Centrum voor Wiskunde en Informatica

# **REPORT***RAPPORT*

The Robinson-Schensted and Schützenberger Algorithms Part II: Geometric Interpretations

M.A.A. van Leeuwen

Department of Analysis, Algebra and Geometry

**AM-R9209 1992**

# The Robinson-Schensted and Schützenberger Algorithms
# Part II: Geometric Interpretations

Marc A. A. van Leeuwen

*CWI*

*P. O. Box 4079, 1009 AB Amsterdam, The Netherlands*

email: `M.van.Leeuwen@cwi.nl`

**Abstract**

There is a correspondence of Young tableaux to irreducible components of the variety $\mathcal{F}_u$ of flags fixed by some unipotent element $u \in \mathbf{GL}_n$, and a correspondence of permutations to relative positions pairs of flags. Using these, Steinberg has given an interpretation of the Robinson-Schensted algorithm; we elaborate this interpretation, and derive a similar interpretation for the Schützenberger algorithm. These interpretations clarify many of the key properties of those algorithms which were treated purely combinatorially in Part I of this paper. Interesting new interpretations of the individual insertion and extraction procedures used in the Robinson-Schensted algorithm, and of their transposed variants are also given. It is also described how these algorithms can be used to obtain explicit information (even if incomplete in general) about the irreducible components of the intersections of unipotent conjugacy classes in $\mathbf{GL}_n$ with a Borel subgroup, the so-called orbital varieties.

*1991 Mathematics Subject Classification:* 05E15, 20G15.

*Key Words and Phrases:* flag variety, unipotent conjugacy classes, orbital varieties.

*Note:* This work was partly supported by the *Schweizerische Nationalfonds*.

### Introduction to Part II.

In the second part of this paper we formulate and prove related geometric interpretations of the Robinson-Schensted and Schützenberger algorithms, of which the former is due to Steinberg [Stb2]. Of these interpretations, the latter will be given first, as is in fact easier to formulate and to prove. Both interpretations are based the fact that (normalised) Young tableaux of shape $\lambda$ correspond bijectively to the irreducible components of the set $\mathcal{F}_u$ of points on the flag manifold fixed by a unipotent transformation $u$, whose Jordan blocks have sizes specified by the parts of $\lambda$. In fact there are are two natural ways to define such a bijection, and this leads to the interpretation of the Schützenberger algorithm. For the interpretation of the Robinson-Schensted algorithm one needs in addition the concept of (generic) relative positions between pairs of flags. The interpretations shall be formulated in such a way that they can be immediately seen to imply the main theorems 3.1, 4.2, and 5.1 of Part I of this paper [vLee3], and also the interesting special case 6.8, which is intimately related to Schützenbergers theory of '*glissements*'. The symmetries expressed by these combinatorial theorems correspond to quite easily understood symmetries of the geometric situation, and in a sense this reveals the "witchcraft operating behind the scenes" (cf. [Kn2], p. 60) of these theorems. The parallel between the geometric interpretation of the algorithms and their (recursive) definitions is so close, that the algorithms could have been deduced from the geometric problems, had they not been known in advance. Indeed, for computing the generic relative positions in the varieties analogous to $\mathcal{F}_u$ for other classical groups than $\mathbf{GL}_n$ (in characteristic $\neq 2$), the author has derived in his thesis [vLee1] analogous (but significantly more complicated) algorithms by similar methods; it was that work which has inspired the current paper.

This part of the paper consists of nine sections, as follows. In the first section we recall notations introduced in Part I which shall also be used in this second part; most importantly we shall use the same recursive definitions of the basic algorithms as were used in Part I. The second section studies the linear algebra of a vector space equipped with a unipotent (or nilpotent) transformation, and the third section introduces the flag manifold, and fixed-point sets on it under a chosen unipotent transformation; it defines the interpretation of Young tableaux that will be central to our approach. The interpretation of the Schützenberger algorithm, whose formulation has become obvious at this point, is proved in §4. The fifth section introduces the concept of relative positions between flags, and various explicit ways to compute them, which enables the interpretations of the Robinson-Schensted algorithm and its transpose to be formulated and proved, which is done in §6. This completes the interpretations of the algorithms; the next two sections discuss some of their possible further applications, and they are of a more tentative nature then the previous sections. In particular we indicate in §8 that these interpretations are useful—although not dicisive—in understanding the irreducible components of the intersections of unipotent orbits in $\mathbf{GL}_n$ with a Borel subgroup (which are also called orbital varieties), and the inclusions among their closures. Finally §9 makes a few concluding remarks.

## §1. Notation.

We shall freely use the notation pertaining to partitions and Young tableaux which was introduced Part I of this paper [vLee3]. References to Part I shall be given by simply enclosing the equation or theorem number in square brackets. For reference we collect here the notations that are used with a brief description.

| *notation* | *description* | *defined where* | *notation* | *description* | *defined where* |
|---|---|---|---|---|---|
| $\lambda_i$ | $i$-th part of $\lambda$ | [§1] | $E(P,s)$ | extract from $P$ clearing square $s$ | [(11)] |
| $\lambda^t$ | transpose of $\lambda$ | [§1] | $R(P,Q)$ | Robinson-Schensted algorithm | [(9)] |
| $Y(\lambda)$ | Young diagram of $\lambda$ | [§1] | $I^t, E^t, R^t$ | transposes of $I, E, R$ | [§2] |
| $\operatorname{sh} T$ | shape of tableau $T$ | [§1] | $P^{\downarrow}$ | $P$ deflated once | [4.1] |
| $\operatorname{ch} T$ | chain in Young lattice | [§1] | $x \parallel y$ | $x$ is adjacent to $y$ | [4.1] |
| $T \sim T'$ | similarity of tableaux | [§1] | $D(P)$ | deflation procedure | [(19)] |
| $\mathcal{T}_\lambda$ | normalised tableaux for $\lambda$ | [§1] | $S(P)$ | Schützenberger algorithm | [(21)] |
| $\lceil T \rceil$ | highest numbered square | [§1] | $\tilde{w}$ | permutation $(n,\ldots,2,1)$ | [5.1] |
| $T^-$ | $T$ with $\lceil T \rceil$ removed | [§1] | $K^\diamond$ | rotate 180° and renumber | [6.6] |
| $I(T,m)$ | insert number $m$ into $T$ | [(12)] | | | |

## §2. Some linear algebra.

Let $k$ be an infinite field, and $V$ a vector space of dimension $n$ over $k$, equipped with a fixed nilpotent transformation $\eta$ of $V$. Choosing a basis of $V$, we identify $\mathbf{GL}_n(k)$ with the group $\mathbf{GL}(V)$ of automorphisms of $V$, and we define $u$ to be the unipotent element $\eta+\mathbf{1}$ of $\mathbf{GL}_n(k)$. By the theory of Jordan normal forms, $V$ can then be decomposed into a direct sum of *Jordan blocks* for $\eta$, or equivalently for $u$, i.e., $u$-stable subspaces that each admit a basis $x_1,\ldots,x_d$ such that $\eta(x_1)=0$ and $\eta(x_i)=x_{i-1}$ for $1<i\leq d$. This decomposition is generally not unique, but the multiset of dimensions of the blocks (i.e., disregarding order but counting multiplicities) is uniquely determined. These dimensions can be arranged into weakly decreasing order, forming partition $\lambda$ of $n$, which we call the *Jordan type* $J(\eta)$ of $\eta$, or equivalently the Jordan type $J(u)$ of $u$. This Jordan type can also be characterised without referring to any particular decomposition into Jordan blocks as follows: the number of squares in the first $i$ columns of the Young diagram $Y(\lambda)$ is equal to $\dim\operatorname{Ker}\eta^i$, and equivalently, the number of squares beyond the $i$-th column is equal to $\dim\operatorname{Im}\eta^i$.

Now consider a $u$-stable subspace $V'$ of $V$. By restriction to $V'$ we obtain nilpotent respectively unipotent transformations $\eta|_{V'}$ and $u|_{V'}$, and in the quotient space $V/V'$ we similarly obtain transformations, denoted by $\eta_{/V'}$ and $u_{/V'}$. To see that the Jordan types of these transformations are contained in $\lambda$ (i.e., all squares of their Young diagrams occur among those of $Y(\lambda)$), we may argue as follows. Considering restriction first, we obviously have $\operatorname{Ker}(\eta|_{V'})^i=V'\cap\operatorname{Ker}\eta^i$, and the codimension of the space within $\operatorname{Ker}\eta^i$ increases in a weakly monotonic way with $i$, from which we deduce that the length of each column of $Y(J(\eta|_{V'}))$ does not exceed the length of the corresponding column of $Y(\lambda)$. For the quotient space, the relevant values are $\dim(V'\cap\operatorname{Im}\eta^i)$, which determine how much $\dim\operatorname{Im}(\eta_{/V'})^i$ is less than $\dim\operatorname{Im}\eta^i$. Since these values clearly form a weakly decreasing sequence as $i$ increases, it follows that no column of $Y(J(\eta_{/V'}))$ can exceed the corresponding column of $Y(\lambda)$ in length. Note that $J(\eta|_{V'})$ is determined by the values $\dim(V'\cap\operatorname{Ker}\eta^i)$ whereas $J(\eta_{/V'})$ depends on $\dim(V'\cap\operatorname{Im}\eta^i)$, whence there is in general no direct relationship between these two partitions. There is one circumstance in which such a direct relationship does exist, namely when all Jordan blocks of $\eta$ have the same dimension $d$: then $\lambda$ is a 'rectangular' partition $(d,d,\ldots,d)$, and we have $\operatorname{Ker}\eta^i=\operatorname{Im}\eta^{(d-i)}$. It follows that if $J(\eta|_{V'})=(\mu_1,\ldots,\mu_m)$, then $J(\eta_{/V'})=(d-\mu_m,\ldots,d-\mu_1)$, where $m$ is the number of parts $d$ of $\lambda$.

We now specialise to the case where $V'$ is either of dimension 1 or of codimension 1. In the former case we are dealing with a $u$-stable line, say $l$, which is an element of the projective space $\mathbf{P}(V)$ of $V$. Since $\eta|_l$ is nilpotent it must be zero, and we consequently have $l\subseteq\operatorname{Ker}\eta$; conversely $u$ fixes $\mathbf{P}(\operatorname{Ker}\eta)$ pointwise, so $l$ can be any of its elements. Now define for each $i>0$ a subspace

$$W_i(u)=\operatorname{Im}\eta^{i-1}\cap\operatorname{Ker}\eta \tag{1}$$

of $\operatorname{Ker}\eta\subseteq V$, and also a subvariety of the projective space $\mathbf{P}(\operatorname{Ker}\eta)\subseteq\mathbf{P}(V)$:

$$U_i(u)=\mathbf{P}(W_i(u))\setminus\mathbf{P}(W_{i+1}(u)). \tag{2}$$

We shall abbreviate $W_i(u)$ and $U_i(u)$ to $W_i$ and $U_i$ respectively. We have $\dim W_i=\lambda^t_i$, and hence $U_i$ is non-empty if and only if $i$ occurs as a (non-zero) part of $\lambda$, and the non-empty $U_i$ form a finite partition of the set $\mathbf{P}(\operatorname{Ker}\eta)$. The centraliser $Z_u$ of $u$ in $\mathbf{GL}_n$ acts on each variety $U_i$, and using a decomposition into Jordan blocks it can easily be shown that these are transitive actions; therefore the non-empty $U_i$ are in fact the orbits in $\mathbf{P}(\operatorname{Ker}\eta)$ under the action of $Z_u$. Let $j$ be such that $l\in U_j$, then it is the minimal value for which $l\not\subseteq\operatorname{Im}\eta^j$, and by the reasoning above we see that the Young diagram of $J(\eta_{/l})$ is obtained from $Y(\lambda)$ be removing a square from the end of the $j$-th column.

For the case of a $u$-stable hyperplane $H\subset V$ the situation is dual to that of a line. Here the transformation $\eta_{/H}$ is zero and consequently $H\supseteq\operatorname{Im}\eta$. To $H$ corresponds a subspace $H^\circ$ of the dual vector space $V^*$ of $V$, consisting of linear forms vanishing on $H$; since $\dim H^\circ=1$ we have $H^\circ\in\mathbf{P}(V^*)$. In this way the set of hyperplanes containing $\operatorname{Im}\eta$ corresponds to the projective subspace $\mathbf{P}(\operatorname{Ker}\eta^*)$ of $\mathbf{P}(V^*)$, where $\eta^*$ is the nilpotent transformation induced in $V^*$ by $\eta$. In $V^*$ and $\mathbf{P}(V^*)$ we have subspaces $W^*_i=W_i(u^*)$ respectively subvarieties $U^*_i=U_i(u^*)$, where $u^*=\eta^*+\mathbf{1}$; we also write $U^\circ_i$

for the set of hyperplanes $H \subset V$ with $H^\circ \in U_i^*$. Note that $J(u^*) = J(u)$, so that $\dim W_i^* = \dim W_i$. Now the Young diagram $Y(J(u|_H))$ is obtained from $Y(\lambda)$ by removing a square from the end of the $j$-th column, where in this case $j$ is such that $H \in U_j^\circ$, which is the minimal $j$ for which $H \not\supseteq \operatorname{Ker} \eta^j$.

## §3. Flags.

A (complete) *flag* $f$ in $V$ is a saturated chain $0 = f_0 \subset f_1 \subset \cdots \subset f_n = V$ of subspaces of $V$. We have $\dim f_i = i$, and the individual spaces $f_i$ are called the *parts* of $f$. We define $\mathcal{F}$ to be the set of all such flags, called the *flag variety* of $V$. It has the structure of a projective algebraic variety (see [Hum], 8.1), and the maps $f \mapsto f_i$ are morphisms onto the respective Grassmann varieties. Of particular interest are the maps giving the line and hyperplane parts: we define $\alpha: \mathcal{F} \to \mathbf{P}(V)$ by $\alpha: f \mapsto f_1$, and $\omega: \mathcal{F} \to \mathbf{P}(V^*)$ by $\omega: f \mapsto f_{n-1}{}^\circ$. The group $\mathbf{GL}_n$ acts on $\mathcal{F}$, and clearly $\alpha$ and $\omega$ are $\mathbf{GL}_n$-equivariant. We define $\mathcal{F}_u$ as the subvariety of flags fixed by $u$ (which is easily seen to be non-empty), and $\alpha_u$ and $\omega_u$ as the restrictions to $\mathcal{F}_u$ of $\alpha$ and $\omega$ respectively. As we have seen above, the image $\operatorname{Im} \alpha_u = \mathbf{P}(\operatorname{Ker} \eta)$, and similarly $\operatorname{Im} \omega_u = \mathbf{P}(\operatorname{Ker} \eta^*)$.

For $l \in \mathbf{P}(V)$, each flag $f \in \alpha^{-1}[l]$ determines a flag $f^\downarrow$ in $V/l$ by $f_i^\downarrow = f_{i+1}/l$ for $0 \leq i < n$; it is easily seen that this defines an isomorphism between $\alpha^{-1}[l]$ and the flag variety of $V/l$. If moreover $f \in \mathcal{F}_u$, then $l$ is $u$-stable and $f^\downarrow \in \mathcal{F}_{u_{/l}}$; this induces an isomorphism between $\alpha_u^{-1}[l]$ and $\mathcal{F}_{u_{/l}}$. Similarly for each hyperplane $H \subset V$, the inverse image $\omega^{-1}[H^\circ]$ is isomorphic to the flag variety of $H$, where the image $f^-$ of $f \in \mathcal{F}$ is obtained by simply omitting the largest part $f_n$ of $f$; this also gives an isomorphism between $\omega_u^{-1}[H^\circ]$ and $\mathcal{F}_{u|_H}$.

The formation of $f^-$ out of $f$ can be repeated, yielding a flag $f^{--}$ in the subspace $f_{n-2}$ which is fixed by $u|_{f_{n-2}}$, and so on. Similarly $f^{\downarrow\downarrow}$ is a flag in $V/f_2$ fixed by $u_{/f_2}$; the operations can also be mixed: $f^{-\downarrow} = f^{\downarrow -}$ is a flag in the subquotient space $f_{n-1}/f_1$ fixed by the unipotent induced by $u$ in this space, which shall be denoted as $u_{f_{n-1}/f_1}$. We may take Jordan types of all these induced unipotents; in particular is we consider the sequence of pure restrictions respectively of pure quotients we obtain two saturated decreasing chains in the Young lattice. These chains determine normalised Young tableaux $r_u(f), q_u(f) \in \mathcal{T}_\lambda$, satisfying

$$\operatorname{ch} r_u(f) = \big(J(u), J(u|_{f_{n-1}}), J(u|_{f_{n-2}}), \ldots, (0)\big) \tag{3}$$

respectively

$$\operatorname{ch} q_u(f) = \big(J(u), J(u_{/f_1}), J(u_{/f_2}), \ldots, (0)\big). \tag{4}$$

In other words, the subtableau of $r_u(f)$ containing entries $\leq i$ has shape $J(u|_{f_i})$, while the subtableau of $q_u(f)$ containing entries $\leq n-i$ has shape $J(u_{/f_i})$. Defining for each flag $f \in \mathcal{F}$ a dual flag $f^*$ in the flag variety of $V^*$ by $f_i^* = f_{n-i}{}^\circ$, we obviously have $q_u(f) = r_{u^*}(f^*)$, and vice versa.

Since we know that there is generally no direct relation between $J(u|_{f_i})$ and $J(u_{/f_i})$, we should not expect a one-to-one correspondence between $r_u(f)$ and $q_u(f)$ either. Indeed such correspondence exists only in the special case noted earlier, when $\lambda$ is a rectangular partition. For that case we obtain

**3.1. Proposition.** *If $u$ has a rectangular Jordan type, then $q_u(f) = r_u(f)^\diamond$ for all $f \in \mathcal{F}_u$.*

*Proof.* Recall that the operation $T \mapsto T^\diamond$ means rotate 180° and renumber the entries in opposite order. Let $\lambda$ consist of $m$ equal parts $d$; it follows from earlier remarks that for any $i \leq md$ if the shape of the subtableau of $r_u(f)$ containing entries $\leq i$ is $\mu$, then the subtableau of $q_u(f)$ containing entries $\leq n-i$ has shape $(d-\mu_m, \ldots, d-\mu_1)$. Combining this information for all $i$ we obtain the stated result. □

We now define for any Young tableau $T$ of shape $\lambda$:

$$\mathcal{F}_{u,T} = \{\, f \in \mathcal{F}_u \mid r_u(f) \sim T \,\} \qquad \text{and} \qquad \mathcal{F}_{u,T}^* = \{\, f \in \mathcal{F}_u \mid q_u(f) \sim T \,\}; \tag{5a, b}$$

a simple inductive argument shows that these sets are non-empty and open in their own closure. The restrictions of $\alpha_u$ and $\omega_u$ to $\mathcal{F}_{u,T}$ and $\mathcal{F}_{u,T}^*$ will be denoted respectively as $\alpha_{u,T}$, $\omega_{u,T}$, $\alpha_{u,T}^*$, and $\omega_{u,T}^*$.

As $T$ ranges over $\mathcal{T}_\lambda$ the sets $\mathcal{F}_{u,T}$ partition $\mathcal{F}_u$ into finitely many subsets, and so do the sets $\mathcal{F}^*_{u,T}$; note however that the expressions $\mathcal{F}_{u,T}$ and $\mathcal{F}^*_{u,T}$ are also defined when $T$ is not normalised.

*Note.* It is a somewhat arbitrary choice to use $r_u$ rather than $q_u$ for the unstarred notation; this choice follows [Stb2], but [Spa], II.5.3 effectively uses $q_u$, not $r_u$. As we have seen, the entries of $r_u(f)$ relate somewhat more directly to the indices of the parts of $f$ than those of $q_u(f)$; also the interpretation of the Robinson-Schensted algorithm will come out more naturally with the chosen convention. On the other hand our choice is slightly unnatural, since the column of the highest numbered square $\lceil r_u(f)\rceil$ corresponds to the set $U_i^*$ containing $\omega(f)$, while the column of $\lceil q_u(f)\rceil$ determines the set $U_i$ containing $\alpha(f)$.

**3.2. Proposition.**
*(a) For each $T \in \mathcal{T}_\lambda$ the sets $\mathcal{F}_{u,T}$ and $\mathcal{F}^*_{u,T}$ are irreducible.*
*(b) $\dim \mathcal{F}_{u,T} = \dim \mathcal{F}^*_{u,T} = \sum_i (i-1)\lambda_i$ independently of $T \in \mathcal{T}_\lambda$.*

*Proof.* We give the proof for $\mathcal{F}^*_{u,T}$, by induction on $|\lambda| = \dim V$; the case $\mathcal{F}_{u,T}$ follows by transition to the dual vector space. Note first that by construction each set $\mathcal{F}^*_{u,T}$ is $Z_u$-stable. For each $j$ occurring as part of $\lambda$, the set $\alpha_u^{-1}[U_j]$ is the union of the sets $\mathcal{F}^*_{u,T}$ for those tableaux $T \in \mathcal{T}_\lambda$ for which $\lceil T\rceil$ lies in column $j$. If $T$ is such a tableau and $l \in U_j$, then under the isomorphism $\alpha_u^{-1}[l] \xrightarrow{\sim} \mathcal{F}_{u_{/l}}$ the subset $C = \alpha^{*\,-1}_{u,T}[l]$ maps isomorphically to $\mathcal{F}^*_{u_{/l},T^-}$, which is irreducible by the induction hypothesis. Because $U_j$ is a $Z_u$-orbit and connected, it is already a $Z_u^\circ$-orbit ($Z_u^\circ$ is the identity component of the group $Z_u$; in fact for $\mathbf{GL}_n$ every unipotent centraliser $Z_u$ is connected, so $Z_u^\circ = Z_u$, but the proof does not use this fact). Since $\mathcal{F}^*_{u,T}$ is the surjective image of $Z_u^\circ \times C$ by the map $(z,f) \mapsto z \cdot f$—we shall abbreviate such an image simply as $Z_u^\circ \cdot C$—it is irreducible, which proves part (a). We also have $\dim \mathcal{F}^*_{u,T} = \dim U_j + \dim \mathcal{F}^*_{u_{/l},T^-}$, so to prove (b) it suffices to show that $\dim U_j = i-1$, where $i$ is the row number of $\lceil T\rceil$. But this is clear since $i = \lambda^t_j$, which equals $\dim W_j$, while $U_j$ is a dense open part of $\mathbf{P}(W_j)$. □

It follows from the proposition that the set of irreducible components of $\mathcal{F}_u$ can be described as the set of the closures $\overline{\mathcal{F}_{u,T}}$ for $T \in \mathcal{T}_\lambda$, but at the same time as the set of closures $\overline{\mathcal{F}^*_{u,T}}$ for $T \in \mathcal{T}_\lambda$. So, although $r_u(f)$ and $q_u(f)$ do not determine one another completely, there does exist a one-to-one correspondence which holds on a dense subset of $\mathcal{F}_u$. In the next section we shall show that it is in fact given by the Schützenberger algorithm:

$$\overline{\mathcal{F}_{u,T}} = \overline{\mathcal{F}^*_{u,S(T)}}. \tag{6}$$

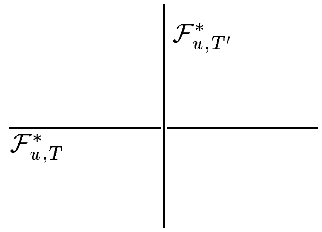
We close this section with an example, illustrating these parametrisations of the irreducible components of $\mathcal{F}_u$ in the simplest non-trivial case, namely for the Jordan type $\lambda = (2,1)$. Then $\mathcal{T}_\lambda$ has 2 elements, namely

$$T = \begin{array}{|c|c|}\hline 1 & 2 \\ \hline 3 \\ \cline{1-1}\end{array} \qquad \text{and} \qquad T' = \begin{array}{|c|c|}\hline 1 & 3 \\ \hline 2 \\ \cline{1-1}\end{array},$$

and hence $\mathcal{F}_u$ has 2 irreducible components, which are 1-dimensional. To be specific, we take

$$u = \begin{pmatrix} 1 & 1 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}.$$

Calling the standard basis vectors $e_1$, $e_2$, $e_3$ we have $W_1 = \langle e_1, e_3\rangle$, $W_2 = \langle e_1\rangle$, and $W_j = 0$ for $j > 2$. There are two orbits of $Z_u$ on $\mathbf{P}(\operatorname{Ker}\eta) = \mathbf{P}(W_1)$, namely the affine line $U_1 = \mathbf{P}(W_1) \setminus \{\langle e_1\rangle\}$ and the point $U_2 = \{\langle e_1\rangle\}$. For any $l \in U_1$ and flag $f \in \alpha_u^{-1}[l]$ we have $f_1 = l$, and since the hyperplane $f_2$ must contain both $f_1$ and $\operatorname{Im}\eta = \langle e_1\rangle$, it can only be $\langle e_1, e_3\rangle$. Therefore $\alpha_u^{-1}[l]$ consists of just one flag for any $l \in U_1$, and $\mathcal{F}^*_{u,T}$ is isomorphic as variety to $U_1$. On the other hand $\mathcal{F}^*_{u,T'}$, which equals the fiber $\alpha_u^{-1}[l]$ for $l = \langle e_1\rangle \in U_2$, is a projective line, since $f_2$ may be chosen to be any plane containing $f_1 = l$. Of these flags in $\mathcal{F}^*_{u,T'}$ there is one that lies in the closure of $\mathcal{F}^*_{u,T}$, namely the one that has $f_2 = \langle e_1, e_3\rangle$. Therefore, the whole variety $\mathcal{F}_u$ can be depicted as




The map $\alpha_u$ corresponds to a vertical projection in this picture, while $\omega_u$ corresponds to a horizontal projection. It follows that $\mathcal{F}_{u,T}$ is the vertical line in the picture, but without the intersection point, and $\mathcal{F}_{u,T'}$ is the horizontal line including that point. Since $S(T) = T'$ we have agreement with (6), and one sees that the closures in that equation cannot be omitted.

## §4. Interpretation of the Schützenberger algorithm.

This section is devoted to the proof of (6). To that end we must consider the line and hyperplane parts of a flag in relation to one another. We first consider for a $u$-stable hyperplane $H$ the spaces $W_i(u|_H)$, which are the analogs of the $W_i$ after restricting to $H$.

**4.1. Proposition.** *For $H \in U_j^\circ$ we have $W_c(u|_H) = W_c$ for all $c \neq j$, while $W_j(u|_H)$ is a subspace of codimension 1 in $W_j$.*

*Proof.* Obviously $W_c(u|_H) \subseteq W_c$ for all $c$, and since $\dim W_c(u|_H)$ is determined by $J(u|_H)$, a dimension argument suffices to prove the claims. □

Suppose we know for $H \in U_j^\circ$ and some irreducible component $C'$ of $\omega_u^{-1}[H^\circ]$ the part $c'$ of $J(u|_H)$ for which $\alpha_u[C'] = \mathbf{P}(W_{c'}(u|_H))$ (because $\alpha_u[C']$ is a closed $Z_{u|_H}$-stable subset of $\mathbf{P}(\operatorname{Ker}\eta|_H)$, such a $c'$ exists). There is a unique irreducible component $C = \overline{Z_u \cdot C'}$ of $\mathcal{F}_u$ determined by $C'$, and we want to know the part $c$ of $\lambda$ for which $\alpha_u[C] = \mathbf{P}(W_c)$. Now $\alpha_u[C]$ is (the closure of) $\bigcup_{H \in U_j^\circ} \mathbf{P}(W_{c'}(u|_H))$, and from 4.1 it follows that we always have $\alpha_u[C] = \mathbf{P}(W_{c'})$, because if $c' = j$ (the only case where the sets in the union actually depend on $H$) we have that $W_{c'+1} = W_{c'+1}(u|_H)$ is strictly contained in $W_{c'}(u|_H)$ (since $c'$ is a part of $J(u|_H)$), so the union must be $W_{c'}$. However, if $j = c'+1$ and $\dim W_j = \dim W_{c'}$, then $W_j = W_{c'}$ and $c'$ in not a part of $\lambda$, so we must have $c = j = c'+1$; in all other cases $c = c'$. We can state the result in a slightly different way, realising that a coordinate pair such as $p = (\dim W_j, j)$ can be interpreted as a square. Then, with $s' = (\dim W_{c'}(u|_H), c')$ and $s = (\dim W_c, c)$ we have $s = s'$ unless $s' \parallel p$, in which case $s = p$. A relation with the deflation procedure $D$ of the Schützenberger algorithm becomes apparent; it will be formulated in the next lemma.

Looking a this situation in another way, our assumption implies that for flags $f$ in a dense open subset of $C$ we have $l \in U_{c'}(u|_H)$ where $l = f_1$ and $H = f_{n-1}$. Then $J(u|_H)$ is obtained from $\lambda$ by decreasing a part $j$, and $J(u_{H/l})$ is obtained from it by further decreasing a part $c'$. The calculated value $c$ is such that $l \in U_c$ on a dense open subset of $C$, and thus is the part of $\lambda$ to be decreased in order to obtain $J(u_{/l})$ on that subset. Note that when $j = c'+1$ but $\dim W_j < \dim W_{c'}$, some flags $f$ in the first mentioned subset of $C$ will have $l \in U_j$, thereby being excluded from the second subset; however this set of aberrant flags is of positive codimension, an hence its complement is still dense. This situation is typical for many of the proofs below.

**4.2. Lemma.** *Let $P$ be a non-empty tableau, from which we compute $(P^\downarrow, s, m) = D(P)$ by the deflation procedure, and let $c$ be the column number of the square $s$. Then $\alpha[\overline{\mathcal{F}_{u,P}}] = \mathbf{P}(W_c)$, and there exists a dense open $Z_u$-stable subset $\mathcal{D}$ of $\mathcal{F}_{u,P}$ such that $f^\downarrow \in \mathcal{F}_{u_{/f_1},P^\downarrow}$ for all $f \in \mathcal{D}$.*

*Proof.* We use induction on the number of squares of $P$; without loss of generality we assume that $P$ is normalised. If $P$ has just one square the lemma is trivially true. Otherwise, let $j$ be the column number of $\lceil P \rceil$, so that we have $\omega_u[\mathcal{F}_{u,P}] = U_j^*$. By [(19)] we have $D(P^-) = (P^{\downarrow -}, s', 1)$ for some square $s'$; let $c'$ be the column number of $s'$. For an arbitrary hyperplane $H \in U_j^\circ$ we have by the induction hypothesis that $\alpha_{u|_H}[\overline{\mathcal{F}_{u|_H,P^-}}] = \mathbf{P}(W_{c'}(u|_H))$. The irreducible component $C' = \overline{\omega_{u,P}^{-1}[H^\circ]}$ of $\omega_u^{-1}[H^\circ]$ is isomorphic to $\overline{\mathcal{F}_{u|_H,P^-}}$ by $f \mapsto f^-$; we therefore find $\alpha_u[C'] = \mathbf{P}(W_{c'}(u|_H))$. We are now in the situation discussed above, and comparing with [(19)] we see that $s$ is computed in the proper way to allow the conclusion $\alpha[\overline{\mathcal{F}_{u,P}}] = \mathbf{P}(W_c)$.

The remaining claim is now easily proved. On one hand we have by the induction hypothesis a dense open subset $\mathcal{D}_H$ of $\omega_{u,P}^{-1}[H^\circ]$ such that $r_{u_{H/f_1}}(f^{-\downarrow}) \sim P^{\downarrow -}$ for all $f \in \mathcal{D}_H$, on the other hand we have for $f$ in the dense open subset $\mathcal{D}' = \alpha_{u,P}^{-1}[U_c]$ of $\mathcal{F}_{u,P}$ that $J(u_{/f_1}) = \operatorname{sh} P^\downarrow$. We can take $\mathcal{D} = (Z_u \cdot \mathcal{D}_H) \cap \mathcal{D}'$ and find for $f \in \mathcal{D}$, using $f^{\downarrow -} = f^{-\downarrow}$, that all partitions in the chain $\operatorname{ch} r_{u_{/f_1}}(f^\downarrow)$ correspond to those in $\operatorname{ch} P^\downarrow$. □

We come to the main theorem of this section.

**4.3. Theorem.** *Let $P \in \mathcal{T}_\lambda$, and let $P^* = S(P)$ be obtained from it by the Schützenberger algorithm, then the intersection $\mathcal{F}_{u,P} \cap \mathcal{F}^*_{u,P^*}$ is dense in both $\mathcal{F}_{u,P}$ and $\mathcal{F}^*_{u,P^*}$.*

*Proof.* By induction on the size of the tableaux, the case of the empty tableau being trivial. Applying the lemma and the definition [(21)] of the algorithm $S$, we see that $J(u_{/f_1}) = \operatorname{sh} P^\downarrow = \operatorname{sh} P^{*-}$ and $f^\downarrow \in \mathcal{F}_{u_{/f_1},P^\downarrow}$ for all $f$ in the subset $\mathcal{D} \subseteq \mathcal{F}_{u,P}$ of the lemma. Of those $f$, a dense open subset has $f^\downarrow \in \mathcal{F}^*_{u_{/f_1},S(P^\downarrow)}$ by the induction hypothesis, and since $S(P^\downarrow) = P^{*-}$ this implies $f \in \mathcal{F}^*_{u,P^*}$. Since all irreducible components of $\mathcal{F}_u$ have the same dimension, this set is also dense in $\mathcal{F}^*_{u,P^*}$. □

The theorem immediately implies (6). Since $f \in \mathcal{F}^*_{u,T}$ is equivalent to $f^* \in \mathcal{F}_{u^*,T}$, and $f^{**} = f$, it also implies $S(S(P)) = P$, providing an alternative proof of [Theorem 4.2]. In combination with 3.1 we find $S(P) = P^\diamond$ for tableaux $P$ of rectangular shape, so we also get an alternative proof for [Corollary 6.8]; this is particularly noteworthy since this proof does not involve the Robinson-Schensted algorithm at all, while our combinatorial proof was based on the relationship between that algorithm and the Schützenberger algorithm. Note that there is one combinatorially obvious property of $S$ which has no clear interpretation geometrically, namely that it commutes with transposition. Indeed we know of no operation which would lead to transposition of Jordan types, which also means that there will be no simple geometric argument relating the interpretations of the Robinson-Schensted algorithm $R$ and its transpose $R^t$.

## §5. Relative positions.

The interpretation we shall give of the Robinson-Schensted algorithm also uses the the correspondence of tableaux to irreducible components of $\mathcal{F}_u$, but in addition needs the concept of relative positions of flags. The relative position of an ordered pair $(f, f')$ of flags in $\mathcal{F}$ describes the orbit of the pair under the diagonal action of $\mathbf{GL}_n$ on $\mathcal{F} \times \mathcal{F}$. Giving the relative position of $f$ and $f'$ is equivalent to giving the values $\dim(f_i \cap f'_j)$ for all $0 < i, j < n$. However, these numbers are not entirely independent; therefore a relative position is better parametrised in a different way, namely by a permutation of $n$, which we shall denote $\pi(f, f')$. If $f$ is the standard flag $\mathbf{f}$ for the basis $e_1, \ldots, e_n$, given by $\mathbf{f}_i = \langle e_1, \ldots, e_i \rangle$, and $f'$ is the standard flag $\mathbf{f}^\sigma$ for this basis permuted by some $\sigma \in \mathbf{S}_n$, given by $\mathbf{f}^\sigma_i = \langle e_{\sigma_1}, \ldots, e_{\sigma_i} \rangle$, then $\pi(\mathbf{f}, \mathbf{f}^\sigma) = \sigma$ by definition. From Bruhat's lemma for $\mathbf{GL}_n$ it follows that together with $\pi(g \cdot f, g \cdot f') = \pi(f, f')$ for all $g \in \mathbf{GL}_n$, this uniquely defines $\pi(f, f')$ for all $f, f' \in \mathcal{F}$, in other words, the general situation can be reduced to the special case by replacing the standard basis $e_1, \ldots, e_n$ by another ordered basis specially adapted to the pair $(f, f')$.

We give some examples. For every $f \in \mathcal{F}$ we have $\pi(f, f) = \mathbf{e}$, the identity permutation. The other extreme occurs when $f, f'$ are generically chosen: then $f_i \cap f'_j$ is zero whenever it can, i.e., whenever $i + j \leq n$, and $\pi(f, f')$ is the order reversing permutation $\tilde{w} \in \mathbf{S}_n$ of [5.1]. In the example with $J(u) = (2, 1)$ given in §3, where $\mathcal{F}_u$ consisted of two intersecting lines, we have $\pi(f, f') = (2, 1, 3)$ for any pair of distinct $f, f' \in \mathcal{F}^*_{u,T}$ (since only their 1-dimensional parts differ), and $\pi(f, f') = (2, 3, 1)$ for any $f \in \mathcal{F}^*_{u,T}$ and any $f' \in \mathcal{F}^*_{u,T'}$ except the flag at the intersection of the two components of $\mathcal{F}_u$ (since $f'_1 \subseteq f_2$ but $f'_2 \not\supseteq f_1$). If $\pi(f, f') = \sigma$, and $\sigma = s_1 \cdots s_l$ is an expression of minimal length for $\sigma$ as product of transpositions $s_i \in \{(1\,2), (2\,3), \ldots, (n-1 \;\, n)\}$, then there is a unique sequence of flags $f^0 = f, f^1, \ldots\ f^l = f'$ such that $\pi(f^{i-1}, f^i) = s_i$, i.e., if $s_i = (d \;\; d+1)$, then $f^{i-1}$ and $f^i$ differ only in their $d$-dimensional part. When

$f, f' \in \mathcal{F}_u$, then all these intermediate flags, and the projective lines connecting successive pairs, also lie in $\mathcal{F}_u$; this follows from the uniqueness of the sequence and the fact that $\mathcal{F}_u$ is a fixed-point variety. We shall not employ such considerations, however. It is clear from the definition that $\pi(f', f) = \pi(f, f')^{-1}$ for all $f, f' \in \mathcal{F}$, and also that $\pi(f^*, f'^*) = \tilde{w}\pi(f, f')\tilde{w}$ (the latter identity comes from the fact that the dual standard flag $\mathbf{f}^*$ is given by $\mathbf{f}_i^* = \langle e_{n+1-i}^*, \ldots, e_n^* \rangle = \langle e_{\tilde{w}_i}^*, \ldots, e_{\tilde{w}_1}^* \rangle$ on the dual standard basis).

We can describe $\sigma = \pi(f, f')$ more explicitly in two ways. First, the permutation $\sigma$ may expressed in terms of the numbers $a_{i,j} = \dim(f_i \cap f_j')$ as $\sigma_j = \min\{\, i \mid a_{i,j} > a_{i,j-1} \,\}$, and its permutation matrix as

$$\delta_{i,\sigma_j} = a_{i,j} - a_{i,j-1} - a_{i-1,j} + a_{i-1,j-1} \qquad \text{for } 1 \leq i, j \leq n; \tag{7}$$

we have conversely that $a_{i,j} = \#\{\, k \leq j \mid \sigma_k \leq i \,\}$. We define a partial ordering on relative positions by putting $\sigma \leq \sigma'$ if and only if $a_{i,j} \geq a'_{i,j}$ for all $i, j$, where the $a'_{i,j}$ are the numbers analogous to $a_{i,j}$, but for $\sigma'$ instead of $\sigma$. This is called the Bruhat order on $\mathbf{S}_n$, and it has the property that the closure in $\mathcal{F} \times \mathcal{F}$ of $\{\, (f, f') \mid \pi(f, f') = \sigma \,\}$ for any fixed $\sigma$ is $\{\, (f, f') \mid \pi(f, f') \leq \sigma \,\}$.

Second, $\pi(f, f')$ can be determined in a recursive way, and it is this form that we shall be using in the sequel. Define the relative position $\pi(f, l)$ of a flag $f$ and a line $l$ by

$$\pi(f, l) = \min\{\, i \mid f_i \supseteq l \,\}. \tag{8}$$

Then the first term in the sequence $\sigma$ is given by $\sigma_1 = \pi(f, l)$ where $l = f_1'$. The remaining values of $\sigma$ are computed from the relative position of a pair of flags in the space $V/l$, namely $f_{/l}$ and $f'^{\downarrow}$, where $f_{/l}$ is defined as follows. For $i < \pi(f, l)$ put $(f_{/l})_i = f_i \oplus l/l$, and for $\pi(f, l) \leq i < n$ put $(f_{/l})_i = f_{i+1}/l$. Now let $\sigma' \in \mathbf{S}_{n-1}$ be the relative position $\pi(f_{/l}, f'^{\downarrow})$, then the remaining values of $\sigma$ are defined by

$$\sigma_i = \begin{cases} \sigma_i' & \text{if } \sigma_i' < \sigma_1 \\ \sigma_i' + 1 & \text{if } \sigma_i' \geq \sigma_1 \end{cases} \qquad (\text{for } i > 1). \tag{9}$$

Note that adding 1 in the second case ascertains that we get a proper permutation, and that in each case the part $(f_{/l})_{\sigma_i'}$ was originally obtained from $f_{\sigma_i}$. The latter remark implies that if we would endow the parts of flags derived from $f$ with numeric labels to indicate which part of $f$ they stem from (setting the label equal to the dimension for parts of $f$ itself, but keeping the label unaltered when dividing out the line $l$) and change the right hand side of (8) so that it returns the label of $f_i$ rather than its dimension $i$, then (9) could simply read $\sigma_i = \sigma_i'$ $(i > 1)$ (but we would not have $\sigma' \in \mathbf{S}_{n-1}$).

The correctness of both explicit descriptions of $\pi(f, f')$ can be verified easily for $\pi(\mathbf{f}, \mathbf{f}^\sigma)$. Incidentally, there is an another recursive description of $\pi(f, f')$, which starts with giving $\sigma_n = \min\{\, i \mid f_i \not\subseteq f_{n-1}' \,\}$, and uses $\pi(f|_H, f'^{-})$ for $H = f_{n-1}'$ and suitably defined $f|_H$ to find the remaining values of $\sigma$. We shall not use that second recursive description however.

## §6. Interpretation of the Robinson-Schensted algorithm.

In terms of relative positions there is a geometric interpretation of the Robinson-Schensted algorithm analogous to that of the Schützenberger algorithm. We need some additional notation. On every irreducible subset $X$ of $\mathcal{F}\times\mathcal{F}$ the relative position $\pi$ assumes its maximal value on a dense subset of $X$; that value is called the generic relative position on $X$, and shall be denoted $\gamma(X)$. We shall prove that the generic relative positions on the irreducible components of $\mathcal{F}_u\times\mathcal{F}_u$ can be expressed as follows:

$$\gamma(\overline{\mathcal{F}_{u,P}}\times\overline{\mathcal{F}_{u,Q}}) = R(P,Q) \tag{10}$$

$$\gamma(\overline{\mathcal{F}^*_{u,P}}\times\overline{\mathcal{F}_{u,Q}}) = \tilde{w}R^t(P,Q) \tag{11}$$

$$\gamma(\overline{\mathcal{F}_{u,P}}\times\overline{\mathcal{F}^*_{u,Q}}) = R^t(P,Q)\tilde{w} \tag{12}$$

$$\gamma(\overline{\mathcal{F}^*_{u,P}}\times\overline{\mathcal{F}^*_{u,Q}}) = \tilde{w}R(P,Q)\tilde{w} \tag{13}$$

A similarity to the combinatorial [Theorem 5.1] is obvious, and indeed that theorem follows when these identities have been proved. We need not prove all four identities, since by $\pi(f^*,f'^*)=\tilde{w}\pi(f,f')\tilde{w}$, the first two identities are equivalent to the last two. We choose to prove last two identities, which choice matches using $\pi(f,l)$ for computing relative positions: computing $\pi(f,l)$ gives the *first* term of a relative position while $R$ and $R^t$ start with computing the *last* term of a permutation; to match these the order reversal obtained by right multiplication by $\tilde{w}$ is most convenient.

Like in the case of the Schützenberger algorithm, most of the work is required in proving an interpretation of the basic step, in the current case the "extraction" procedures $E^t$ and $E$. Despite the fact that (12) has less symmetry than (13), it will be slightly easier to interpret $E^t$ than $E$; this is due to the fact that the interpretation of tableaux in $\mathcal{F}_{u,T}$ and $\mathcal{F}^*_{u,T}$ emphasises column numbers rather than row numbers. We denote the stabiliser $(Z_u)_l$ of $l$ in $Z_u$ simply by $Z_{u,l}$.

**6.1. Lemma.** *Let a tableau $P\in\mathcal{T}_\lambda$, a corner $s$ of $\lambda$ in column $c$, and a line $l\in U_c$ be given. Using the transpose extraction procedure compute $(T,m)=E^t(P,s)$; then the $Z_{u,l}$-stable subset of $\mathcal{F}_{u,P}$ defined by $\mathcal{D}_l=\{\,f\in\mathcal{F}_{u,P}\mid \pi(f,l)=m\wedge f_{/l}\in\mathcal{F}_{u_{/l},T}\,\}$ is dense and open.*

*Proof.* Let $f\in\mathcal{F}_{u,P}$, let $H=f_{n-1}$ be its hyperplane part, and let the square $\lceil P\rceil$ appear in column $j$ so that $H\in U^\circ_j$. We examine in which circumstances we can have $H\not\supseteq l$, which is equivalent to $\pi(f,l)=n$. If either $H\supseteq\operatorname{Ker}\eta$ or $\operatorname{Im}\eta\supseteq l$ we readily conclude $H\supseteq l$, so it is only possible to have $\pi(f,l)=n$ if $j=c=1$. In that case, we have $T=P^-$ and $m=n$ by [(11$^t$)] (the transposed version of [(11)]), and $U_1\cap\mathbf{P}(H)=U_1(u|_H)$, which is either empty or of codimension 1 in $U_1$. So for $f$ in a dense open subset of $\mathcal{F}_{u,P}$ we then indeed have $H\not\supseteq l$, so that $\pi(f,l)=m$. For such $f$ we may furthermore decompose $V=H\oplus l$ as direct sum of $u$-stable subspaces, giving an isomorphism $H\xrightarrow{\sim}V/l$ which transforms $u|_H$ into $u_{/l}$, and also $f^-$ into $f_{/l}$; it follows that $f_{/l}\in\mathcal{F}_{u_{/l},T}$, completing the proof for this case.

The remaining cases use induction on $|\lambda|$; for $|\lambda|=1$ we are always in the case $j=c=1$ already treated. Knowing that $f^-\in\mathcal{F}_{u|_H,P^-}$, we wish to apply the induction hypothesis to $u|_H$ and $P^-$. Thereto we need to find the number $c'$ such that $l\in U_{c'}(u|_H)$; we also wish to find $j'$ such that $H/l\in U^\circ_{j'}(u_{/l})$, which is the column number of $\lceil r_{u_{/l}}(f_{/l})\rceil$. Both questions are equivalent to determining $J(u_{H/l})$, since $J(u|_H)$ and $J(u_{/l})$ are already known, and differ from $J(u_{H/l})$ by squares in columns $c'$ and $j'$ respectively. It follows from 4.1 that unless $c=j$ we have $c'=c$ and hence $j'=j$ (this also follows from the fact that $J(u_{H/l})$ is contained in both $J(u|_H)$ and $J(u_{/l})$). When $c=j>1$ we have that $W_c(u|_H)$ has codimension 1 in $W_c$ while $W_c\subseteq W_{c-1}=W_{c-1}(H)$, so for $H$ in a dense open subset of $U^\circ_j$ we shall have $l\in U_{c-1}(u|_H)$ and hence $c'=c-1=j'$. It will suffice to prove for such $H$, and $f$ in a dense open subset of $\omega^{-1}_{u,P}[H^\circ]$, that $\pi(f,l)=m$ and $f_{/l}\in\mathcal{F}_{u_{/l},T}$, because $\mathcal{D}_l$ will then be dense, and $\mathcal{F}_{u_{/l},T}$ being open in its closure, also open in $\mathcal{F}_{u,P}$. Comparing with [(11$^t$)], we find that $(T^-,m)=E^t(P^-,s')$ where $s'$ is the corner of $\operatorname{sh}P^-$ in column $c'$. We can therefore apply the induction hypothesis to $u|_H$, $P^-$ and $s'$, and conclude that for $f$ in a dense open subset of $\omega^{-1}_{u,P}[H^\circ]$ we have $\pi(f^-,l)=m$ and $f^-_{/l}\in\mathcal{F}_{u_{H/l},T^-}$. Since $\pi(f,l)=\pi(f^-,l)$ and $J(u_{/l})=\operatorname{sh}T$, this completes the proof. □

Contrary to what was the case for the interpretation of the Schützenberger algorithm, we cannot proceed immediately to prove our main identity, (12). This is due to the fact that in general the set

$\{ f_{/l} \mid f \in \mathcal{D}_l \}$, although contained in $\mathcal{F}_{u_{/l},T}$, is not dense in that set. So even though we now know the generic value of $\pi(f, f'_1)$ for $(f, f') \in \mathcal{F}_{u,P} \times \mathcal{F}^*_{u,Q}$, we would only get an upper bound (in the Bruhat order on $\mathbf{S}_n$) for $\pi(f, f')$. We could resolve the difficulty, as is done in [Stb2], by observing that such an inequality is in fact sufficient to imply identity, in view of the fact known from the general study of the unipotent variety that every permutation occurs as generic relative position for some $(u, P, Q)$ (see [Spr1], 3.8, [Spr2], 4.4.1 or [Stb1], 3.5, 3.6), together with the invertibility of the Robinson-Schensted algorithm.

We shall proceed however in a manner that is in many ways more satisfactory, and that does not depend on this external knowledge, but employs the same kind of methods used until now. The basic auxiliary result used is that $Z_{u_{/l}} \cdot \{ f_{/l} \mid f \in \mathcal{D}_l \}$ *is* dense in $\mathcal{F}_{u_{/l},T}$ (note the use of $Z_{u_{/l}}$ rather than $Z_{u,l}$). In order to prove this, we view $V/l$ equipped with $u_{/l}$ abstractly as a vector space with unipotent transformation, i.e., we forget the quotient construction used to obtain them. Then, for a given flag in $\mathcal{F}_{u_{/l},T}$ we try to reconstruct the quotient situation by extending the vector space, in such a way that the given flag corresponds to some appropriate $\hat{f}_{/l}$. Extending a vector space to a larger one is a bit more technical than restricting to a subspace or forming a quotient, but as we shall see the conditions that are to be fulfilled are quite similar to those we have met before: one of the interesting aspects of this construction is that it involves an interpretation of the *insertion* procedure in addition to the interpretation of the extraction procedure given in 6.1. Observe first that the initial claim of 6.1 can be formulated as follows.

**6.2. Corollary.** *Let a tableau $P$ and a corner $s$ of $\operatorname{sh} P$ in column $c$ be given; put $(T, m) = E^t(P, s)$, and let $d$ be the number of entries of $P$ that are $\leq m$. Then $d = \min\{ i \mid \forall f \in \mathcal{F}_{u,P} \colon f_i \supseteq W_c \}$.*

*Proof.* Without loss of generality we may take $P$ to be normalised, so that $d = m$. For $l \in U_c$ the set $\{ f \in \mathcal{F}_{u,P} \mid f_d \supseteq l \}$ is closed in $\mathcal{F}_{u,P}$, and by 6.1 it contains a dense subset of $\mathcal{F}_{u,P}$, whence it must be all of $\mathcal{F}_{u,P}$; combining this for all $l$ we get $f_d \supseteq W_c$ for all $f \in \mathcal{F}_{u,P}$. On the other hand any $f \in \mathcal{D}_l$ has $f_{d-1} \not\supseteq l$ and hence $f_{d-1} \not\supseteq W_c$, proving the minimality of $d$. □

It is natural to consider this matter also in the opposite direction, by asking for given $d$ what is the smallest $c$ such that $W_c \subseteq f_d$ for all $f \in \mathcal{F}_{u,P}$ (or in other words, determining $\operatorname{Ker}\eta \cap \bigcap_{f \in \mathcal{F}_{u,P}} f_d$). Note that we are now asking for *minimal* values of $c$, so that the possible outcomes are the column numbers of the *cocorners* of $\operatorname{sh} P$; indeed the requested intersection might be the zero space, in which case the answer will be $c = \lambda_1 + 1$. The question may of course be answered by applying $E^t$ for all corners of $\lambda$ and interpolating the results using the 6.2; however the answer can be obtained more directly by using the procedure $I^t$:

**6.3. Lemma.** *Let $(P, s) = I^t(T, m)$ where $m$ exceeds exactly $d$ of the entries of $T$, and let $c$ be the column number of $s$. Then $c = \min\{ i \mid \forall f \in \mathcal{F}_{u,T} \colon W_i \subseteq f_d \}$.*

*Proof.* If $m$ exceeds all entries of $T$, then $f_d$ is the whole vector space and $c = 1$, so the lemma is trivially satisfied. Otherwise, let the square $\lceil T \rceil$ appear in column $j$ so that $\omega_u[\mathcal{F}_{u,T}] = U^*_j$. Applying induction for $T^-$ in place of $T$ we find that for $(P', s') = I^t(T^-, m)$ the column number $c'$ of $s'$ satisfies $c' = \min\{ i \mid \forall f \in \mathcal{F}_{u|_H,T^-} \colon W_i(u|_H) \subseteq f_d \}$ for all $H \in U^\circ_j$. Since for $f \in \mathcal{F}_{u,T}$ we have $f^- \in \mathcal{F}_{u|_H,T^-}$ where $H = \omega(f)$, and of course $f^-{}_d = f_d$, we may conclude $c' = \min\{ i \mid \forall f \in \mathcal{F}_{u,T} \colon W_i(u|_{\omega(f)}) \subseteq f_d \}$. Now if $c' \neq j$ then [$(12^t)$] gives $c = c'$, and $W_c(u|_H) = W_c$ for all $H \in U^\circ_j$ by 4.1, whence the lemma holds. When $c' = j$ then [$(12^t)$] gives $c = c' + 1$, while $W_{c'}(u|_H)$ has codimension 1 in $W_{c'}$ by 4.1, so $W_{c'} \not\subseteq f_d$ for certain flags $f \in \mathcal{F}_{u,T}$, but $W_c = W_c(u|_H) \subseteq W_{c'}(u|_H)$. This proves the lemma in this case as well. □

We now come to the process of reconstructing for a given flag $f'$ in a vector space $V'$ a situation where $V' \cong V/l$ and where $f'$ corresponds to $f_{/l}$ for suitable $V$, $l$, and $f$. More precisely, let a (not necessarily normalised) tableau $T$ of shape $\mu$ be given, and a vector space $V'$ with an unipotent transformation $u'$ such that $J(u') = \mu$; we shall consider flags $f' \in \mathcal{F}_{u',T}$. Also let a number $m$ not occurring as entry of $T$, and exceeding exactly $d$ of the entries of $T$, be given, for which we compute $(P, s) = I^t(T, m)$; as usual $c$ denotes the column number of $s$. We construct the vector space $V = k \times V'$, of which we denote the line forming the first factor by $l$, and we identify $V' = V/l$. We wish to equip $V$ with a unipotent transformation $\hat{u}$ stabilising $l$ and inducing the given unipotent $u'$ in $V'$, from which it follows that $\hat{u}$ must be given by a block matrix

$$\hat{u} = \begin{pmatrix} 1 & \phi \\ 0 & u' \end{pmatrix} \tag{14}$$

where $\phi \in V'^*$. We do not fix this linear form, in order to allow ourselves the freedom to adapt it to the situation, which shall be as follows: a flag $f' \in \mathcal{F}_{u',T}$ is given and we wish to find a corresponding flag $\hat{f} \in \mathcal{F}_{\hat{u},P}$ for which $\pi(\hat{f}, l) = d + 1$ and $f' = \hat{f}_{/l}$. This last pair of conditions will be met if we take the flag $\hat{f}$ defined by

$$\hat{f}_i = \begin{cases} f'_i & \text{if } i \leq d \\ l \oplus f'_{i-1} & \text{if } i > d \end{cases} \tag{15}$$

The parts $\hat{f}_i$ with $i > d$ are automatically $\hat{u}$-stable, but those with $i \leq d$ are not necessarily so. In order to have $\hat{u}$-stability for these parts it will be sufficient that $\hat{f}_d$ is $\hat{u}$-stable, which is equivalent to $f'_d \subseteq \operatorname{Ker}\phi$. Therefore we define the variety

$$\mathcal{V} = \{\, (f', \phi) \in \mathcal{F}_{u',T} \times V'^* \mid f'_d \subseteq \operatorname{Ker}\phi \,\} \tag{16}$$

which is a vector bundle over the irreducible variety $\mathcal{F}_{u',T}$, and hence irreducible.

**6.4. Proposition.** *On a dense open subset of $\mathcal{V}$ we have $J(\hat{u}) = \operatorname{sh} P$.*

*Proof.* Because $Y(\operatorname{sh} P)$ is obtained from $Y(\mu)$ by adding the square $s$, which lies in column $c$, we have $J(\hat{u}) = \operatorname{sh} P$ if and only if $l \in U_c(\hat{u})$. One easily checks that $l \subseteq W_1(\hat{u})$ always holds, and $l \subseteq W_i(\hat{u})$ is equivalent to $W_{i-1}(u') \nsubseteq \operatorname{Ker}\phi$ for $i > 1$. Therefore the condition $l \in U_c(\hat{u})$ is equivalent to $c = \min\{\, i \mid W_i(u') \subseteq \operatorname{Ker}\phi \,\}$. It follows from 6.3 that $W_c(u') \subseteq \operatorname{Ker}\phi$ holds on all of $\mathcal{V}$ and if $c > 1$ then we have $W_{c-1}(u') \nsubseteq \operatorname{Ker}\phi$ for some $(f', \phi) \in \mathcal{V}$. Since any condition $W_i(u') \subseteq \operatorname{Ker}\phi$ defines a closed subset of $\mathcal{V}$ the proposition follows. □

**6.5. Proposition.** *On a dense open subset of $\mathcal{V}$ we have $\hat{f} \in \mathcal{F}_{\hat{u},P}$.*

*Proof.* By induction on the number of entries of $T$ exceeding $m$. If there are no such entries then $\hat{f}^- = f'$ and $P^- = T$ so the proposition follows immediately from 6.4. Otherwise let $j$ be the column number of $\lceil T \rceil$; for arbitrary hyperplane $H \in U_j^\circ$ we apply the isomorphism $\omega_{u',T}^{-1}[H^\circ] \cong \mathcal{F}_{u'|_H,T^-}$, and construct a vector bundle $\mathcal{V}^-$ over $\mathcal{F}_{u'|_H,T^-}$ analogous to $\mathcal{V}$. The part of the bundle $\mathcal{V}$ lying above $\omega_{u',T}^{-1}[H^\circ]$ maps onto $\mathcal{V}^-$ by $(f', \phi) \mapsto (f'^-, \phi|_H)$. We apply the induction hypothesis to $\mathcal{V}^-$, and find that for $(f'^-, \phi|_H)$ in a dense open subset of $\mathcal{V}^-$ we have $r_{\hat{u}|_H}(\hat{f}^-) \sim P^-$. Applying $Z_{u'}$ to the preimage of this subset of $\mathcal{V}^-$ and intersecting with the dense open subset of 6.4 we obtain the required dense open subset of $\mathcal{V}$. □

We can now prove the announced converse of 6.1.

**6.6. Lemma.** *In the situation of 6.1, $Z_{u_{/l}} \cdot \{\, f_{/l} \mid f \in \mathcal{D}_l \,\}$ is dense in $\mathcal{F}_{u_{/l},T}$.*

*Proof.* Choose any complementary subspace to $l$ in $V$, by means of which we identify $V$ with $l \times V/l$. Now apply 6.5 with $u' = u_{/l}$; it will suffice to prove for any $v$ in the dense subset of $\mathcal{V}$ of that proposition, that its projection $f'$ on $\mathcal{F}_{u_{/l},T}$ lies in $Z_{u_{/l}} \cdot \{\, f_{/l} \mid f \in \mathcal{D}_l \,\}$. Construct $\hat{u}$ and $\hat{f}$ according to $v$, so that $\hat{f} \in \mathcal{F}_{\hat{u},P}$. Since $J(u) = J(\hat{u}) = \operatorname{sh} P$, the unipotents $u$ and $\hat{u}$ are conjugate in $\mathbf{GL}(V)$; moreover since $l$ lies both in $U_c(u)$ and $U_c(\hat{u})$, where $c$ is the column number of $\lceil P \rceil$, there is even some element $g$ in the stabiliser of $l$ in $\mathbf{GL}(V)$ such that $g\hat{u}g^{-1} = u$, and hence $g \cdot \hat{f} \in \mathcal{F}_{u,P}$. Since we have by construction $\hat{u}_{/l} = u_{/l}$ and $\hat{f}_{/l} = f' \in \mathcal{F}_{u_{/l},T}$, we have $g \cdot \hat{f} \in \mathcal{D}_l$. Now $g$ induces a transformation $g_{/l}$ of $V/l$; and since we also have $g_{/l}\hat{u}_{/l}g_{/l}^{-1} = u_{/l}$, we conclude that $g_{/l} \in Z_{u_{/l}}$. As $f' = g_{/l}^{-1} \cdot (g \cdot \hat{f})_{/l}$, this proves the lemma. □

Combining 6.1 and 6.6 we come to the first main result of this section, which reformulates (12).

**6.7. Theorem.** *Let $P,Q \in \mathcal{T}_\lambda$, and compute $\sigma = R^t(P,Q)$ by the transpose Robinson-Schensted algorithm. Then for $(f,f')$ in a dense open $Z_u$-stable subset of $\mathcal{F}_{u,P} \times \mathcal{F}^*_{u,Q}$ one has $\pi(f,f') = \sigma\tilde{w}$.*

*Proof.* By induction on $|\lambda|$, with $|\lambda| = 0$ as trivial starting case. In view of [$(9^t)$] and the fact that $\sigma\tilde{w}$ corresponds to the sequence $(\sigma_n,\ldots,\sigma_1)$, let $(T,\sigma_1) = E^t(P,\lceil Q\rceil)$, and let $c$ be the column number of $\lceil Q\rceil$. We choose a line $l \in \alpha[\mathcal{F}^*_{u,Q}] = U_c$. We may restrict ourselves to $f' \in \alpha_u^{-1}[l]$: once we have found an appropriate dense subset of $\mathcal{F}_{u,P} \times \alpha_u^{-1}[l]$ we can apply the $Z_u$-action, which is transitive on $U_c$. We apply 6.1 with $s = \lceil Q\rceil$ and find for $f$ in a dense subset $\mathcal{D}_l$ of $\mathcal{F}_{u,P}$ that $\pi(f,l) = \sigma_1$ and $f_{/l} \in \mathcal{F}_{u_{/l},T}$. From $T$ we obtain the similar normalised tableau $\bar{T}$ by decreasing by 1 all entries exceeding $\sigma_1$. We invoke the induction hypothesis for the pair $(\bar{T},Q^-)$, and find a dense open $Z_{u_{/l}}$-stable subset $\Delta$ of $\mathcal{F}_{u_{/l},T} \times \mathcal{F}_{u_{/l},Q^-}$ such that any pair $(\bar{f},\bar{f}') \in \Delta$ has $\pi(\bar{f},\bar{f}') = \tau \in \mathbf{S}_{n-1}$ where $\tau$ and $\sigma\tilde{w}$ are related as $\sigma'$ and $\sigma$ in (9). To complete the proof it therefore suffices to show that the subset of $\mathcal{D}_l \times \alpha^{*\,-1}_{u,Q}[l]$ of pairs $(f,f')$ with $(f_{/l},f'^{\downarrow}) \in \Delta$ is open and dense. Since $\Delta$ is open and $\mathcal{D}_l \times \alpha^{*\,-1}_{u,Q}[l]$ is irreducible, our claim will be proved as soon the mentioned subset is non-empty, and this is equivalent to the existence of $f \in \mathcal{D}_l$ for which $f_{/l}$ lies in the projection $\Delta_1$ of $\Delta$ on the first factor. Now $\Delta_1$ is dense and open in $\mathcal{F}_{u_{/l},P^-}$, so it meets the dense subset $Z_{u_{/l}} \cdot \{\, f_{/l} \mid f \in \mathcal{D}_l \,\}$ of 6.6, but since $\Delta_1$ is also $Z_{u_{/l}}$-stable, it must already meet $\{\, f_{/l} \mid f \in \mathcal{D}_l \,\}$, and this completes the proof. □

This establishes the interpretation of the transpose Robinson-Schensted algorithm; we now proceed to the interpretation of the Robinson-Schensted algorithm itself. As was remarked earlier, we cannot expect an interpretation to be immediately implied by that of its transposed version, lacking a good interpretation for transposition of Jordan types. We could use [Theorem 5.1] to derive one interpretation from the other, but doing so would invalidate one of our main objectives, which is "explain" that theorem from the geometric interpretations. But although there is no formal connection, there is a great analogy between the interpretations of the two versions of the algorithm, which we shall emphasise by our formulations. We first need an analogue for 4.1, describing the behaviour of the sets $U_i$ under a projection $V \to V/l$.

**6.8. Proposition.** *Let $l \in U_c$ be given, and let $p\colon \mathbf{P}(\operatorname{Ker}\eta) \setminus \{l\} \to \mathbf{P}(\operatorname{Ker}\eta_{/l})$ be the map induced by the natural projection $V \to V/l$. For all $j \neq c$ we have $p[U_j] \subseteq U_j(u_{/l})$, while $p[U_c \setminus \{l\}] = \mathbf{P}(W_c(u_{/l}))$; moreover $\dim(U_c \cap p^{-1}[U_i(u_{/l})]) = \dim U_i(u_{/l}) + 1$ for all $i \geq c$.*

*Proof.* The projection maps each $W_j$ into $W_j(u_{/l})$, and for $j > c$ as well as for $j < c$ one easily shows that the induced maps $W_j/W_{j+1} \to W_j(u_{/l})/W_{j+1}(u_{/l})$ are isomorphisms, proving the first statement. For $i > c$ and a line $h \in U_i$ put $h' = p(h) = h \oplus l/l$. We have $h' \in U_i(u_{/l})$ and $p^{-1}[h'] = \mathbf{P}(h \oplus l)$, of which projective line all points except $h$ lie in $U_c$. Together with $p[U_c \setminus \mathbf{P}(W_{c+1} \oplus l)] \subseteq U_c(u_{/l})$, this proves the remainder of the proposition. □

**6.9. Lemma.** *Let a tableau $P \in \mathcal{T}_\lambda$, a corner $s$ of $\lambda$ in column $c$, and a line $l \in U_c$ be given. Using the extraction procedure compute $(T,m) = E(P,s)$, and put $m' = \tilde{w}_m = n+1-m$ where $n = |\lambda|$. Then the $Z_{u,l}$-stable set $\{\, f \in \mathcal{F}^*_{u,P} \mid \pi(f,l) = m' \wedge f_{/l} \in \mathcal{F}^*_{u_{/l},T} \,\}$ is dense and open in $\mathcal{F}^*_{u,P}$.*

*Proof.* Let $f \in \mathcal{F}^*_{u,P}$, let $h = f_1$ be its line part, and let the square $\lceil P\rceil$ appear in column $j$ so that $h \in U_j$. Obviously, if $j \neq c$ then $h \neq l$, and even if $j = c$ we can avoid having $h = l$, unless $U_c$ consists of a single projective point, i.e., unless $\dim W_c = 1$. Consequently, it is only in this case that we have $\pi(f,l) = 1$ for all $f$; moreover, we then also have $T = P^-$ and $m = n$ by [(11)], implying $m' = 1$, and since $f_{/l} = f^{\downarrow} \in \mathcal{F}^*_{u_{/l},T}$, we obtain the lemma for this case.

The remaining cases use induction on $|\lambda|$; for $|\lambda| = 1$ we are always in the case already treated. We restrict to the open subset of those $f \in \mathcal{F}^*_{u,P}$ for which $h \neq l$, so that $h \oplus l$ is a plane in $\operatorname{Ker}\eta$. Knowing that $f^{\downarrow} \in \mathcal{F}^*_{u_{/h},P^-}$, we wish to apply the induction hypothesis to $u_{/h}$ and $P^-$. Thereto we need to find the number $c'$ such that $l' \in U_{c'}(u_{/h})$ where $l' = l \oplus h/h$; we also wish to find $j'$ such that $h' \in U_{j'}(u_{/l})$ where $h' = h \oplus l/l$, which is the column number of $\lceil q_{u_{/l}}(f_{/l})\rceil$. Both questions are equivalent to determining $J(u_{/h\oplus l})$, since $J(u_{/h})$ and $J(u_{/l})$ are already known, and differ from $J(u_{/h\oplus l})$ by squares in columns $c'$ and $j'$ respectively. Now $h' = p(h)$ with $p$ as in 6.8, and it follows that for $j \neq c$ we have $j' = j$ and hence $c' = c$ (this also follows from the fact that $J(u_{/h\oplus l})$ is contained in both $J(u_{/h})$ and $J(u_{/l})$). When $j = c$ and $\dim W_c > 1$, we (further) restrict $h$ to the subset $U_c \cap p^{-1}[U_i(u_{/l})]$ of maximal dimension, i.e., for the smallest value $i \geq c$ for which $U_i(u_{/l})$ is non-empty; this defines a dense open subset of $\mathcal{F}^*_{u,P}$ in which $j' = c' = i$ holds. Note that the square $s' = (\dim W_i(u_{/l}), i)$ is the corner of $\operatorname{sh} P^-$ the row numbered $\dim W_c - 1$; comparing with [(11)], we find that $(T^-, m) = E^t(P^-, s')$. We can therefore apply the induction hypothesis to $u_{/h}$, $P^-$ and $s'$, and conclude that for $f$ in a dense open subset of $\mathcal{F}^*_{u,P}$ we have $\pi(f^\downarrow, l') = n - m = m' - 1$ and $f^\downarrow_{/l'} \in \mathcal{F}_{u_{/h\oplus l}, T^-}$. Since $\pi(f, l) = \pi(f^\downarrow, l') + 1$ and $J(u_{/l}) = \operatorname{sh} T$, this proves the lemma. □

Note that together with 4.3 and 6.1, this lemma already implies [Lemma 5.2], from which [Theorem 5.1] followed by a relatively simple combinatorial argument. We can also deduce from 4.3, 6.1 and 6.9, that the subset of $\mathcal{F}_{u,S(P)}$ defined as $\mathcal{D}_l$ in 6.1 intersects the set $\{ f \in \mathcal{F}^*_{u,P} \mid \pi(f,l) = m' \wedge f_{/l} \in \mathcal{F}^*_{u_{/l},T} \}$ of 6.9in a subset which is dense in either of them. Therefore we effortlessly obtain from 6.6:

**6.10. Lemma.** *The set* $Z_{u_{/l}} \cdot \{ f_{/l} \mid f \in \mathcal{F}^*_{u,P} \wedge \pi(f,l) = m' \wedge f_{/l} \in \mathcal{F}^*_{u_{/l},T} \}$ *is dense in* $\mathcal{F}^*_{u_{/l},T}$. □

This brings us to our second main theorem, which reformulates (13). We omit the proof, which is entirely analogous to 6.7.

**6.11. Theorem.** *Let* $P, Q \in \mathcal{T}_\lambda$, *and compute* $\sigma = R(P,Q)$ *by the Robinson-Schensted algorithm. Then for* $(f, f')$ *in a dense open* $Z_u$*-stable subset of* $\mathcal{F}^*_{u,P} \times \mathcal{F}^*_{u,Q}$ *one has* $\pi(f, f') = \tilde{w}\sigma\tilde{w}$. □

This completes the proof of the equations (10)–(13) stated at the beginning of this section. For reference we conclude this section by stating without proof direct interpretations of the extraction and insertion procedures, analogous to the ones we proved for their transposed counterparts.

**6.12. Proposition.** *Let a tableau* $P$ *and a corner* $s$ *of* $\operatorname{sh} P$ *in column* $c$ *be given; put* $(T, m) = E(P, s)$, *and let* $d$ *be the number of entries of* $P$ *that are* $\geq m$. *Then* $d = \min \{ i \mid \forall f \in \mathcal{F}^*_{u,P}\colon f_i \supseteq W_c \}$.

**6.13. Proposition.** *Let* $(P, s) = I(T, m)$ *where* $m$ *is exceeded by exactly* $d$ *of the entries of* $T$, *and let* $c$ *be the column number of* $s$. *Then* $c = \min \{ i \mid \forall f \in \mathcal{F}^*_{u,T}\colon W_i \subseteq f_d \}$.

## §7. Some further observations.

In the preceding sections we have shown how the Schützenberger and Robinson-Schensted algorithms provide basic information about the varieties $\mathcal{F}_u$. There are on one hand a number of further points that follow from these facts, and on the other hand numerous related questions arising naturally, for which the given analysis gives no (complete) answers. In this section and the next, we shall elaborate our results in several directions, and also mention some questions that remain open for further investigation. One obvious instance is the question of a more complete geometric description of $\mathcal{F}_u$ and its irreducible components. Having consistently considered only generically chosen elements, we have little detailed information about the varieties as a whole. For instance, we have no general method for computing the dimension of the intersection of a given pair of irreducible components of $\mathcal{F}_u$, or even for telling whether or not this intersection is empty. Although it is not too difficult to give a complete geometric description of $\mathcal{F}_u$ in individual cases for which its dimension is small, such exercises hardly provide a clue for a general approach.

Let us start with remarking that the "pictorial versions" of [3.1] and [4.2] have an obvious geometric significance. For the picture corresponding to the computation of $\sigma = R(P,Q)$, the partition ascribed to the point $(i,j)$ equals $J(u|_{f_i \cap f'_j})$ for generically chosen $(f, f') \in \mathcal{F}_{u,P} \times \mathcal{F}_{u,Q}$; in case $(i,j) \notin \Sigma_\sigma$ (i.e., the point $(i,j)$ was left vacant) this is the partition ascribed to the maximal point $< (i,j)$ of $\Sigma_\sigma$. For the picture corresponding to the computation of $S(P)$, the partition ascribed to the point $(i,j)$ equals $J(u_{f_j/f_{i-1}})$ for generically chosen $f \in \mathcal{F}_{u,P}$.

It is interesting to consider what combinatorial consequences can be derived from the given interpretations, in addition to the already mentioned main theorems of [vLee3]. For instance, the statements 6.2, 6.3, 6.12, and 6.13 imply a weak monotonicity of the procedures $E^t$, $I^t$, $E$ and $I$. To facilitate the formulation, define a partial ordering $\leq_{\swarrow}$ on the set of squares, transverse to the natural ordering, by $(r,c) \leq_{\swarrow} (r',c') \iff r \geq r' \wedge c \leq c'$; this induces a total ordering on the set of corners and cocorners of any fixed partition. Now, when $(T,m) = E(P,s)$ and $(T',m') = E(P,s')$, then $s \leq_{\swarrow} s'$ implies $m \leq m'$, and conversely if $(P,s) = I(T,m)$ and $E(P',s') = I(T,m')$, then $m \leq m'$ implies $s \leq_{\swarrow} s'$; for $E^t$ instead of $E$ one should replace $m \leq m'$ by $m \geq m'$. Direct combinatorial proofs of these facts can also easily be given using Schensted's description of $I$, $E$, and their transposes.

Another interesting observation can be made by combining 6.12 and 6.13. If $P$, $s$, $c$, $T$, $m$ and $d$ are as in 6.12, then for all $f \in \mathcal{F}^*_{u,P}$ we have $f_d \supseteq W_c$, while $f_{d-1} \not\supseteq W_c$ for some such $f$. Now if $m'$ does not occur as entry of $P$, so that we can compute $(P',s') = I(P,m')$, then by 6.13 the column number $c'$ of $s'$ equals $\min\{\, i \mid \forall f \in \mathcal{F}^*_{u,P}\colon W_i \subseteq f_{d'} \,\}$, where $d'$ is the number of entries of $P$ exceeding $m'$. Now if $m' < m$, then $d' \geq d$ which implies $c' \leq c$ and hence $s' <_{\swarrow} s$, while if on the other hand $m' > m$ then $d' < d$ which implies $c' > c$ hence $s <_{\swarrow} s'$; in other words the distinct squares $s, s'$ satisfy $s' <_{\swarrow} s \iff m' < m$. Note that in this situation $P'$ can be obtained from $T$ by the successive insertion of the numbers $m$ and $m'$, so this situation actually occurs during two successive insertion steps of the Robinson-Schensted algorithm. In that case, if this were say the $i$-th and $i+1$-st insertion steps, then the squares $s, s'$ are recorded in the right tableau produced by the algorithm as the squares containing $i$ and $i+1$. Denoting the square containing $j$ in a tableau $Q$ by $Q[j]$, we conclude that when $(P,Q) = R^{-1}(\sigma)$, the order with respect to $\leq_{\swarrow}$ of the squares $Q[i]$ and $Q[i+1]$ will coincide with the order of the successive terms $\sigma_i$ and $\sigma_{i+1}$ of $\sigma$. This combinatorial statement is due to Knuth, [Kn1], Theorem 1. By [3.1] it follows also that $P[i] <_{\swarrow} P[i+1]$ if and only if $\sigma^{-1}_i < \sigma^{-1}_{i+1}$, i.e., the entries $i$ and $i+1$ of $P$ occur in the same order (with respect to $<_{\swarrow}$) as the terms $i$ and $i+1$ occur in the sequence $\sigma$. One consequence of this, which is already noted (without proof) by Schensted ([Sche], Part II), is that we can extend the Robinson-Schensted algorithm to deal with equal entries in $\sigma$ and $P$, if we stipulate that for the purpose of comparison of entries we use their *positions* (ordered by $<_{\swarrow}$) in case their values are equal; the tableau condition then should exclude equal entries in the same column, but allow them in the same row. The computations then mimic the situation for the ordinary Robinson-Schensted algorithm where each set of equal numbers in $\sigma$ of $P$ is "pulled apart" into distinct numbers, their order determined to be increasing by their positions. A generalisation where the left tableau $Q$ is allowed the same liberties as $P$, is given in [Kn1].

The condition $Q[i+1] <_{\swarrow} Q[i]$ also has a geometric significance: it is equivalent to the fact that for all $f \in \mathcal{F}_{u,Q}$ the subspace $f_{i-1}$ of codimension 2 in $f_{i+1}$ contains $\operatorname{Im}(\eta|_{f_{i+1}})$ (recall that the hyperplane $f_i$ in $f_{i+1}$ necessarily does so), and hence that is we vary $f_i$ but no other part of $f$, we stay inside $\mathcal{F}_u$. The set of flags so obtained is called the (projective) *line of type $i$ in $\mathcal{F}$ through $f$*, and if for all $f \in \mathcal{F}_{u,Q}$ these lines are contained in $\mathcal{F}_u$, then they are already contained in $\overline{\mathcal{F}_{u,Q}}$, lest their union would form an irreducible subset of $\mathcal{F}_u$ of dimension $\dim \mathcal{F}_u + 1$. The combinatorial statements above lead to the following geometric fact. For any $f \in \mathcal{F}_{u,P}$ and $f' \in \mathcal{F}_{u,Q}$ in relative position $\pi(f,f') = \sigma = R(P,Q)$, and for any reduced expression $\sigma = s_1 s_2 \cdots s_l$, the unique sequence of lines of respective types $s_1, \ldots, s_l$ linking $f$ and $f'$ (as was mentioned in §5) is such that the first line is entirely contained in $\overline{\mathcal{F}_{u,P}}$, and the last line in $\overline{\mathcal{F}_{u,Q}}$. None of the facts above are new, but we mention them to illustrate the useful connections that can be made between combinatorics, linear algebra and geometry; quite possibly doing so for some of the many other known properties could lead to some new insights.

We have focussed our study on the situation restricted to $\operatorname{Ker}\eta$, as is illustrated for instance by statements like 6.2 and 6.3; by dualisation we also get information modulo $\operatorname{Im}\eta$. When we go outside of $\operatorname{Ker}\eta$, the situation becomes more complicated, for instance $Z_u$ need not act on all of $\mathbf{P}(V)$ with finitely many orbits. Nevertheless we can get some additional information by reducing modulo $\operatorname{Ker}\eta$, or dually, by restricting to $\operatorname{Im}\eta$. For $f \in \mathcal{F}_u$ we can define $f_{/\operatorname{Ker}\eta}$ in analogy to $f_{/l}$, by collecting all the distinct subspaces among the $(f_i + \operatorname{Ker}\eta)/\operatorname{Ker}\eta$. Since the Young diagram of the Jordan type $J(u_{/\operatorname{Ker}\eta})$ is related to that of $J(u)$ by removal of the first column, we find that $f \in \mathcal{F}_{u,T}$ implies $f_{/\operatorname{Ker}\eta} \in \mathcal{F}_{u_{/\operatorname{Ker}\eta},T'}$, where $T'$ is obtained from $T$ by removing the first column. If $f|_{\operatorname{Im}\eta}$ is similarly defined by collecting all the distinct subspaces among the $f_i \cap \operatorname{Im}\eta$, then for $f \in \mathcal{F}^*_{u,T}$ we also have $f|_{\operatorname{Im}\eta} \in \mathcal{F}^*_{u|_{\operatorname{Im}\eta},T'}$. Admittedly, the information obtained in this way is rather limited, since knowing $f|_{\operatorname{Ker}\eta}$ and $f_{/\operatorname{Ker}\eta}$ is certainly not enough to determine $f$; for instance this method does not allow us to extend 6.3 to a complete description of $\bigcap_{f\in\mathcal{F}_{u,P}} f_d$. There is however an amusing combinatorial consequence: let $\chi$ stand for the operation of chopping off the first column of a tableau and renormalising, then

$$S(\chi(S(\chi(T)))) = \chi(S(\chi(S(T)))) \tag{17}$$

for all Young tableaux $T$, since both describe the tableau $T'$ such that for $f \in \mathcal{F}_{u,T}$ we generically have $f_{\mathrm{Im}\,\eta/(\mathrm{Ker}\,\eta\cap\mathrm{Im}\,\eta)} \in \mathcal{F}^*_{u',T'}$, where $u' = u_{\mathrm{Im}\,\eta/(\mathrm{Ker}\,\eta\cap\mathrm{Im}\,\eta)}$. Not too surprisingly, this identity does not appear to have been noted before.

## §8. Varying the unipotent element.

Other interesting questions arise when we no longer keep the unipotent $u$ fixed. Let $U$ be the variety of all unipotents in $\mathbf{GL}_n$, on which $\mathbf{GL}_n$ acts by conjugation, and let $\widetilde{U} = \{(u,f) \in U \times \mathcal{F} \mid f \in \mathcal{F}_u\}$. Then $\mathbf{GL}_n$ also acts on $\widetilde{U}$, and the projections onto $U$ and $\mathcal{F}$ are equivariant maps. Since the fibres of the second projection are isomorphic to an affine space (the unipotent radical of a Borel subgroup), and the action of $\mathbf{GL}_n$ on $\mathcal{F}$ is transitive, we see that $\widetilde{U}$ is a smooth variety. The projection onto $U$ is a resolution of singularities of $U$, and it this fact which has originally led to the study of its fibres $\mathcal{F}_u$. We have $\dim\widetilde{U} = \dim U$: on a dense open subset of $U$ we have $J(u) = (n)$, and the fibre at such a generic unipotent consists of a single point. Put $U_\lambda = \{u \in U \mid J(u) = \lambda\}$, and denote by $\widetilde{U}_\lambda$ its inverse image in $\widetilde{U}$. The irreducible components of $\widetilde{U}_\lambda$ are in bijection with those of $\mathcal{F}_u$ for $u \in U_\lambda$, via the operation of intersecting with that fibre $\mathcal{F}_u$; this is because in $\mathbf{GL}_n$ the centraliser $Z_u$ of $u$ is always connected. Therefore the irreducible components of $\widetilde{U}_\lambda$ can be parametrised by $\mathcal{T}_\lambda$: denote the component corresponding to $\overline{\mathcal{F}_{u,T}}$ by $\widetilde{U}_T$. An obvious question in this context is to describe the partial ordering on $\bigcup_{\lambda\in\mathcal{P}_n}\mathcal{T}_\lambda$ defined by $P \le T$ if and only if $\widetilde{U}_P \subseteq \overline{\widetilde{U}_T}$. To our knowledge no effective and general answer to this question is known, but we shall indicate how the combinatorial algorithms treated above, in particular the Schützenberger algorithm, can help to study this question, and give answers in many particular cases.

The easier question of determining the closures of the orbits in $U$ (i.e., of the unipotent orbits in $\mathbf{GL}_n$), has the following answer. We continue to write $\eta$ for the nilpotent $u - \mathbf{1}$, even when $u$ varies; we have seen that if $J(\eta) = \lambda$, then $\dim \mathrm{Im}\,\eta^i$ (i.e., the rank of $\eta^i$) is equal to the number of squares in $Y(\lambda)$ strictly beyond the $i$-th column. Now any condition $\mathrm{rk}(\eta^i) \le r$ for fixed $i$ and $r$ defines a closed irreducible subset of $U$ stable under conjugation by $\mathbf{GL}_n$; we call such conditions *power-rank conditions* on $u$ (even though they are expressed in terms of $\eta$). Define a partial ordering on $\mathcal{P}_n$ by putting $\lambda \le \mu$ if each power-rank condition that is satisfied when $J(u) = \mu$ is also satisfied when $J(u) = \lambda$, i.e., if for all $i$ we have $\sum_{c>i}\lambda^t_c \le \sum_{c>i}\mu^t_c$, or equivalently, if for all $i$ we have $\sum_{r\le i}\lambda_r \le \sum_{r\le i}\mu_r$. This partial ordering is anti-symmetric with respect to transposition: $\lambda \le \mu \iff \mu^t \le \lambda^t$, and is generated (not minimally) by pairs of partitions whose Young diagrams differ by the position of one square only. If $\lambda < \mu$ is such a pair one easily sees, by looking at an appropriate 1-dimensional subset of $U$ in which all Jordan blocks but two are constant, that the closure of the unipotent class parametrised by $\mu$ contains unipotents of Jordan type $\lambda$. We conclude that $\lambda \le \mu$ is a necessary and sufficient condition for $U_\lambda \subseteq \overline{U_\mu}$, and the closure of the orbit of any $u \in U$ is the set of common solutions of all power-rank conditions satisfied by $u$.

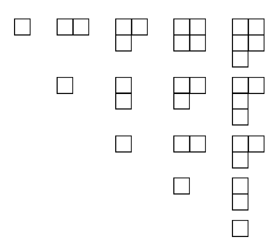
For the question of the closures of the $\widetilde{U}_T$ it is convenient to restrict attention to a fibre of the projection $\widetilde{U} \to \mathcal{F}$, which makes no essential difference, since the action of $\mathbf{GL}_n$ on $\mathcal{F}$ is transitive with connected stabilisers, namely the Borel subgroups. So let $\mathbf{f}$ as before be the standard flag, $\mathbf{B}$ its stabiliser, the Borel subgroup of upper triangular matrices, and $\mathbf{U}$ its subset of unipotent elements (the unipotent radical of $\mathbf{B}$) which we identify with the fibre in $\widetilde{U}$ above $\mathbf{f}$. Put $\mathbf{U}_\lambda = \widetilde{U}_\lambda \cap \mathbf{U}$ (which is also the intersection of the $\mathbf{GL}_n$-conjugacy class $U_\lambda$ with $\mathbf{B}$), and $\mathbf{U}_T = \overline{\widetilde{U}_T} \cap \mathbf{U}$ (which is the closure of one of its irreducible components, if $\lambda = \mathrm{sh}\,T$). The sets $\{u - \mathbf{1} \mid u \in \widetilde{U}_T \cap \mathbf{U}\}$ are called orbital varieties, and have been studied for instance in [Jos], [Benl], [Meln]. Our partial ordering on tableaux can now be expressed as $P \le T \iff \mathbf{U}_P \subseteq \mathbf{U}_T$.

We can view the tableau $r_u(\mathbf{f})$ as a function of $u$, and $\mathbf{U}_T$ coincides with the closure of the set $\{u \in \mathbf{U} \mid r_u(\mathbf{f}) = T\}$. Now if $r_u(\mathbf{f}) = T$, then $u|_{\mathbf{f}_i}$ satisfies the power-rank conditions corresponding to the subtableau of $T$ with entries $\leq i$, however, it is *not* generally true that $\mathbf{U}_T$ is the set of common solutions of all these conditions. For instance for $T = \begin{smallmatrix}1&2\\3\end{smallmatrix}$, the shapes of $T^-$ and $T^{--}$ correspond to the dense unipotent classes in $\mathbf{GL}_2$ and $\mathbf{GL}_1$, and therefore only contribute trivial power-rank conditions. Hence *all* unipotents of Jordan type $(2,1)$ satisfy the given power-rank conditions, including those with $r_u(\mathbf{f}) = \begin{smallmatrix}1&3\\2\end{smallmatrix}$, which in general do not lie in $\mathbf{U}_T$. The difficulty is that by restricting to $\mathbf{U}$, the power-rank conditions are no longer irreducible (this is directly related to the falling apart of unipotent orbits into irreducible components upon intersecting with $\mathbf{U}$), and the set $\{\, u \in \mathbf{U} \mid r_u(\mathbf{f}) = T \,\}$ is not only determined by the power-rank conditions corresponding to $T$ and its subtableaux, but also by the negation of all strictly stronger power-rank conditions. In order to get equations for $\mathbf{U}_T$ one needs to replace these negated conditions by positive (closed) ones.

In the given example $T = \begin{smallmatrix}1&2\\3\end{smallmatrix}$ it is easy to see which closed condition is missing, namely $\operatorname{Im}\eta \subseteq \mathbf{f}_1$, or equivalently $J(u_{/\mathbf{f}_1}) = (1,1)$ (i.e., $\eta_{/\mathbf{f}_1} = 0$). Now by (6), power-rank conditions for $u_{/\mathbf{f}_i}$ can be obtained from subtableaux of $S(T)$, and an obvious improvement is to include these conditions as well for specifying $\mathbf{U}_T$. While this is useful, and suffices for the case of $\mathbf{GL}_3$, it fails for $\mathbf{GL}_4$, since the tableaux $T = \begin{smallmatrix}1&2\\3&4\end{smallmatrix}$ and $T' = \begin{smallmatrix}1&3\\2&4\end{smallmatrix}$ are both fixed under $S$, but the power-rank conditions for $u|_{\mathbf{f}_i}$ fail to show that $\mathbf{U}_{T'} \not\subseteq \mathbf{U}_T$, for the same reasons as in the previous example. What is needed here is condition on $\eta_{\mathbf{f}_3/\mathbf{f}_1}$ (namely that it is zero), and in general we may add to our repertoire all applicable power-rank conditions for $u_{\mathbf{f}_j/\mathbf{f}_{i-1}}$ with $1 \leq i \leq j \leq n$ (or for $i < j$, since the condition for $i = j$ is always trivial). The matrix for $u_{\mathbf{f}_j/\mathbf{f}_{i-1}}$ is the square submatrix of that of $u$ with rows and columns numbered $< i$ or $> j$ removed (hence having its main diagonal along that of $u$, and with upper right-hand corner in position $(i,j)$), which shows that such conditions do indeed define closed subsets of $\mathbf{U}_T$. The generic Jordan type of $u_{\mathbf{f}_j/\mathbf{f}_{i-1}}$ can be found as $T^{[i,j]} \stackrel{\text{def}}{=} \operatorname{sh} T^{\downarrow\cdots\downarrow-\cdots-}$, where the entries $< i$ are removed from $T$ by successive applications of $\cdot^{\downarrow}$, and the entries $> j$ by $\cdot^{-}$. It turns out that the pictorial rendering of the proof of [4.2] displays all relevant partitions in an upper triangular area, and—by a happy coincidence—each partition appears in the proper place, namely $T^{[i,j]}$ at position $(i,j)$, which is the upper right-hand corner of the submatrix it applies to!

We now define a partial ordering '$\leq_S$' on $\bigcup_{\lambda\in\mathcal{P}_n} \mathcal{T}_\lambda$ by $P \leq_S T$ if and only if $P^{[i,j]} \leq T^{[i,j]}$ for all $i,j$ with $1 \leq i < j \leq n$; this ordering can be determined combinatorially. From the reasoning above it follows that $P \leq_S T$ is a necessary condition for $P \leq T$. We call a tableau $T$ power-rank complete if the power-rank conditions derived from all $T^{[i,j]}$ (each applied to the correct submatrix of $u \in \mathbf{U}$) together define an irreducible variety, which then must be $\mathbf{U}_T$; if this is the case then $P \leq_S T$ is also a sufficient condition for $P \leq T$. An example may illustrate how explicit equations for $\mathbf{U}_T$ can be found if $T$ is power-rank complete (such equations in fact provide much more information than just allowing us to determine the inclusions between the varieties $\mathbf{U}_T$). Let $n = 5$, $\lambda = (2,2,1)$ and $T = \begin{smallmatrix}1&2\\3&4\\5\end{smallmatrix}$. A general matrix $\eta = u - \mathbf{1}$ for $u \in \mathbf{U}$, and the pictorial version of the Schützenberger algorithm for this case are

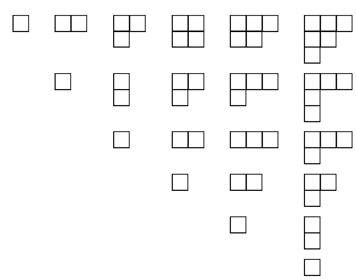
$$\eta = \begin{pmatrix} 0 & a & b & c & d \\ 0 & 0 & e & f & g \\ 0 & 0 & 0 & h & k \\ 0 & 0 & 0 & 0 & l \\ 0 & 0 & 0 & 0 & 0 \end{pmatrix} \qquad \text{respectively}$$



For each Young diagram consisting of a single column, the corresponding power-rank condition states the its submatrix of $\eta$ is zero, thereby annihilating certain indeterminates. All diagonal entries are already 0, and in the example the additional conditions of this kind are due to $T^{[2,3]} = T^{[4,5]} = (1,1)$: the conditions are $e = l = 0$. Of the remaining Young diagrams only a few typically contribute non-trivial conditions: if $T^{[i,j]}$ has gained a square in its Young diagram to the right of column $c$ with respect to either of its neighbours $T^{[i-1,j]}$ and $T^{[i,j-1]}$, then its $c$-th power-rank condition is are already implied by the similar condition for that neighbour, because the rank of the $c$-th power cannot increase by more than 1 due to adding a single row and column. In the present example we only get new conditions from $T^{[1,4]} = (2,2)$, from $T^{[2,5]} = (2,1,1)$ and from $T^{[1,5]} = (2,2,1)$. For position $(1,4)$ the condition is that the square of the corresponding matrix is 0; since we already know $e = 0$ this just means that $af + bh = 0$. For position $(2,5)$ the condition is that the corresponding matrix has rank $\leq 1$, which means $\left|\begin{smallmatrix} f & g \\ h & k \end{smallmatrix}\right| = 0$, and for the corner position $(1,5)$ the condition is again that the square of the corresponding matrix is 0, which means that $af + bh = 0$ may be extended to $(a\ b)\left(\begin{smallmatrix} f & g \\ h & k \end{smallmatrix}\right) = (0\ 0)$. Observe that the last equation does not decrease the dimension of the solution set, as $\left(\begin{smallmatrix} f \\ h \end{smallmatrix}\right)$ and $\left(\begin{smallmatrix} g \\ k \end{smallmatrix}\right)$ are linearly dependent by the previous equation, but it serves to remove the irreducible component given by $f = h = 0$ from the set defined by the previous two equations. It is not difficult to see that the set defined by all these conditions is irreducible, and hence that $T$ is power-rank complete. We can also see that we could not have done with less equations than the 5 given ones, and since the codimension of $\mathbf{U}_T$ in $\mathbf{U}$ is 4, this means $\mathbf{U}_T$ is not a complete intersection in $\mathbf{U}$ (this circumstance is rather exceptional among the cases for $n \leq 5$). In general this codimension of $\mathbf{U}_T$ in $T$ can be shown to be equal to $\dim \mathcal{F}_u$ for $J(u) = \operatorname{sh} T$. In the example one may observe that, while in general power-rank conditions appear to be of a forbidding complexity, they usually (at least when $n$ is not too large) reduce to quite simple forms or disappear altogether, when the conditions coming from $T^{[i,j]}$ for smaller $i,j$ are taken into account.

From small examples one gets the impression that all tableaux are power-rank complete—indeed they are for $n \leq 5$—but unfortunately for $n = 6$ there are two counterexamples, which are of shape $\lambda = (3,2,1)$. The failure is most easily detected by the existence of pairs $(P,T)$ with $\operatorname{sh} P = \operatorname{sh} T$ and $P <_S T$; in such a case $T$ cannot be power-rank complete, because $\dim \mathbf{U}_P = \dim \mathbf{U}_T$ immediately implies $P \not< T$. (When for some tableau $T$ no such $P$ exists, like in the example above, and for the vast majority of small $T$, then the power-rank conditions for $\mathbf{U}_T$ define a closed subvariety of $\overline{\mathbf{U}_\lambda}$, which contains none of the irreducible components of $\overline{\mathbf{U}_\lambda}$ except $\mathbf{U}_T$. This condition does not guarantee that $T$ is power-rank complete, although it does turn out to be true in all such cases with $n \leq 6$.) The two mentioned counterexamples, are witnessed by the pair $P = \begin{smallmatrix} 1 & 2 & 5 \\ 3 & 6 & \\ 4 & & \end{smallmatrix} <_S T = \begin{smallmatrix} 1 & 2 & 5 \\ 3 & 4 & \\ 6 & & \end{smallmatrix}$ and the pair obtained from it by application of $S$: $P^* = \begin{smallmatrix} 1 & 3 & 6 \\ 2 & 4 & \\ 5 & & \end{smallmatrix} <_S T^* = \begin{smallmatrix} 1 & 3 & 4 \\ 2 & 6 & \\ 5 & & \end{smallmatrix}$. To see what goes wrong we again display a general matrix $\eta$ for $u \in \mathbf{U}$, and the pictorial computation of $S(T)$:

$$\eta = \begin{pmatrix} 0 & a & b & c & d & e \\ 0 & 0 & f & g & h & k \\ 0 & 0 & 0 & l & m & n \\ 0 & 0 & 0 & 0 & p & q \\ 0 & 0 & 0 & 0 & 0 & r \\ 0 & 0 & 0 & 0 & 0 & 0 \end{pmatrix} \qquad \text{and}$$



The power-rank conditions for this case are $f = r = 0$, and (from $T^{[1,4]}$, $T^{[2,6]}$, and $T^{[1,6]}$):

$$ag + bl = 0, \qquad \begin{vmatrix} g & h & k \\ l & m & n \\ 0 & p & q \end{vmatrix} = 0, \qquad \operatorname{rk}\begin{pmatrix} ah+bm & ak+bn \\ gp & gq \\ lp & lq \end{pmatrix} \leq 1.$$

The terms $cp$ and $cq$ have been dropped from the first row of the last matrix since this does not affect any of the $2 \times 2$ minors. In fact the last condition is entirely redundant: the $2 \times 2$ minor formed by the last two rows clearly vanishes, and with a bit more effort one sees that the other two such minors lie in the ideal generated by the polynomials in the two previous conditions. Nevertheless these minors are useful, since they obviously factor into a term $g$ respectively $l$ and a term $D \stackrel{\text{def}}{=} \left|\begin{smallmatrix} ah+bm & ak+bn \\ p & q \end{smallmatrix}\right|$; since none of these lie in the mentioned ideal, that ideal is not prime, and $T$ is not power-rank complete. In fact, the variety defined by the ideal has two irreducible components: one on which $g = l = 0$ and another on which $D = 0$. Since $g = l = 0$ clearly does not hold on all of $\mathbf{U}_T$ (or we would have had $T^{[2,4]} = (1,1,1)$), it is the component on which $D = 0$ which coincides with $\mathbf{U}_T$. The fact that the equation $D = 0$ is not implied by the power-rank conditions is what causes $T$ to be problematic, and we can also see why $P <_S T$ despite $P \not< T$: the variety $\mathbf{U}_P$ is given by the linear equations $f = g = l = r = 0$ (these do follow from its power-rank conditions), which correspond to the spurious irreducible component. As said before, we know that $P < T$ must be false for dimension reasons, but there might be other tableaux $Q$

for which $\dim \mathbf{U}_Q < \dim \mathbf{U}_T$ and $Q \not< T$ despite the fact that $Q <_S T$. Such $Q$ necessarily has $Q < P$, but it turns out that all such $Q$ also have $Q < T$, since $\mathbf{U}_Q$ satisfies in addition to $f = g = l = r = 0$ other equations which imply $D = 0$; the two most interesting cases being

$$Q = \begin{smallmatrix}1&2\\3&5\\4&6\end{smallmatrix} \quad \text{for which} \quad (a\ b\ c)\begin{pmatrix} h & k \\ m & n \\ p & q \end{pmatrix} = (0\ 0) \qquad \text{and} \qquad Q = \begin{smallmatrix}1&2&5\\3\\4\\6\end{smallmatrix} \quad \text{for which} \quad \operatorname{rk}\begin{pmatrix} h & k \\ m & n \\ p & q \end{pmatrix} \le 1$$

So for $n = 6$ the combinatorial test $P \le_S T$ does not correspond to $P \le T$ for only two of all $76^2 = 5776$ possible pairs $(P, T)$, and for those two pairs dimension considerations show that $P \not< T$. For $n = 7$ however, the test $P \le_S T$ fails more seriously. Combinatorially no fundamentally new phenomena appear: although there are now 16 equal shaped pairs $(P, T)$ of distinct tableaux for which $P <_S T$, these all appear to be essentially "enlarged versions" of similar pairs for $n = 6$. However, contrary to those cases, we can now find $Q < P$ with $Q \not< T$, as exemplified by $Q = \begin{smallmatrix}1&2&5\\3&6&7\\4\end{smallmatrix} < P = \begin{smallmatrix}1&2&5&7\\3&6\\4\end{smallmatrix} <_S T = \begin{smallmatrix}1&2&5&7\\3&4\\6\end{smallmatrix}$. Even more disturbingly, there are tableaux $T$ which are not power-rank complete, despite the fact that there is *no* $P <_S T$ of the same shape. Such a situation is implied by the existence of a tableau $Q$ with $\operatorname{sh} Q < \operatorname{sh} T$ for which $Q <_S T$, but $Q \not< T$, as for instance $Q = \begin{smallmatrix}1&3&6\\2&4\\5\\7\end{smallmatrix} <_S T = \begin{smallmatrix}1&3&4\\2&6\\5&7\end{smallmatrix}$. The lack of equal shaped $P$ with $P <_S T$ means the solution set of the power-rank conditions for $T$ has one or more spurious irreducible components of *lower* dimension than $\mathbf{U}_T$; in the current case this is in fact $\mathbf{U}_Q$. By merely computing the relation '$\le_S$' such cases cannot be detected. We must conclude that the combinatorial test $P \le_S T$ is useful, but definitely insufficient to decide the partial order '$\le$' for $n \ge 7$.

For tableaux $T$ which are not power-rank complete, our claims about $P \le T$ could only be established by close inspection of the polynomials giving the power-rank conditions. The number and complexity of these polynomials is already considerable in many cases, so that we had much benefit from the use of Computer Algebra (we used the package *Maple*) for deciding ideal membership; even so, we were not able to automate the process of isolating irreducible components (indeed, it is not even clear that this process can always be done independently of the ground field). A theoretical possibility that should be mentioned here is to include in addition to power-rank conditions, which state upper bounds for the ranks of powers of submatrices, also conditions that state these ranks should not be strictly less than these upper bounds (thereby eliminating the unwanted irreducible components). Although these are not closed conditions, they can still be rendered in polynomial form by introducing new variables: to express that all polynomials in some finite set (viz. the set of all minors of given size of a matrix) do not vanish simultaneously, multiply each of them with a fresh variable, subtract one from each of these products and multiply everything together. If the ideal generated by the collection of polynomials obtained in this way inside the polynomial ring generated by all (old and new) variables, is intersected with the polynomial ring generated by the old variables only, an ideal is obtained for which the spurious components have been removed from its solution set; such an intersection can be determined by computing a Gröbner basis for the ideal of the larger polynomial ring with respect to a pure lexicographic order on the monomials in which the old variables are considered less than the new ones. We have indeed been able to extract "hidden" conditions such as $D = 0$ above in this way (and in fact it is in this case sufficient to consider only conditions corresponding to positions $(1, j)$ along the top row of the matrix), but conditions giving lower bounds for ranks had to be carefully selected, since indiscriminate use of all available conditions would lead to systems whose the Gröbner basis computation does not terminate in any acceptable amount of time, due to the sheer number of polynomials and indeterminates.

Not only the Schützenberger algorithm, but also the Robinson-Schensted algorithm can be used to obtain useful information pertaining to the current problem. In order to see how, we consider instead of the variety $\widetilde{U}$ the variety $Y = \{(u, f, f') \in U \times \mathcal{F} \times \mathcal{F} \mid f \in \mathcal{F}_u \wedge f' \in \mathcal{F}_u\}$. This variety is not irreducible: its irreducible components are the closures of the inverse images under the projection onto $\mathcal{F} \times \mathcal{F}$ of the (diagonal) $\mathbf{GL}_n$ orbits in that set. As we have seen those $\mathbf{GL}_n$ orbits are characterised by the relative positions, and hence in bijection with $\mathbf{S}_n$; since the stabiliser of a point $(f, f')$ of $\mathcal{F} \times \mathcal{F}$ is the intersection of two Borel subgroups, in which the subgroup of unipotent elements always has codimension $n$, all irreducible components of $Y$ have the same dimension $n^2 - n$. Denoting the projection

$Y \to U$ by $\Phi$, the irreducible components of $Y$ are also the closures of the irreducible components of the sets $\Phi^{-1}[U_\lambda]$ for $\lambda \in \mathcal{P}_n$, and intersection with $\Phi^{-1}[u]$ for $u \in U_\lambda$ defines a bijection from the irreducible components of $\Phi^{-1}[U_\lambda]$ to those of $\Phi^{-1}[u] \cong \mathcal{F}_u \times \mathcal{F}_u$ (again since $Z_u$ is connected). The correspondence between these two ways of describing the irreducible components of $Y$ is just the interpretation given for the Robinson-Schensted algorithm: the component corresponding to $\overline{\mathcal{F}_{u,P}} \times \overline{\mathcal{F}_{u,Q}} \subseteq \mathcal{F}_u \times \mathcal{F}_u$ is the closure of the inverse image in $Y$ of $\{\, (f, f') \in \mathcal{F} \times \mathcal{F} \mid \pi(f, f') = R(P,Q) \,\}$. A dense subset of that component is given by $Y_{P,Q} = \{\, (u, f, f') \in Y \mid r_u(f) = P \wedge r_u(f') = Q \wedge \pi(f, f') = R(P,Q) \,\}$.

Now let a tableau $P$ be given, choose an arbitrary tableau $Q$ of the same shape, and put $\sigma = R(P,Q)$. Under the projection $Y \to \widetilde{U}$ given by $(u, f, f') \mapsto (u, f)$, the image of $Y_{P,Q}$ is dense in $\widetilde{U}_P$, and intersecting with $\mathbf{U}$ gives a dense subset of $\mathbf{U}_P$. Taking the inverse image of this set in $Y_{P,Q}$ and projecting to $\mathcal{F}$ by $(u, \mathbf{f}, f') \mapsto f'$, the image is the set $\{\, f' \in \mathcal{F} \mid \pi(\mathbf{f}, f') = \sigma \,\}$, which is equal to the $\mathbf{B}$-orbit $\mathbf{B} \cdot \mathbf{f}^\sigma$ of the permuted standard flag $\mathbf{f}^\sigma$. Denoting the stabiliser of $\mathbf{f}^\sigma$ by $\mathbf{B}^\sigma$, the fibre in $Y$ above $(\mathbf{f}, \mathbf{f}^\sigma) \in \mathcal{F} \times \mathcal{F}$ is isomorphic (by projection on $U$) with $U \cap \mathbf{B} \cap \mathbf{B}^\sigma$ (the unipotent radical of an intersection of two Borel subgroups), and a dense part of it lies in $Y_{P,Q}$. Writing $\mathbf{U}(\sigma) = U \cap \mathbf{B} \cap \mathbf{B}^\sigma$, we may conclude that $\mathbf{U}_P = \overline{\mathbf{U}(\sigma)^{\mathbf{B}}}$ where $\mathbf{U}(\sigma)^{\mathbf{B}} \stackrel{\text{def}}{=} \{\, bub^{-1} \mid b \in \mathbf{B}, u \in \mathbf{U}(\sigma) \,\}$.

The subset $\mathbf{U}(\sigma)$ of $\mathbf{U}$ is easy to describe: the matrix coefficient of $\eta$ at position $(i,j)$ is unrestricted if both $i < j$ and $\sigma_i^{-1} < \sigma_j^{-1}$, and zero otherwise. So we get in this way a quite explicit description of $\mathbf{U}_P$, in fact several such descriptions, since we obtain a permutation $\sigma$ for each choice of the tableau $Q$. The difficulty however is, that it is hard to describe in concrete terms the effect of the conjugation action of $\mathbf{B}$, and in particular to decide for permutations $\sigma, \tau$ whether $\mathbf{U}(\sigma) \subseteq \overline{\mathbf{U}(\tau)^{\mathbf{B}}}$. Nevertheless there is an easy sufficient condition for this, namely $\mathbf{U}(\sigma) \subseteq \mathbf{U}(\tau)$, which can be established combinatorially: it means that whenever $i < j$ and $i$ precedes $j$ in the sequence $\sigma$, then $i$ also precedes $j$ in the sequence $\tau$; if this condition is satisfied, we write $\sigma \leq_R \tau$. For tableaux $P$, $T$ we write $P \preceq_R T$ if there exist tableaux $Q$, $Q'$ of the appropriate shapes for which $R(P,Q) \leq_R R(T,Q')$; this is a sufficient condition for $P \leq T$. As a consequence of the freedom of choosing $Q$ and $Q'$ the relation '$\preceq_R$' is not transitive in general, so denote by '$\leq_R$' its transitive closure.

In the definition of $P \preceq_R T$ we may clearly limit ourselves to the most useful choices for $Q$ and $Q'$, namely those for which $R(P,Q)$ is a minimal element with respect to '$\leq_R$' within the subset $R(P,*) \stackrel{\text{def}}{=} \{\, R(P,Q) \mid Q \in \mathcal{T}_{\mathrm{sh}\,P} \,\}$ of $\mathbf{S}_n$, and $R(T,Q')$ a maximal element of $R(T,*)$. Typically these sets $R(P,*)$ contain only a few minimal elements (often just one), and likewise for maximal elements; when computing many tests $P \preceq_R T$, it is useful to precompute these sets of extremes for each tableau involved. Furthermore it is not difficult to show that $\sigma \preceq_R \tau$ is equivalent to each of $\tilde{w}\sigma\tilde{w} \preceq_R \tilde{w}\tau\tilde{w}$, $\tau\tilde{w} \preceq_R \sigma\tilde{w}$, and $\tilde{w}\tau \preceq_R \tilde{w}\sigma$, which implies by [3.1] that $P \leq_R T$ is equivalent to each of $S(P) \leq_R S(T)$, $T^t \leq_R P^t$ and $S(T^t) \leq_R S(P^t)$.

Being a sufficient condition for $P \leq T$, the condition $P \leq_R T$ forms a useful complement to $P \leq_S T$ (which is a necessary condition), and together they can settle the question whether or not $P \leq T$ in the vast majority of the cases. It should be noted however that deciding whether or not $P \leq_R T$ holds is computationally harder to establish than $P \leq_S T$, first of all because one needs to find the extremal permutations with respect to '$\leq_R$' in each set $R(P,*)$ involved, and also because the transitive closure of '$\preceq_R$' has to be computed. In practice the following procedure proved useful: first test whether $P \leq_S T$, then if it holds, test whether $P \preceq_R T$, and if this fails, compute the transitive closure of '$\preceq_R$' within the subset $\{\, Q \in \mathcal{T} \mid P \leq_S Q \leq_S T \,\}$ in order decide whether or not $P \leq_R T$. Only if this last test fails does one have to resort to actually computing and analysing the power-rank conditions to decide whether $P \leq T$. The following data give an indication of the effectivity of this method. Up to $n = 4$ the relations '$\leq_S$' and '$\preceq_R$' coincide with each other, and therefore with '$\leq$'. For $n = 5$ their are already two pairs of tableaux for which the transitive closure of '$\preceq_R$' has to be invoked (one such pair is $P = \begin{array}{|c|c|}\hline 1&4\\\hline 2&5\\\hline 3\\\cline{1-1}\end{array}$, $T = \begin{array}{|c|c|c|}\hline 1&2&5\\\hline 3&4\\\cline{1-2}\end{array}$), but '$\leq_S$' still coincides with '$\leq_R$' and hence with '$\leq$'. For $n = 6$ there are 40 pairs for which computing $P \leq_S T$ and $P \preceq_R T$ does not suffice, and after transitive closure of '$\preceq_R$' there remain 6 undecided pairs; among these are the two pairs mentioned above for which $P <_S T$ despite $P \not< T$, and for the other four one has $P < T$ despite $P \not<_R T$. For $n = 7$ there are 442 pairs (from a total of 53824) for which the transitive closure of '$\preceq_R$' needs to be determined, and after that has been done 90 pairs remain undecided. Only 48 distinct tableaux are involved in these pairs, and it was possible to determine a complete set of equations for $\mathbf{U}_T$ in each case, from which the remaining values of $P \le Q$ could be found: in 46 of the 90 cases it did in fact hold (in accordance with '$\le_S$') while in the other 44 cases it did not (in accordance with '$\le_R$').

Apart from these statistics, the calculations showed a pattern that is worth mentioning: we were unable to find any pairs of tableaux $P, Q$ with $P < Q$ but $Q^t \not< P^t$, which would have contradicted the partial order on tableaux being anti-symmetric with respect to transposition. The mentioned anti-symmetry does obviously hold for '$\le_S$' and '$\le_R$', but where these two partial orders differ, the nature of the calculation of '$\le$' gives no reason to expect that anti-symmetry would hold for it as well; for instance the property of being power-rank complete has no such symmetry whatsoever. The fact that the anti-symmetry does in fact appear to hold for '$\le$' might indicate that there is a combinatorial description of '$\le$' yet to be found, more subtle than the combination of '$\le_S$' and '$\le_R$', but posessing the same anti-symmetry.

## References.


[Benl] E. Benlolo, *Etude sur let variétés orbitales dans* $\mathfrak{sl}_n(\mathbf{C})$, Thesis (Haifa, Israel), (1990).

[Hum] J. E. Humphreys, *Linear Algebraic Groups*, Graduate Texts in Mathematics 21, Springer Verlag, 1975.

[Jos] A. Joseph, "On the variety of a highest weight module", *Journal of Algebra* **88**, (1984), 238–278.

[Kn1] D. E. Knuth, "Permutations, matrices and generalized Young tableaux", *Pacific Journal of Math.* **34**, (1970), 709–727.

[Kn2] D. E. Knuth, *The art of Computer programming, Vol. III Sorting and Searching*, pp. 48–72, Addison-Wesley, 1975.

[vLee1] M. A. A. van Leeuwen, *A Robinson-Schensted algorithm in the geometry of flags for Classical Groups*, Thesis (Rijksuniversiteit Utrecht, the Netherlands), (1989).

[vLee2] M. A. A. van Leeuwen, "The Robinson-Schensted and Schützenberger algorithms and interpretations", pp. 65–88 in *Computational aspects of Lie group representations and related topics* (A. M. Cohen, ed.), CWI Tract 84, 1991.

[vLee3] M. A. A. van Leeuwen, *The Robinson-Schensted and Schützenberger algorithms, Part I: New combinatorial proofs*, CWI report AM-R9208, (1992).

[Meln] A. Melnikov, *Geometric Interpretation of Robinson-Schensted Procedure and Related Orders on Young Tableaux*, Thesis (Weizmann Institute, Israel), (1992).

[Sche] C. Schensted, "Longest increasing and decreasing subsequences", *Canadian Journal of Math.* **13**, (1961), 179–191.

[Schü1] M. P. Schützenberger, "Quelques remarques sur une construction de Schensted", *Math. Scandinavica* **12**, (1963), 117–128.

[Schü2] M. P. Schützenberger, "La correspondence de Robinson", pp. 59–113 in *Combinatoire et Représentation du Groupe Symétrique* (D. Foata, ed.), Lecture Notes in Mathematics 579, 1976.

[Spa] N. Spaltenstein, *Classes Unipotentes et Sous-groupes de Borel*, Lecture Notes in Mathematics 946, Springer-Verlag, 1982.

[Spr1] T. A. Springer, "Geometric questions arising in the study of unipotent elements", pp. 255–264 in *The Santa Cruz conference on Finite Groups*, Proceedings of Symposia in Pure Mathematics 37, 1980.

[Spr2] T. A. Springer, "Conjugacy classes in Algebraic Groups", pp. 175–209 in *Group Theory, Beijing*, Lecture Notes in Mathematics 1185, 1984.

[Stb1] R. Steinberg, "On the desingularisation of the unipotent variety", *Inventiones Math.* **36**, (1976), 209–224.

[Stb2] R. Steinberg, "An Occurrence of the Robinson-Schensted Correspondence", *Journal of Algebra* **113**, (1988), 523–528.