)$ at a point w such that $\{p, f^{-m}(y), q_L, w\}$ are the vertices of a rectangle in $\mathcal{R}_R$.*

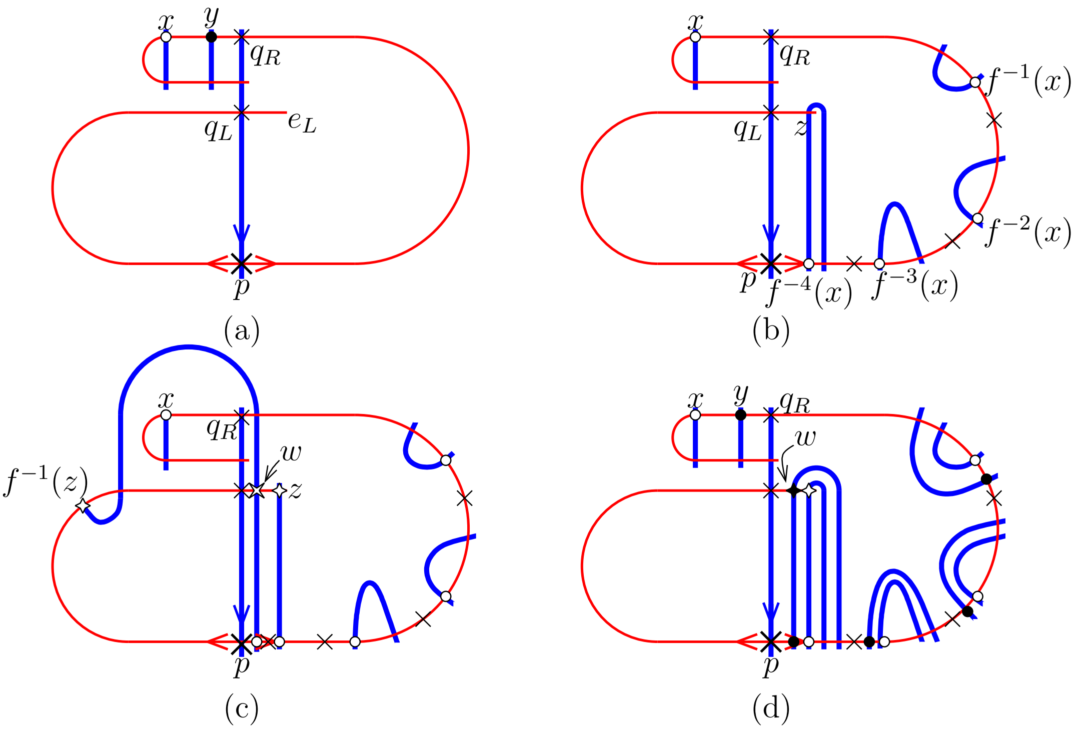

Figure 11: Constructing new homoclinic orbits.

Proof.

1. Let $S = W^S(x, \tilde{x})$. By the Lambda-lemma, $f^{-n}(S)$ converges to W^S as $n \to \infty$ in the C^1-topology. Hence, there exists m such that $f^{-m}(S)$ transversely intersects $T^U(q_L, e_L)$ at a point z, and by taking sufficiently large m, we can choose z so that $W^S(f^{-m}(x), z)$ does not intersect T^U.

2. Let S be the segment $T^S[f^{-m}(x), z]$. Since $f^{-(m+1)}(x) \in T^U(p, f^{-m}(x))$, $f^{-1}(z) \in T^U(p, q_L)$, and f is orientation-preserving, the arc $f^{-1}(S)$ crosses $W^U(q_L, x)$. Since $f^{-1}(S)$ cannot cross $T^U[p, f^{-m}(x)$, there must exist $w \in W^U(q_L, x) \cap f^{-1}(S)$ such that $W^S(f^{-(m+1)}(x), w)$ does not intersect T^U.

3. Since $x, \tilde{x} \in W^U(y, \tilde{y})$, the arc $W^S(f^{-m}(y), f^{-m}(\tilde{y}))$ lies "outside" of $W^S(f^{-m}(x), f^{-x}(\tilde{r}))$ and so must intersect $T^U(q_L, z)$.

□

The same argument shows that a similar result holds on interchanging the L and R subscripts.

We use the following naming scheme for homoclinic orbits. We let H_{L^∞} be the orbit of q_L, and H_{R^∞} be the orbit of q_R. If x is a point of a homoclinic orbit formed from an orbit $H_{R^\infty,\ldots}$ by the construction of Lemma 3.8, then we say that x is a point of the orbit $H_{L^\infty,R^{m+1},\ldots}$. We use $m+1$ for compatibility with the binary star graph labelling used in Section 3.1.

Theorem 3.9. *Let T be a trellis for a diffeomorphism f such that T has a fixed point $p \in T^P$ such that each branch of $T^U(p)$ intersects a branch of $T^S(p)$. Then $[T; f]$ is universal.*

Proof. Let H_L be the orbit of q_L and H_R be the orbit of q_R. By Lemma 3.8 there exists m_r such that there is a homoclinic orbit of type $H_{L^\infty R^{m_r}}$, and n_r such that there is an orbit of type $H_{R^\infty L^{n_r} R^{m_r}}$. Similarly, there exist m_l, n_l such that there is a homoclinic orbit of type $H_{L^\infty R^{n_l} L^{m_l}}$. If we let $m = \max\{m_l, m_r\}$ and $n = \max\{n_l, n_r\}$, then by Lemma 3.8 there is an orbit of type $H_{R^\infty L^n R^m}$ and an orbit of type $H_{L^\infty R^n L^m}$. We now take subsets of W^U and W^S containing points of these orbits, and prune to obtain a trellis type $[T_{m,n}; f_{m,n}]$ forced by $H_{R^\infty L^n R^m}$ and $H_{L^\infty R^n L^m}$ as shown in Figure 12. We now show that $\mathcal{T}_{m,n}$ is topologically universal.

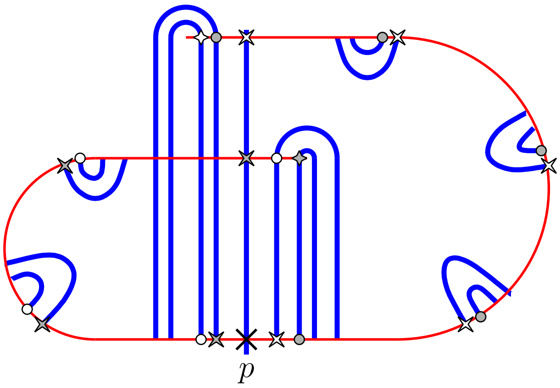

Figure 12: A topologically universal trellis $\mathcal{T}_{m,n}$.

The graph representative $g_{m,n}$ of $[T_{m,n}; f_{m,n}]$ is a binary star graph map and is shown in Figure 13.

By Theorem 3.2, the suspension of the graph map contains a universal template. This means that for any knot or link L, there is a braid B in the suspension of the graph map with knot type L. Since $g_{m,n}$ is the graph representative of $[T_{m,n}; f_{m,n}]$, this means that any diffeomorphism in the trellis mapping class has a collection of periodic orbits with braid B in the untwisted suspension flow by Theorem 3.4. Since $[T_{m,n}; f_{m,n}]$ is obtained from $[f^k(T^U), f^{-k}(T^S); f]$ by a pruning isotopy, it follows from Theorem 3.5 that f has a periodic orbit of braid B in the untwisted suspension flow. Hence f is universal. □

4 Conclusion

In this paper, we have shown that the existence of a pair of transverse homoclinic orbits to a hyperbolic saddle point which lie on different branches is sufficient to imply the presence of all

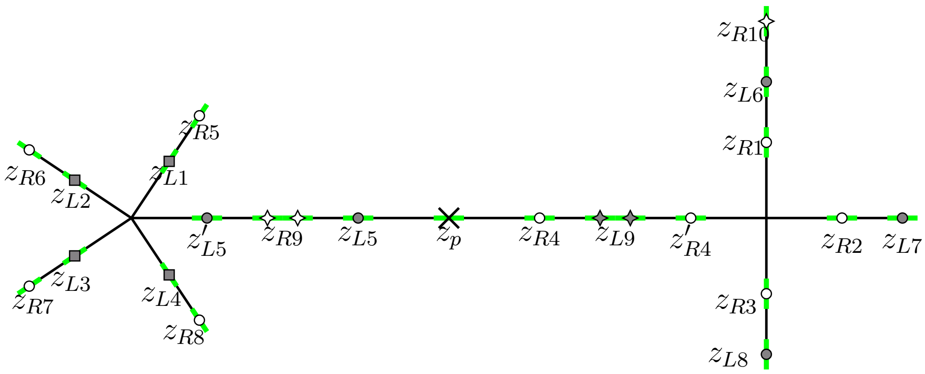

Figure 13: The graph representative $\}_{m,n}$ of $[T_{m,n}; f_{m,n}]$. The control edges are mapped $z_{Lm} \mapsto z_{L(m-1)}$ and $z_{Rm} \mapsto z_{R(m-1)}$.

knots and links in the natural suspension flow.

The paper combines ideas of template theory, embedded graph maps, trellises and homoclinic orbits, together with a number of constructions to pass from one object to another.

We pass from the homoclinic orbits to a template in a number of steps, first using the Lambda lemma to construct extra homoclinic orbits, a pruning isotopy to reduce to a known class of trellises, a deformation retract to collapse to the graph representative of the trellis type, and a suspension of the graph representative to obtain a template. This template can then shown to be universal by using results of [GK04].

The trellis type $[T; f]$ shown here is particularly simple, as it only has two homoclinic intersections. We conjecture that any universal diffeomorphism contains a pair of homoclinic orbits with with homoclinic braid type of q_L and q_R, making $[T; f]$ a "universal" universal trellis. and not some iterate.

Acknowledgement This work was carried out during a three-month stay with the Department of Mathematics at Kyoto University, Japan. The author would like to thank Hiroshi Kokubu for the invitation to visit, and Eiko Kin for many fruitful discussions which were invaluable in the writing of this paper. The author was partially supported by the 21-st century COE program "Formation of an International Center of Excellence in the Frontier of Mathematics and Fostering of Researchers in Future Generations".

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