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High-order time-accuracy schemes for parabolic singular perturbation problems with convection

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Abstract — The first boundary value problem for a singularly perturbed parabolic PDE with convection is considered on an interval. For the case of sufficiently smooth data, it is easy to construct a standard finite difference operator and a piecewise uniform mesh condensing in the boundary layer, which gives an ε-uniformly convergent difference scheme. The order of convergence for such a scheme is exactly one and close to one up to a small logarithmic factor with respect to the time and space variables, respectively. In this paper we construct high-order time-accurate ε-uniformly convergent schemes by a defect-correction technique. The efficiency of the new defect-correction scheme is confirmed by numerical experiments.

We consider the first boundary value problem for a singularly perturbed parabolic PDE with convection on an interval. The highest derivative in the equation is multiplied by an arbitrarily small parameter ε. When the parameter ε tends to zero, boundary layers may appear, which leads to difficulties when classical discretization methods are applied, because the error in the approximate solution depends on the value of ε. The appropriate location of the nodes is needed to ensure that the error is independent of the parameter value and depends only on the number of nodes in the mesh. Special schemes with this property are called ε-uniformly convergent. In [1–5] we introduced and analysed ε-uniformly convergent difference schemes for singularly perturbed boundary value problems for elliptic and parabolic equations. If the problem data is sufficiently smooth, for the parabolic equations with convection terms, the order of ε-uniform convergence for the scheme studied is exactly one and up to a small logarithmic factor one with respect to the time and space variables, respectively, i.e. $O(N^{-1}\ln^2 N + K^{-1})$, where N and K are the number of intervals in the space and time discretization. Because the amount of the computational work is proportional to the number K, the higher-order accuracy in time can considerably reduce the computational cost. Therefore it is of interest to develop methods for which the order of convergence with respect to the time variable is increased.

For equations without convective terms the improvement of the accuracy in time, preserving ε-uniform convergence, by means of a defect-correction technique was also studied in [4, 5]. In this paper we develop schemes for which the order of

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convergence in time can be arbitrarily large if the solution is sufficiently smooth. They are also based on the defect-correction principle, but for a new class of singular perturbation problems, i.e. for equations with convective terms. In contrast to our previous papers [4, 5], here we use the new experimental technique for the determination of convergence orders. As a result, we carry out a sufficiently accurate analysis of the errors in the numerical solutions, which strongly supports the nontrivial theoretical results.

1. THE STUDIED CLASS OF BOUNDARY VALUE PROBLEMS

On the domain $G=(0,1)\times(0,T]$ with the boundary $S=\overline{G}\setminus G$ we consider the following singularly perturbed parabolic equation with the Dirichlet boundary conditions:

$$\begin{aligned} L_{(1.1)}u(x,t) &\equiv \left\{\varepsilon\, a(x,t)\frac{\partial^2}{\partial x^2}+b(x,t)\,\frac{\partial}{\partial x}-c(x,t)-p(x,t)\frac{\partial}{\partial t}\right\}u(x,t) \\ &= f(x,t), \quad (x,t)\in G \qquad (1.1a) \\ u(x,t) &= \varphi(x,t), \quad (x,t)\in S. \qquad (1.1b) \end{aligned}$$

Hereafter the notation is such that the operator $L_{(a.b)}$ is first introduced in equation $(a.b)$:

For $S=S_0\cup S^L$ we distinguish the lateral boundary $S^L=\{(x,t):\ x=0 \text{ or } x=1,\ 0<t\leqslant T\}$ and the initial boundary $S_0=\{(x,t):\ x\in[0,1],\ t=0\}$. In (1.1b) $a(x,t)$, $b(x,t)$, $c(x,t)$, $p(x,t)$, $f(x,t)$, $(x,t)\in\overline{G}$, and $\varphi(x,t)$, $(x,t)\in S$, are sufficiently smooth and bounded functions which satisfy

$$0<a_0\leqslant a(x,t),\ 0<b_0\leqslant b(x,t),\ 0<p_0\leqslant p(x,t),\ c(x,t)\geqslant 0,\ (x,t)\in\overline{G}. \qquad (1.1c)$$

The real parameter ε from (1.1a) may take any values from the half-open interval:

$$\varepsilon\in(0,1]. \qquad (1.1d)$$

When the parameter ε tends to zero, the solution exhibits a layer in a neighbourhood of the set $S_1^L=\{(x,t):\ x=0,\ 0\leqslant t\leqslant T\}$, i.e. the left side of the lateral boundary. This layer is described by an ordinary differential equation (an ordinary boundary layer).

2. DIFFERENCE SCHEME ON AN ARBITRARY MESH

To solve problem (1.1) we first consider a classical finite difference method. On the set $\overline{G}$ we introduce the rectangular mesh

$$\overline{G}_h=\overline{\omega}\times\overline{\omega}_0 \qquad (2.1)$$

where $\overline{\omega}$ is the (possibly) non-uniform mesh of nodal points x^i on $[0,1]$, $\overline{\omega}_0$ is a uniform mesh on the interval $[0,T]$, N and K are the numbers of intervals in the meshes $\overline{\omega}$ and $\overline{\omega}_0$, respectively. We define $\tau=T/K$, $h^i=x^{i+1}-x^i$, $h=\max_i h^i$, $h\leqslant M/N$, $G_h=G\cap\overline{G}_h$, $S_h=S\cap\overline{G}_h$.

In the following we denote by M (or m) sufficiently large (or small) positive constants which do not depend on the value of the parameter ε or on the difference operators.

For problem (1.1) we use the difference scheme [9]

$$\Lambda_{(2.2)}\,z(x,t)=f(x,t), \qquad (x,t)\in G_h \qquad (2.2a)$$

$$z(x,t)=\varphi(x,t), \qquad (x,t)\in S_h. \qquad (2.2b)$$

Here

$$\begin{aligned} \Lambda_{(2.2)}\,z(x,t) &\equiv \{\varepsilon\, a(x,t)\delta_{\overline{x}\widehat{x}}+b(x,t)\delta_x-c(x,t)-p(x,t)\delta_{\overline{t}}\}\,z(x,t) \\ \delta_{\overline{x}\widehat{x}}z(x^i,t) &= 2(h^{i-1}+h^i)^{-1}\left(\delta_x z(x^i,t)-\delta_{\overline{x}}z(x^i,t)\right) \\ \delta_{\overline{x}}z(x^i,t) &= (h^{i-1})^{-1}\left(z(x^i,t)-z(x^{i-1},t)\right) \\ \delta_x z(x^i,t) &= (h^i)^{-1}\left(z(x^{i+1},t)-z(x^i,t)\right) \\ \delta_{\overline{t}}z(x^i,t) &= \tau^{-1}\left(z(x^i,t)-z(x^i,t-\tau)\right), \end{aligned}$$

where $\delta_x z(x,t)$ and $\delta_{\overline{x}}z(x,t)$, $\delta_{\overline{t}}z(x,t)$ are the forward and backward differences, and the difference operator $\delta_{\overline{x}\widehat{x}}z(x,t)$ is an approximation of the operator $(\partial^2/\partial x^2)u(x,t)$ on the non-uniform mesh.

From [9] we know that the difference scheme (2.2), (2.1) is monotone. Using the maximum principle and taking into account the estimates of the derivatives (see Theorem 8.1 in the Appendix) we find that the solution of the difference scheme (2.2), (2.1) converges for a fixed value of the parameter ε:

$$|\,u(x,t)-z(x,t)\,|\leqslant M(\varepsilon^{-2}N^{-1}+\tau), \quad (x,t)\in\overline{G}_h. \qquad (2.3)$$

This error bound for the classical difference scheme is clearly not ε-uniform.

The proof of (2.3) follows the lines of the classical convergence proof for monotone difference schemes (see [9, 10]). Taking into account the above *a priori* estimates for the solution, this results in the following theorem.

Theorem 2.1. *Suppose that for the functions in equation* (1.1) *we have* a, b, c, p, $f\in H^{(\alpha+2n-2)}(\overline{G})$, $\varphi\in H^{(\alpha+2n)}(\overline{G})$, $\alpha>4$, $n=0$, *and let the condition* (1.1) *with* $n=0$ *be satisfied. Then for a fixed value of the parameter* ε, *the solution of* (2.2), (2.1) *converges to the solution of* (1.1) *with an error bound given by* (2.3).

3. THE ε-UNIFORMLY CONVERGENT SCHEME

Here we consider ε-uniformly convergent method for (1.1) using a special mesh condensed in the neighbourhood of the boundary layer. The location of the nodes is derived from the *a priori* estimates of the solution and its derivatives. The way to construct the mesh for problem (1.1) is the same as in [4, 5, 11]. More specifically, we take

$$\overline{G}_h^* = \overline{\omega}^*(\sigma) \times \overline{\omega}_0 \tag{3.1}$$

where $\overline{\omega}_0$ is the uniform mesh with step-size $\tau = T/K$, i.e. $\overline{\omega}_0 = \overline{\omega}_{0(2.1)}$, and $\overline{\omega}^* = \overline{\omega}^*(\sigma)$ is a special *piecewise* uniform mesh depending on the parameter $\sigma \in \mathbf{R}$, which depends on ε and N. We take $\sigma = \sigma_{(3.1)}(\varepsilon, N) = \min(1/2, m^{-1}\varepsilon \ln N)$, where $m = m_{(3.1)}$ is an arbitrary number from the interval $m_0 = \min_{\overline{G}}[a^{-1}(x,t)b(x,t)]$. The mesh $\overline{\omega}^*(\sigma)$ is constructed as follows. The interval $[0,1]$ is divided into two parts $[0,\sigma]$ and $[\sigma,1]$, $\sigma \leqslant 1/2$. In each part we use a uniform mesh with $N/2$ subintervals both on $[0,\sigma]$ and $[\sigma,1]$.

Theorem 3.1. *Let the conditions of Theorem* 2.1 *hold. Then the solution of* (2.2), (3.1) *converges ε-uniformly to the solution of* (1.1) *and the following estimate holds:*

$$|u(x,t) - z(x,t)| \leqslant M(N^{-1}\ln N + \tau), \quad (x,t) \in \overline{G}_h^*. \tag{3.2}$$

The proof of this theorem can be found in [10, 12].

4. NUMERICAL RESULTS FOR SCHEME (2.2), (3.1)

To see the effect of the special mesh in practice, we take the model problem

$$L_{(4.1)}u(x,t) \equiv \left\{\varepsilon\frac{\partial^2}{\partial x^2} + \frac{\partial}{\partial x} - \frac{\partial}{\partial t}\right\} u(x,t) = f(x,t), \quad (x,t) \in G \tag{4.1}$$

$$u(x,t) = \varphi(x,t), \quad (x,t) \in S$$

where

$$f(x,t) = -4t^3, \quad (x,t) \in \overline{G}, \qquad \varphi(x,t) = 0, \quad (x,t) \in S; \quad T = 1.$$

For the approximation of problem (4.1) we use the scheme (2.2), (3.1), where $m = 1/2$, $\overline{G}_h = \overline{G}_h^*$.

Since the exact solution of this problem is unknown, we replace it by the numerical solution U_ε^{2048} computed on the finest available mesh $\overline{G}_h$ with $N = K = 2048$ for each value of ε. Then the maximum pointwise error computed is defined by

$$E(N,K,\varepsilon) = \max_{(x,t)\in\overline{G}_h} |z(x,t) - u^*(x,t)|. \tag{4.2}$$

Here $u^*(x,t)$ is the linear interpolation obtained from the reference solution U_ε^{2048} corresponding to the numerical solution $z(x,t)$ of problem (2.2), (3.1). We compute

Table 1.
Errors $E(N = K, \varepsilon)$ for model problem (4.1) with the special scheme (2.2), (3.1).

$\varepsilon \setminus N$	16	32	64	128	256	512
1.0	2.178-03	1.179-03	6.046-04	2.989-04	1.410-04	6.073-05
2^{-1}	6.460-03	3.555-03	1.840-03	9.126-04	4.312-04	1.859-04
2^{-2}	1.533-02	8.465-03	4.402-03	2.188-03	1.035-03	4.467-04
2^{-3}	2.950-02	1.639-02	8.544-03	4.257-03	2.017-03	8.708-04
2^{-4}	4.819-02	3.275-02	2.238-02	1.148-02	5.510-03	2.399-03
2^{-5}	6.342-02	3.601-02	2.334-02	1.454-02	8.192-03	4.061-03
2^{-6}	7.341-02	4.263-02	2.409-02	1.498-02	8.460-03	4.192-03
2^{-7}	7.763-02	4.651-02	2.495-02	1.521-02	8.601-03	4.269-03
2^{-8}	7.939-02	4.819-02	2.618-02	1.534-02	8.669-03	4.307-03
2^{-9}	8.015-02	4.893-02	2.673-02	1.540-02	8.707-03	4.326-03
2^{-10}	8.050-02	4.927-02	2.699-02	1.543-02	8.728-03	4.336-03
2^{-18}	8.082-02	4.984-02	2.730-02	1.547-02	8.749-03	4.345-03
$\overline{E}(N)$	8.082-02	4.984-02	2.730-02	1.547-02	8.749-03	4.345-03

$E(N,K,\varepsilon)$ for various values of ε, N, K. Note that no special interpolation is needed along the t-axis.

The results are given in Table 1. From the analysis of the numerical results we conclude that in accordance with (3.2) the order of convergence for large $N = K$ is $O(N^{-1}\ln N + K^{-1})$, i.e. almost one with respect to the space and time variables which corresponds to the theoretical results.

In Table 1 the function $E(N,K,\varepsilon)$ is defined by (4.2). Here $K = N$. In the bottom row $\overline{E}(N)$ gives the computed maximum pointwise errors for each column.

5. IMPROVED ACCURACY IN TIME

5.1. A scheme based on defect correction

In this section we construct a new discrete method based on defect correction, which also converges ε-uniformly to the solution of the boundary value problem, but with an order of accuracy (with respect to τ) higher than that in (3.2).

To improve time-accuracy we use the technique based on the one proposed in [4, 5]. For the difference scheme (2.2), (3.1) the error in the approximation of the partial derivative $(\partial/\partial t)\,u(x,t)$ is due to the divided difference $\delta_{\bar t}z(x,t)$ and is associated with the truncation error given by the relation

$$\frac{\partial}{\partial t}u(x,t) - \delta_{\bar t}u(x,t) = \frac{1}{2}\tau\frac{\partial^2}{\partial t^2}u(x,t) - \frac{1}{6}\tau^2\frac{\partial^3}{\partial t^3}u(x,t-\vartheta), \quad \vartheta \in [0,\tau]. \tag{5.1}$$

Therefore we now use for the approximation of $(\partial/\partial t)\,u(x,t)$ the expression

$$\delta_{\bar t}u(x,t) + \tau\delta_{\bar t\bar t}u(x,t)/2$$

where $\delta_{\bar t\bar t}\, u(x,t) \equiv \delta_{t\bar t}\, u(x,t-\tau)$. Notice that $\delta_{t\bar t}\, u(x,t)$ is the second central divided difference. We can evaluate a better approximation than (2.2a) by defect correction

$$\Lambda_{(2.2)} z^c(x,t) = f(x,t) + \frac{1}{2} p(x,t)\tau \frac{\partial^2}{\partial t^2} u(x,t) \tag{5.2}$$

with $x \in \overline{\omega}$ and $t \in \overline{\omega}_0$, where $\overline{\omega}$ and $\overline{\omega}_0$ are given in (2.1); τ is the step-size of the mesh $\overline{\omega}_0$; $z^c(x,t)$ is the 'corrected' solution. Instead of $(\partial^2/\partial t^2)\, u(x,t)$ we shall use $\delta_{\bar t\bar t}\, z(x,t)$, where $z(x,t)$, $(x,t) \in G_{h(3.1)}$, is the solution of the difference scheme (2.2), (3.1). We may expect that the new solution $z^c(x,t)$ has a consistency error $O(\tau^2)$. This is true, as will be shown in Subsection 5.3.

Moreover, in a similar way we can construct an ε-uniform difference approximation with a convergence order higher than two (with respect to the time variable) and $O(N^{-1} \ln N)$ with respect to the space variable.

5.2. The defect-correction scheme of second-order accuracy in time

We denote by $\delta_{k\bar t} z(x,t)$ the backward difference of order k:

$$\delta_{k\bar t}\, z(x,t) = \left(\delta_{k-1\,\bar t}\, z(x,t) - \delta_{k-1\,\bar t}\, z(x,t-\tau)\right)/\tau, \quad t \geqslant k\tau, \quad k \geqslant 1$$
$$\delta_{0\bar t}\, z(x,t) = z(x,t), \quad (x,t) \in \overline{G}_h.$$

To construct the difference schemes of second-order accuracy in τ in (5.2), instead of $(\partial^2/\partial t^2)u(x,t)$ we use $\delta_{2\bar t}\, z(x,t)$, i.e. the second divided difference of the solution to the discrete problem (2.2), (3.1). On the mesh $\overline{G}_h$ we write the finite difference scheme (2.2) in the form

$$\begin{aligned} \Lambda_{(2.2)} z^{(1)}(x,t) &= f(x,t), \quad (x,t) \in G_h \\ z^{(1)}(x,t) &= \varphi(x,t), \quad (x,t) \in S_h \end{aligned} \tag{5.3}$$

where $z^{(1)}(x,t)$ is the uncorrected solution. For the corrected solution $z^{(2)}(x,t)$ we solve the problem for $(x,t) \in G_h$:

$$\begin{aligned} \Lambda_{(2.2)} z^{(2)}(x,t) &= f(x,t) + \left\{ \begin{array}{ll} p(x,t)\frac{1}{2}\tau\frac{\partial^2}{\partial t^2}u(x,0), & t = \tau \\ p(x,t)\frac{1}{2}\tau\delta_{2\bar t}\, z^{(1)}(x,t), & t \geqslant 2\tau \end{array} \right\}, \quad (x,t) \in G_h \\ z^{(2)}(x,t) &= \varphi(x,t), \quad (x,t) \in S_h. \end{aligned} \tag{5.4}$$

Here the derivative $(\partial^2/\partial t^2)u(x,0)$ is obtained from the equation (1.1a). We say that $z^{(2)}(x,t)$ is the solution of the difference scheme (5.4), (5.3), (3.1) [or briefly, (5.4), (3.1)].

For simplicity, in the remainder of this subsection we suppose that the coefficients $a(x,t)$, $b(x,t)$ do not depend on t:

$$a(x,t) = a(x), \quad b(x,t) = b(x), \quad (x,t) \in \overline{G} \tag{5.5}$$

and we take the homogeneous initial condition:

$$\varphi(x,0) = 0, \quad x \in [0,1]. \tag{5.6}$$

Under the conditions (5.5), (5.6), the following estimate holds for the solution of problem (5.4), (3.1):

$$\left| u(x,t) - z^{(2)}(x,t) \right| \leqslant M\left(N^{-1}\ln N + \tau^2\right), \quad (x,t) \in \overline{G}_h. \tag{5.7}$$

Theorem 5.1. *Suppose the conditions* (5.5), (5.6) *hold and for the functions in equation* (1.1) *we have* a, b, c, p, $f \in H^{(\alpha+2n-2)}(\overline{G})$, $\varphi \in H^{(\alpha+2n)}(\overline{G})$, $\alpha > 4$, $n = 1$. *Let the condition* (8.1) *with* $n = 1$ *be satisfied. Then the estimate* (5.7) *holds for the solution of the difference scheme* (5.4), (3.1).

Proof. The proof of Theorem 5.1 is given in the Appendix, see Subsection 8.2.

5.3. The defect-correction scheme of third-order accuracy in time

The above procedure can be used to obtain an arbitrarily high order of accuracy in time. Here we only show how to construct the difference scheme of third-order accuracy. On the grid $\overline{G}_h$ we consider the difference scheme

$$\begin{aligned} &\Lambda_{(2.2)}\, z^{(3)}(x,t) \qquad (5.8a) \\ &= f(x,t) + \left\{ \begin{array}{ll} p(x,t)\left(C_{11}\tau\frac{\partial^2}{\partial t^2}u(x,0) + C_{12}\tau^2\frac{\partial^3}{\partial t^3}u(x,0)\right), & t = \tau \\ p(x,t)\left(C_{21}\tau\frac{\partial^2}{\partial t^2}u(x,0) + C_{22}\tau^2\frac{\partial^3}{\partial t^3}u(x,0)\right), & t = 2\tau \\ p(x,t)\left(C_{31}\tau\delta_{2\bar t}z^{(2)}(x,t) + C_{32}\tau^2\delta_{3\bar t}z^{(1)}(x,t)\right), & t \geqslant 3\tau \end{array} \right\} \\ &\quad (x,t) \in G_h \\ &z^{(3)}(x,t) = \varphi(x,t), \quad (x,t) \in S_h. \end{aligned}$$

Here $z^{(1)}(x,t)$ and $z^{(2)}(x,t)$ are the solutions of problems (5.3), (3.1) and (5.4), (3.1), respectively, the derivatives $(\partial^2/\partial t^2)u(x,0)$, $(\partial^3/\partial t^3)u(x,0)$ are obtained from equation (1.1a) as before. The coefficients C_{ij} are chosen such that they satisfy the conditions:

$$\frac{\partial}{\partial t}u(x,t) = \delta_{\bar t}u(x,t) + C_{11}\tau\frac{\partial^2}{\partial t^2}u(x,t-\tau) + C_{12}\tau^2\frac{\partial^3}{\partial t^3}u(x,t-\tau) + O(\tau^3)$$
$$\frac{\partial}{\partial t}u(x,t) = \delta_{\bar t}u(x,t) + C_{21}\tau\frac{\partial^2}{\partial t^2}u(x,t-2\tau) + C_{22}\tau^2\frac{\partial^3}{\partial t^3}u(x,t-2\tau) + O(\tau^3)$$
$$\frac{\partial}{\partial t}u(x,t) = \delta_{\bar t}u(x,t) + C_{31}\tau\delta_{2\bar t}u(x,t) + C_{32}\tau^2\delta_{3\bar t}u(x,t) + O(\tau^3).$$

It follows that

$$C_{11} = C_{21} = C_{31} = \frac{1}{2}, \quad C_{12} = C_{32} = \frac{1}{3}, \quad C_{22} = \frac{5}{6}. \tag{5.8b}$$

By $z^{(3)}(x,t)$ we denote the solution of the difference scheme (5.8), (3.1) and, as before, for simplicity we introduce the homogeneous initial conditions

$$\varphi(x,0) = 0, \quad f(x,0) = 0, \quad x \in [0,1]. \tag{5.9}$$

Under the conditions (5.5), (5.9) the following estimate holds for the solution of the difference scheme (5.8), (3.1):

$$\left| u(x,t) - z^{(3)}(x,t) \right| \leqslant M \left(N^{-1} \ln N + \tau^3 \right), \quad (x,t) \in \overline{G}_h. \tag{5.10}$$

Theorem 5.2. *Suppose the conditions* (5.9) *hold and for the functions in equation* (1.1) *we have* $a, b, c, p, f \in H^{(\alpha+2n-2)}(\overline{G})$, $\varphi \in H^{(\alpha+2n)}(\overline{G})$, $\alpha > 4$, $n = 2$. *Let the condition* (8.1) *with* $n = 2$ *be satisfied. Then the estimate* (5.10) *is valid for the solution of the scheme* (5.8), (3.1).

Proof. The proof of Theorem 5.2 is given in the Appendix, see Subsection 8.3.

In a similar way we could construct difference schemes with an arbitrarily high order of accuracy

$$O(N^{-1} \ln N + \tau^{n+1}), \quad n > 2.$$

6. NUMERICAL RESULTS FOR THE TIME-ACCURATE SCHEMES

We find the solution of the boundary value problem

$$L_{(4.1)} u(x,t) = 0, \quad 0 < x < 1, \quad 0 < t \leqslant T, \quad T = 1 \tag{6.1}$$

$$u(0,t) = t^4, \quad 0 < t \leqslant T, \quad u(x,t) = 0, \quad (x,t) \in S, \quad x > 0.$$

It should be noted that the solution of this problem is singular.

The idea of using the analytical solution of problem (6.1) to compute errors in the approximate solution, as was done in [4, 5], is appealing. But here the suitable (for computation) representation of the solution $u(x,t)$ is unknown. It is possible to use, as the exact solution, the solution of the grid problem on a very fine mesh with a large number of nodes. But this method is not efficient because the analysis of the order of accuracy for a defect-correction scheme requires a very dense mesh, which leads to high computational costs and, besides, to large round-off errors.

Here we use the method from [6], which is different from the above techniques. The solution of problem (6.1) is represented in the form of the sum:

$$u(x,t) = V^{(1)}(x,t) + v(x,t), \quad (x,t) \in \overline{G} \tag{6.2}$$

where $V^{(1)}(x,t)$ is the main singular part (two first terms) of the asymptotic expansion of the solution of problem (6.1), and $v(x,t)$ is the remainder term, which is a sufficiently small smooth function. The function $V^{(1)}(x,t)$ has a sufficiently simple analytical representation:

$$V^{(1)}(x,t) = V_0(x,t) + V_1(x,t), \quad (x,t) \in \overline{G}$$

where

$$V_0(x,t) = t^4 \Psi(x), \quad \Psi(x) = \left(\exp(-\varepsilon^{-1}x) - \exp(-\varepsilon^{-1})\right) / \left(1 - \exp(-\varepsilon^{-1})\right)$$

$$V_1(x,t) = -4t^3 x \exp(-x/\varepsilon) \,/\, (1 - \exp(-1/\varepsilon))$$

$$|V_0(x,t)| \leqslant M, \quad |V_1(x,t)| \leqslant M\varepsilon, \quad (x,t) \in \overline{G}.$$

The function $v(x,t)$ is the solution of the problem

$$L_{(4.1)} v(x,t) = f_0(x,t), \quad (x,t) \in G \tag{6.3}$$

$$v(0,t) = 0, \quad v(1,t) = -V_1(1,t), \quad 0 < t \leqslant T, \qquad v(x,0) = 0, \quad 0 < x < 1.$$

Here

$$f_0(x,t) = -4t^2 \left(t \exp(-1/\varepsilon) + 3x \exp(-x/\varepsilon) \right) / \left(1 - \exp(-1/\varepsilon)\right).$$

For the function $v(x,t)$ the estimate holds:

$$\left| \frac{\partial^{k+k_0}}{\partial x^k \partial t^{k_0}} v(x,t) \right| \leqslant M\varepsilon^2 (1 + \varepsilon^{-k}), \quad (x,t) \in \overline{G}, \quad k + 2k_0 \leqslant 4, \quad k \leqslant 3.$$

Then the function $v(x,t)$ and the product $\varepsilon^2 (\partial^4 / \partial x^4) v(x,t)$ are ε-uniformly bounded. Thus, we can consider $v(x,t)$ as the regular part of this solution.

(1) We solve the grid problem, which approximates the boundary value problem (6.3), on the finest available mesh $\overline{G}_h = \overline{G}_{(*)} h(3.1)$ for $N = K = 2048$ and for the chosen value of ε. It is not difficult to find the function $v(x,t) = v_\varepsilon^{2048}(x,t)$ and the reference solution

$$u(x,t)_{(6.2)} = u_\varepsilon^{2048}(x,t) = V^{(1)}(x,t) + v_\varepsilon^{2048}(x,t).$$

(2) Further, for solving problem (6.1) we use successively the scheme (5.3), (3.1) and the defect-correction schemes (5.4), (3.1) and (5.8), (3.1) to find the functions $z^{(1)}(x,t)$, $z^{(2)}(x,t)$ and $z^{(3)}(x,t)$, respectively. Note that $z^{(1)}(x,t)$ is the uncorrected solution, $z^{(2)}(x,t)$ and $z^{(3)}(x,t)$ are the corrected solutions. In these cases we compute the maximum pointwise errors $E(N,K,\varepsilon)$ by formula (4.2), where $u^*(x,t)$ is the linear interpolation obtained from the reference solution $u_\varepsilon^{2048}(x,t)$ corresponding to the numerical solution $z^{(k)}(x,t)$, $k = 1,2,3$, for the values $N = 2^i$, $i = 2,3,\dots,10$, $K = 2^j$, $j = 2,3,\dots,10$.

Table 2.
Errors $E(N,K)$ for $\varepsilon = 2^{-10}$.

$K \setminus N$	4	8	16	32	64	128	256	512	1024
					$z^{(1)}$				
4	**1.51-1**	1.42-1	1.03-1	6.86-2	4.39-2	2.67-2	1.58-2	9.21-3	5.36-3
8	1.39-1	**1.38-1**	1.02-1	6.82-2	4.36-2	2.65-2	1.56-2	9.00-3	5.15-3
16	1.32-1	1.36-1	**1.02-1**	6.80-2	4.35-2	2.63-2	1.55-2	8.88-3	5.03-3
32	1.29-1	1.35-1	1.01-1	**6.79-2**	4.34-2	2.63-2	1.54-2	8.81-3	4.97-3
64	1.27-1	1.35-1	1.01-1	6.79-2	**4.34-2**	2.62-2	1.54-2	8.78-3	4.93-3
128	1.26-1	1.35-1	1.01-1	6.78-2	4.34-2	**2.62-2**	1.53-2	8.76-3	4.92-3
256	1.26-1	1.35-1	1.01-1	6.78-2	4.34-2	2.62-2	**1.53-2**	8.75-3	4.91-3
512	1.26-1	1.34-1	1.01-1	6.78-2	4.34-2	2.62-2	1.53-2	**8.75-3**	4.91-3
1024	1.26-1	1.34-1	1.01-1	6.78-2	4.34-2	2.62-2	1.53-2	8.75-3	**4.90-3**
					$z^{(2)}$				
4	**1.33-1**	1.37-1	1.02-1	6.81-2	4.35-2	2.64-2	1.55-2	8.89-3	5.05-3
8	1.28-1	**1.35-1**	1.01-1	6.79-2	4.34-2	2.62-2	1.54-2	8.79-3	4.94-3
16	1.26-1	1.35-1	**1.01-1**	6.78-2	4.34-2	2.62-2	1.53-2	8.76-3	4.91-3
32	1.26-1	1.34-1	1.01-1	**6.78-2**	4.34-2	2.62-2	1.53-2	8.75-3	4.90-3
64	1.25-1	1.34-1	1.01-1	6.78-2	**4.34-2**	2.62-2	1.53-2	8.75-3	4.90-3
128	1.25-1	1.34-1	1.01-1	6.78-2	4.34-2	**2.62-2**	1.53-2	8.75-3	4.90-3
256	1.25-1	1.34-1	1.01-1	6.78-2	4.34-2	2.62-2	**1.53-2**	8.75-3	4.90-3
512	1.25-1	1.34-1	1.01-1	6.78-2	4.34-2	2.62-2	1.53-2	**8.75-3**	4.90-3
1024	1.25-1	1.34-1	1.01-1	6.78-2	4.34-2	2.62-2	1.53-2	8.75-3	**4.90-3**
					$z^{(3)}$				
4	**1.28-1**	1.35-1	1.01-1	6.79-2	4.34-2	2.62-2	1.54-2	8.78-3	4.94-3
8	1.26-1	**1.34-1**	1.01-1	6.78-2	4.34-2	2.62-2	1.53-2	8.75-3	4.91-3
16	1.25-1	1.34-1	**1.01-1**	6.78-2	4.34-2	2.62-2	1.53-2	8.75-3	4.90-3
32	1.25-1	1.34-1	1.01-1	**6.78-2**	4.34-2	2.62-2	1.53-2	8.75-3	4.90-3
64	1.25-1	1.34-1	1.01-1	6.78-2	**4.34-2**	2.62-2	1.53-2	8.75-3	4.90-3
128	1.25-1	1.34-1	1.01-1	6.78-2	4.34-2	**2.62-2**	1.53-2	8.75-3	4.90-3
256	1.25-1	1.34-1	1.01-1	6.78-2	4.34-2	2.62-2	**1.53-2**	8.75-3	4.90-3
512	1.25-1	1.34-1	1.01-1	6.78-2	4.34-2	2.62-2	1.53-2	**8.75-3**	4.90-3
1024	1.25-1	1.34-1	1.01-1	6.78-2	4.34-2	2.62-2	1.53-2	8.75-3	**4.90-3**

The computational process (1) and (2) is repeated for all values of $\varepsilon = 2^{-n}$, $n = 0,2,4,\dots,12$. As a result, we get $E(N,K,\varepsilon)$ for various values of ε, N, K for each of the functions $z^{(1)}(x,t)$, $z^{(2)}(x,t)$, $z^{(3)}(x,t)$. Analysing these results, we observe the convergence of the solutions for increasing $N = K$ for any of the functions $z^{(1)}$, $z^{(2)}$, $z^{(3)}$ and for all used values of ε. In order to illustrate this result we give Table 2 for $\varepsilon = 2^{-10}$. The analogous tables for other values of ε are similar.

In Table 2 the values of $E(N,K)$ are given for the functions $z^{(1)}$, $z^{(2)}$, and $z^{(3)}$. For each of them we see decreasing errors for $N = K$, i.e. we have ε-uniform convergence. But the order of convergence, which we observe, is approximately equal to one for all functions. All the errors corresponding to the same values of N, K but to different $z^{(k)}$ are similar.

We know that the error of approximation consists of two parts. One part is due

Table 3.
Space errors $E^{(s)}(N,K)$ for $\varepsilon = 2^{-10}$.

$K \setminus N$	8	16	32	64	128	256	512
				$z^{(1)}$			
4	3.85-2	3.47-2	2.47-2	1.72-2	1.09-2	6.59-3	3.85-3
8	3.61-2	3.40-2	2.46-2	1.72-2	1.09-2	6.59-3	3.85-3
16	3.48-2	3.36-2	2.45-2	1.72-2	1.09-2	6.59-3	3.85-3
32	3.41-2	3.34-2	2.45-2	1.72-2	1.09-2	6.58-3	3.85-3
64	3.38-2	3.32-2	2.45-2	1.72-2	1.09-2	6.58-3	3.85-3
128	3.36-2	3.32-2	2.45-2	1.72-2	1.09-2	6.58-3	3.84-3
256	3.35-2	3.32-2	2.45-2	1.72-2	1.09-2	6.58-3	3.84-3
512	3.35-2	3.31-2	2.45-2	1.72-2	1.09-2	6.58-3	3.84-3
1024	3.35-2	3.31-2	2.45-2	1.72-2	1.09-2	6.58-3	3.84-3
				$z^{(2)}$			
4	3.50-2	3.36-2	2.45-2	1.72-2	1.09-2	6.59-3	3.85-3
8	3.39-2	3.33-2	2.45-2	1.72-2	1.09-2	6.58-3	3.85-3
16	3.36-2	3.32-2	2.45-2	1.72-2	1.09-2	6.58-3	3.84-3
32	3.35-2	3.31-2	2.45-2	1.72-2	1.09-2	6.58-3	3.84-3
64	3.35-2	3.31-2	2.45-2	1.72-2	1.09-2	6.58-3	3.84-3
128	3.34-2	3.31-2	2.45-2	1.72-2	1.09-2	6.58-3	3.84-3
...	...	...	...	...	...	...	...
1024	3.34-2	3.31-2	2.45-2	1.72-2	1.09-2	6.58-3	3.84-3
				$z^{(3)}$			
4	3.41-2	3.33-2	2.45-2	1.72-2	1.09-2	6.58-3	3.85-3
8	3.35-2	3.32-2	2.45-2	1.72-2	1.09-2	6.58-3	3.84-3
16	3.35-2	3.31-2	2.45-2	1.72-2	1.09-2	6.58-3	3.84-3
32	3.34-2	3.31-2	2.45-2	1.72-2	1.09-2	6.58-3	3.84-3
...	...	...	...	...	...	...	...
1024	3.34-2	3.31-2	2.45-2	1.72-2	1.09-2	6.58-3	3.84-3

to the discretization of the space derivatives and the second is due to the time discretisation. For brevity, these components will be referred to as the space error and the time error. Since by the defect correction we improve only the accuracy with respect to the time, we expect a decreasing time error. It can be much smaller than the space error and therefore the observed error in Table 2 corresponds only to the space error. In order to show this we split the combined error into the space error (Table 3) and the time error (Table 5). The structure of Table 3 is similar to that of Table 2.

Table 3 presents the values of the space errrors computed from the formula

$$E^{(s)}(N_i,K) = E(N_i,K) - E(N_{i+1},K), \quad i = 3,4,\dots,9, \quad N_i = 2^i.$$

We see that the errors are the same for all different K. The orders of the errors in Table 2 and Table 3 are the same.

From Table 3 we construct Table 4, where the ratios of the space errors are given by

$$R^{(s)}(N_i,K) = E^{(s)}(N_i,K)/E^{(s)}(N_{i+1},K), \quad i = 3,4,\dots,8.$$

Table 4.
Ratios of space errors $R^{(s)}(N,K)$ for $\varepsilon = 2^{-10}$.

$K \setminus N$	8	16	32	64	128	256
			$z^{(1)}$			
4	1.11	1.40	1.44	1.58	1.65	1.71
8	1.06	1.38	1.43	1.58	1.65	1.71
16	1.04	1.37	1.43	1.58	1.65	1.71
32	1.02	1.36	1.43	1.58	1.65	1.71
64	1.02	1.36	1.43	1.58	1.65	1.71
128	1.01	1.36	1.43	1.58	1.65	1.71
...	...	...	...	...	...	...
1024	1.01	1.36	1.43	1.58	1.65	1.71
			$z^{(2)}$			
4	1.04	1.37	1.43	1.58	1.65	1.71
8	1.02	1.36	1.43	1.58	1.65	1.71
16	1.01	1.36	1.43	1.58	1.65	1.71
...	...	...	...	...	...	...
1024	1.01	1.35	1.43	1.58	1.65	1.71
			$z^{(3)}$			
4	1.02	1.36	1.43	1.58	1.65	1.71
8	1.01	1.36	1.43	1.58	1.65	1.71
...	...	...	...	...	...	...
1024	1.01	1.35	1.43	1.58	1.65	1.71

In Table 4 we see the first order of convergence with respect to the space up to a small logarithmic factor.

In a similar way we construct Table 5 for the time errror:

$$E^{(t)}(N,K_j) = E(N,K_j) - E(N,K_{j+1}), \quad j = 2,3,...,9$$

and Table 6 for their ratios:

$$R^{(t)}(N,K_j) = E^{(t)}(N,K_j)/E^{(t)}(N,K_{j+1}), \quad j = 2,3,4,...,8, \quad K_j = 2^j.$$

And now we see very interesting results in Table 5:

(1) we see that the time error is essentially smaller than the space error. This explains the fact that we do not see the influence of the time error in Table 2;

(2) the errors for $z^{(1)}$ are larger than those for $z^{(2)}$ and the errors for $z^{(2)}$ are larger than those for $z^{(3)}$;

(3) we see that approximately the same error ($\approx 10^{-6}$) is obtained for $z^{(1)}$ at $K = 512$, for $z^{(2)}$ at $K = 32$, and for $z^{(3)}$ at $K = 16$. Because the computational cost is proportional to K, we see that the computational cost is reduced by the defect correction;

(4) Table 6 actually confirms the order of convergence, 6 which is theoretically obtained in Section 5. In theory the solution $z^{(1)}(x,t)$ of problem (5.3), (3.1) converges with rate $O(\tau)$ [estimate (3.2) and Theorem 3.1]. The solution $z^{(2)}(x,t)$ of problem (5.4), (3.1), where $z^{(1)}(x,t)$ is the solution of problem (5.3), (3.1), converges

Table 5.
Time errors $E^{(t)}(N,K)$ for $\varepsilon = 2^{-10}$.

$K \setminus N$	4	8	16	32	64	128	256	512	1024
					$z^{(1)}$				
4	1.24-2	3.46-03	1.08-03	3.64-04	2.44-04	2.23-04	2.15-04	2.11-04	2.08-04
8	6.63-3	1.91-03	6.14-04	2.08-04	1.39-04	1.27-04	1.23-04	1.21-04	1.19-04
16	3.42-3	1.00-03	3.26-04	1.11-04	7.44-05	6.80-05	6.56-05	6.44-05	6.37-05
32	1.73-3	5.12-04	1.68-04	5.73-05	3.84-05	3.51-05	3.39-05	3.32-05	3.29-05
64	8.73-4	2.59-04	8.54-05	2.91-05	1.95-05	1.78-05	1.72-05	1.69-05	1.67-05
128	4.38-4	1.30-04	4.30-05	1.47-05	9.84-06	8.98-06	8.67-06	8.51-06	8.42-06
256	2.19-4	6.54-05	2.16-05	7.37-06	4.94-06	4.51-06	4.35-06	4.27-06	4.22-06
512	**1.10-4**	**3.27-05**	**1.08-05**	**3.69-06**	**2.47-06**	**2.26-06**	**2.18-06**	**2.14-06**	**2.12-06**
					$z^{(2)}$				
4	5.39-3	1.64-03	5.47-04	1.86-04	1.24-04	1.13-04	1.09-04	1.07-04	1.06-04
8	1.55-3	4.82-04	1.61-04	5.36-05	3.54-05	3.23-05	3.12-05	3.06-05	3.03-05
16	4.14-4	1.31-04	4.36-05	1.43-05	9.41-06	8.58-06	8.27-06	8.12-06	8.03-06
32	**1.07-4**	**3.40-05**	**1.13-05**	**3.69-06**	**2.42-06**	**2.21-06**	**2.13-06**	**2.09-06**	**2.07-06**
64	2.73-5	8.68-06	2.88-06	9.35-07	6.14-07	5.59-07	5.40-07	5.29-07	5.24-07
128	6.87-6	2.19-06	7.28-07	2.36-07	1.55-07	1.41-07	1.36-07	1.33-07	1.32-07
256	1.73-6	5.51-07	1.83-07	5.91-08	3.88-08	3.53-08	3.41-08	3.34-08	3.31-08
512	4.32-7	1.38-07	4.58-08	1.48-08	9.71-09	8.85-09	8.53-09	8.37-09	8.28-09
					$z^{(3)}$				
4	2.67-3	7.37-04	2.09-04	5.74-05	3.53-05	3.19-05	3.08-05	3.02-05	2.99-05
8	3.15-4	9.26-05	2.66-05	7.17-06	4.41-06	3.99-06	3.85-06	3.77-06	3.73-06
16	**4.13-5**	**1.23-05**	**3.45-06**	**9.00-07**	**5.51-07**	**4.99-07**	**4.81-07**	**4.72-07**	**4.67-07**
32	5.36-6	1.60-06	4.38-07	1.13-07	6.88-08	6.23-08	6.01-08	5.90-08	5.83-08
64	6.84-7	2.04-07	5.51-08	1.41-08	8.60-09	7.79-09	7.51-09	7.37-09	7.29-09
128	8.64-8	2.57-08	6.91-09	1.76-09	1.08-09	9.74-10	9.39-10	9.21-10	9.11-10
256	1.09-8	3.23-09	8.65-10	2.20-10	1.34-10	1.22-10	1.17-10	1.15-10	1.14-10
512	1.36-9	4.04-10	1.08-10	2.75-11	1.68-11	1.52-11	1.47-11	1.44-11	1.42-11

Table 6.
Ratios of time errors $R^{(t)}(N,K)$ for $\varepsilon = 2^{-10}$.

$K\backslash N$	4	8	16	32	64	128	256	512	1024
					$z^{(1)}$				
4	1.87	1.81	1.77	1.75	1.75	1.75	1.75	1.75	1.75
8	1.94	1.91	1.88	1.87	1.87	1.87	1.87	1.87	1.87
16	1.97	1.95	1.94	1.94	1.94	1.94	1.94	1.94	1.94
32	1.99	1.98	1.97	1.97	1.97	1.97	1.97	1.97	1.97
64	1.99	1.99	1.98	1.98	1.98	1.98	1.98	1.98	1.98
128	2.00	1.99	1.99	1.99	1.99	1.99	1.99	1.99	1.99
256	2.00	2.00	2.00	2.00	2.00	2.00	2.00	2.00	2.00
					$z^{(2)}$				
4	3.49	3.40	3.39	3.47	3.50	3.51	3.51	3.51	3.51
8	3.73	3.69	3.70	3.75	3.77	3.77	3.77	3.77	3.77
16	3.86	3.84	3.85	3.88	3.89	3.89	3.89	3.89	3.89
32	3.93	3.92	3.93	3.94	3.94	3.94	3.94	3.94	3.94
64	3.97	3.96	3.96	3.97	3.97	3.97	3.97	3.97	3.97
128	3.98	3.98	3.98	3.99	3.99	3.99	3.99	3.99	3.99
256	3.99	3.99	3.99	3.99	3.99	3.99	3.99	3.99	3.99
					$z^{(3)}$				
4	8.49	7.96	7.85	8.00	8.01	8.00	8.00	8.00	8.00
8	7.62	7.52	7.72	7.97	8.00	8.00	8.00	8.00	8.00
16	7.71	7.71	7.87	7.99	8.00	8.00	8.00	8.00	8.00
32	7.84	7.85	7.94	8.00	8.00	8.00	8.00	8.00	8.00
64	7.92	7.92	7.97	8.00	8.00	8.00	8.00	8.00	8.00
128	7.96	7.96	7.99	8.00	8.00	8.00	8.00	8.00	8.00
256	7.98	7.98	7.99	8.00	8.00	8.00	8.00	8.00	8.00

with rate $O(\tau^2)$ [estimate (5.7) and Theorem 5.1]. The solution $z^{(3)}(x,t)$ of problem (5.8), (3.1), where $z^{(2)}(x,t)$ and $z^{(1)}(x,t)$ are the solutions of problems (5.4), (3.1) and (5.3), (3.1), respectively, converges with rate $O(\tau^3)$ [estimate (5.10) and Theorem 5.2]. The corresponding reduction factors can easily be obtained from Table 6.

7. CONCLUSION

In this paper we have shown the possibilities of the defect-correction procedure used to improve the time-accuracy for a parabolic PDE, which, besides, ensures ε-uniform first-order accuracy in the space discretization.

The error of the approximation consists of two parts. One part is due to the discretization of the space derivatives and the second is due to the time discretisation. We use the defect-correction process only for the improvement of the accuracy with respect to the time and it does not change the error with respect to the space variable. Applying the new experimental technique for the determination of the convergence orders, we have investigated separately the numerical results for the time and space error components. We emphasize that the time error was found as the value of order 10^{-4}–10^{-11} whereas the space error is the value of order 10^{-2}–10^{-3}. Thus, the time error is considerably smaller than the space error, and the total error is essentially equal to the space error.

The numerical results confirm that the order of convergence with respect to the space variable is close to one. By defect correction we are able to increase considerably the time accuracy of the approximate solution, i.e. from the 1st to the 2nd and the 3rd order. The numerical experiments confirm this fact. As a result, we can essentially decrease the number of time intervals and therefore the computational cost.

8. APPENDIX

8.1. Estimates of the solution and its derivatives

In this Appendix we use the *a priori* estimates for the solution of problem (1.1) on the domain $G = D \times [0,T]$ and for its derivatives as derived for elliptic and parabolic equations in [10, 11, 13].

We denote by $H^{(\alpha)}(\overline{G}) = H^{\alpha,\alpha/2}(\overline{G})$ the Hölder space, where α is an arbitrary positive number [7]. We suppose that the functions $f(x,t)$ and $\varphi(x,t)$ satisfy compatibility conditions at the corner points, so that the solution of the boundary value problem is smooth for every fixed value of the parameter ε.

For simplicity we assume that at the corner points $S_0 \cap \overline{S}_1$ the following conditions hold:

$$\begin{aligned} &\frac{\partial^k}{\partial x^k}\varphi(x,t) = \frac{\partial^{k_0}}{\partial t^{k_0}}\varphi(x,t) = 0, \qquad k+2k_0 \leqslant [\alpha]+2n \\ &\frac{\partial^{k+k_0}}{\partial x^k \partial t^{k_0}} f(x,t) = 0, \qquad k+2k_0 \leqslant [\alpha]+2n-2 \end{aligned} \tag{8.1}$$

where $[\alpha]$ is the integer part of a number α, $\alpha > 0$, $n \geqslant 0$ is an integer. We also suppose that $[\alpha]+2n \geqslant 2$.

Using the interior *a priori* estimates and estimates up to the boundary for the regular function $\tilde{u}(\xi,t)$ (see [7]), where $\tilde{u}(\xi,t) = u(x(\xi),t)$, $\xi = x/\varepsilon$, we find for $(x,t) \in \overline{G}$ the estimate

$$\left|\frac{\partial^{k+k_0}}{\partial x^k \partial t^{k_0}} u(x,t)\right| \leqslant M\varepsilon^{-k}, \quad k+2k_0 \leqslant 2n+4, \quad n \geqslant 0. \tag{8.2}$$

This estimate holds, for example, for

$$u \in H^{(2n+4+\nu)}(\overline{G}), \quad \nu > 0 \tag{8.3}$$

where ν is some small number.

For example, the fulfilment of the condition (8.3) is ensured for the solution of (1.1) if the coefficients satisfy $a \in H^{(\alpha+2n-1)}(\overline{G})$, $c, p, f \in H^{(\alpha+2n-2)}(\overline{G})$, $\varphi \in H^{(\alpha+2n)}(\overline{G})$, $\alpha > 4$, $n \geqslant 0$, and the condition (8.1) is satisfied.

In fact, we need a more accurate estimate than (8.2). Therefore we represent the solution of the boundary value problem (1.1) in the form of the sum

$$u(x,t) = U(x,t) + W(x,t), \quad (x,t) \in \overline{G} \tag{8.4}$$

where $U(x,t)$ represents the regular part and $W(x,t)$ the singular part, i.e. the parabolic boundary layer. The function $U(x,t)$ is the smooth solution of the equation (1.1a) satisfying the condition (1.1b) for $t = 0$. For example, under suitable assumptions for the data of the problem, we can consider the solution of the Dirichlet boundary value problem for equation (1.1a) smoothly extended to the domain $\overline{G}^*$ beyond S_1^L ($\overline{G}^*$ is a sufficiently large neighbourhood of $\overline{G}$ beyond S_1^L). On the domain $\overline{G}$ the coefficients and the initial value of the extended problem are the same as for (1.1). Then the function $U(x,t)$ is the restriction (on $\overline{G}$) of the solution to the extended problem, and $U \in H^{(2n+4+\nu)}(\overline{G})$, $\nu > 0$. The function $W(x,t)$ is the solution of the boundary value problem for the parabolic equation:

$$L_{(1.1a)} W(x,t) = 0, \quad (x,t) \in G, \quad W(x,t) = u(x,t) - U(x,t), \quad (x,t) \in S. \tag{8.5}$$

If (8.3) is true, then $W \in H^{(2n+4+\nu)}(\overline{G})$. Now we derive the estimates for the functions $U(x,t)$ and $W(x,t)$:

$$\left| \frac{\partial^{k+k_0}}{\partial x^k \partial t^{k_0}} U(x,t) \right| \leqslant M \tag{8.6}$$

$$\left| \frac{\partial^{k+k_0}}{\partial x^k \partial t^{k_0}} W(x,t) \right| \leqslant M \varepsilon^{-k} \exp(-m_{(8.7)} \varepsilon^{-1} r(x,0)) \tag{8.7}$$

$$(x,t) \in \overline{G}, \quad k + 2k_0 \leqslant 2n+2$$

where $r(x,0)$ is the distance between the point $x \in [0,1]$ and the endpoint $x = 0$ at which the boundary layer occurs, $m_{(8.7)}$ is a sufficiently small positive number. The estimates (8.6) and (8.7) hold, for example, when

$$U, \, W \in H^{(2n+4+\nu)}(\overline{G}), \quad \nu > 0. \tag{8.8}$$

The inclusions (8.8) hold if $a \in H^{(\alpha+2n-1)}(\overline{G})$, c, p, $f \in H^{(\alpha+2n-2)}(\overline{G})$, $\varphi \in H^{(\alpha+2n)}(\overline{G})$, $\alpha > 4$, $n \geqslant 0$, and the condition (8.1) is satisfied. We summarise the above results in the following theorem.

Theorem 8.1. *Suppose that for the functions in equation* (1.1) *we have* a, b, c, p, $f \in H^{(\alpha+2n-2)}(\overline{G})$, $\varphi \in H^{(\alpha+2n)}(\overline{G})$, $\alpha > 4$, $n \geqslant 0$. *Let the condition* (8.1) *be satisfied. Then from this it follows that* u, U, $W \in H^{(\alpha+2n)}(\overline{G})$ *and that the estimates* (8.2), (8.6), (8.7) *hold for the solution* $u(x,t)$ *of problem* (1.1) *and for its components in the representation* (8.4).

The proof of the theorem is similar to the proof in [10], where the equation

$$\varepsilon a(x,t) \frac{\partial^2}{\partial x^2} u(x,t) + b(x,t) \frac{\partial}{\partial x} u(x,t) - c(x,t) u(x,t) - p(x,t) \frac{\partial}{\partial t} u(x,t) = f(x,t)$$

was considered.

8.2. The proof of Theorem 5.1

Let us show that the function $\delta_{\bar t} z(x,t)$, where $z(x,t) = z_{(5.3)}(x,t)$ is the solution of the difference problem (5.3), approximates the function $\delta_{\bar t} u(x,t)$ ε-uniformly. For simplicity we assume $a(x,t)$, $b(x,t)$ to be constant on $\overline{G}$. The function $\delta_{\bar t} z(x,t)$ is the solution of the difference problem

$$\Lambda_{(8.9)} \delta_{\bar t} z(x,t) = f_{(8.9)}(x,t), \qquad (x,t) \in G_h^{[1]} \tag{8.9a}$$

$$\delta_{\bar t} z(x,t) = \varphi_{(8.9)}(x,t), \qquad (x,t) \in S_h^{[1]}. \tag{8.9b}$$

Here

$$\overline{G}_h^{[k]} = \overline{G}_h \cap \{t \geqslant k\tau\}, \quad G_h^{[k]} = G_h \cap \{t > k\tau\}, \quad S_h^{[k]} = \overline{G}_h^{[k]} \setminus G_h^{[k]}, \quad k \geqslant 1$$

$$\Lambda_{(8.9)} \delta_{\bar t} z(x,t) \equiv \{ \varepsilon a \delta_{x\bar x} + b \delta_x - \check c(x,t) - p_{\bar t}(x,t) - \check p(x,t) \delta_{\bar t} \} \delta_{\bar t} z(x,t)$$

$$f_{(8.9)}(x,t) = f_{\bar t}(x,t) + c_{\bar t}(x,t) z(x,t)$$

$$\varphi_{(8.9)}(x,t) = \varphi_{\bar t}(x,t), \quad x = 0, d, \quad (x,t) \in S_h^{[1]}$$

$$\varphi_{(8.9)}(x,t) = \varphi^0_{(8.9)}(x) \equiv \tau^{-1} (z(x,\tau) - \varphi(x,0)), \quad t = \tau, \quad (x,t) \in S_h^{[1]}$$

$\check v(x,t) = v(x,t-\tau)$ where $\check v(x,t)$ *is one of the functions* $\check c(x,t)$, $\check p(x,t)$.

The function $\delta_{\bar t} u(x,t) \equiv (u(x,t) - u(x,t-\tau))/\tau$, $(x,t) \in \overline{G}$, $t \geqslant \tau$, is the solution of the differential problem

$$L_{(8.10)} \delta_{\bar t} u(x,t) \;=\; f_{(8.10)}(x,t), (x,t) \in G^{[1]} \tag{8.10a}$$

$$\delta_{\bar t} u(x,t) \;=\; \varphi_{(8.10)}(x,t), (x,t) \in S^{[1]}. \tag{8.10b}$$

Here

$$\overline{G}^{[k]} = \overline{G} \cap \{t \geqslant k\tau\}, \quad G^{[k]} = G \cap \{t > k\tau\}, \quad S^{[k]} = \overline{G}^{[k]} \setminus G^{[k]}, \quad k \geqslant 1$$

$$L_{(8.10)} \delta_{\bar{t}} u(x,t) \equiv \varepsilon a \frac{\partial^2}{\partial x^2} + b \frac{\partial}{\partial x} - \check{c}(x,t) - p_{\bar{t}}(x,t) - \check{p}(x,t) \frac{\partial}{\partial t} \delta_{\bar{t}} u(x,t)$$

$$f_{(8.10)}(x,t) = f_{\bar{t}}(x,t) + c_{\bar{t}}(x,t)\, u(x,t) + p_{\bar{t}}(x,t) \left(\frac{\partial}{\partial t} u(x,t) - \delta_{\bar{t}} u(x,t) \right)$$

$$\varphi_{(8.10)}(x,t) = \varphi_{\bar{t}}(x,t), \quad x = 0, d, \quad (x,t) \in S^{[1]}$$

$$\varphi_{(8.10)}(x,t) = \varphi^0_{(8.10)}(x) \equiv \tau^{-1} \left(u(x,\tau) - \varphi(x,0) \right), \quad t = \tau, \quad (x,t) \in S^{[1]}.$$

Let us estimate

$$\varphi^0_{(8.10)}(x) - \varphi^0_{(8.9)}(x) = \tau^{-1} \omega(x,\tau)$$

where

$$\omega(x,t) = u(x,t) - z(x,t), \quad (x,t) \in \overline{G}_h.$$

The function $\omega(x,t)$ is the solution of the problem

$$\Lambda_{(5.3)}\, \omega(x,t) = \left(\Lambda_{(5.3)} - L_{(1.1)} \right) u(x,t), \quad (x,t) \in G_h, \quad \omega(x,t) = 0, \quad (x,t) \in S_h.$$

The above assumptions and Theorem 8.1 lead to the estimates of the truncation error (the derivation technique for these estimates is shown, for example, in [8, 10]):

$$\left| \left(\Lambda_{(5.3)} - L_{(1.1a)} \right) U(x,t) \right| \leqslant M \left(N^{-1} \ln N + \tau \right), \quad (x,t) \in G_h$$

$$\left| \left(\Lambda_{(5.3)} - L_{(1.1a)} \right) W(x,t) \right| \leqslant M \left(\varepsilon^{-1} N^{-1} \ln N \exp(-m \varepsilon^{-1} x) + \tau \right)$$

$$(x,t) \in G_h, \quad x \leqslant \sigma$$

where $U(x,t)$ and $W(x,t)$ are the regular and singular parts of the solution from (8.4); $\sigma = \sigma_{(3.1)}$, $m = m_{(8.7)}$. For the components $W(x,t)$ and $W^h(x,t)$ the following estimate holds:

$$|W(x,t)|, \left| W^h(x,t) \right| \leqslant M N^{-1}, \quad (x,t) \in \overline{G}_h, \quad x \geqslant \sigma.$$

Here $W^h(x,t)$ is the solution of the problem

$$\Lambda_{(5.3)} W^h(x,t) = 0, \quad (x,t) \in G_h, \quad W^h(x,t) = W(x,t), \quad (x,t) \in S_h.$$

Using the maximum principle, we estimate $\omega(x,t)$:

$$|\omega(x,t)| \leqslant M \left(N^{-1} \ln N + \tau \right) t, \quad (x,t) \in \overline{G}_h.$$

Further, for the derivatives we proceed in a similar way. On the boundary we have

$$|\delta_{\bar{t}} u(x,\tau) - \delta_{\bar{t}} z(x,\tau)| = \left| \varphi^0_{(8.10)}(x) - \varphi^0_{(8.9)}(x) \right| \leqslant M \left(N^{-1} \ln N + \tau \right)$$

$$(x,t) \in S_h^{[1]}, \quad t = \tau$$

i.e. the function $\delta_{\bar{t}} z(x,\tau)$ approximates $\delta_{\bar{t}} u(x,\tau)$ ε-uniformly. Now it is easy to see that the solution of the difference problem (8.9) approximates the solution of the differential problem (8.10) for the divided difference. Thus, using the same argument as above, we derive the estimate

$$|\delta_{\bar{t}} u(x,t) - \delta_{\bar{t}} z(x,t)| \leqslant M \left(N^{-1} \ln N + \tau \right), \quad (x,t) \in \overline{G}_h^{[1]}.$$

Now for the second difference derivative we show that under the condition (5.6) the function $\delta_{2\bar{t}} z(x,t)$ approximates the function $\delta_{2\bar{t}} u(x,t)$ ε-uniformly on the set $\overline{G}_h^{[2]}$. Thus, the functions $\delta_{2\bar{t}} z(x,t)$ and $\delta_{2\bar{t}} u(x,t)$ are the solutions of the equations

$$\Lambda_{(8.11)} \delta_{2\bar{t}} z(x,t) = f_{(8.11)}(x,t), \quad (x,t) \in G_h^{[2]} \tag{8.11a}$$

$$L_{(8.12)} \delta_{2\bar{t}} u(x,t) = f_{(8.12)}(x,t), \quad (x,t) \in G_h^{[2]}. \tag{8.12a}$$

These equations are found by applying the operator $\delta_{\bar{t}}$ to the equations (8.9a), (8.10a). On the left and right boundaries the following conditions are satisfied:

$$\delta_{2\bar{t}} z(x,t) = \varphi_{(8.11)}(x,t), \quad (x,t) \in S_h^{[2]} \tag{8.11b}$$

$$\delta_{2\bar{t}} u(x,t) = \varphi_{(8.12)}(x,t), \quad (x,t) \in S_h^{[2]} \tag{8.12b}$$

where

$$\varphi_{(8.11)}(x,t) = \varphi_{2\bar{t}}(x,t), \quad x = 0, d, \quad (x,t) \in S_h^{[2]} \tag{8.11c}$$

$$\varphi_{(8.11)}(x,t) = \varphi^0_{(8.11)}(x) \equiv \delta_{2\bar{t}} z_{(5.3)}(x,t), \quad t = 2\tau, \quad (x,t) \in S_h^{[2]}$$

$$\varphi_{(8.12)}(x,t) = \varphi_{2\bar{t}}(x,t), \quad x = 0, d, \quad (x,t) \in S^{[2]}, \tag{8.12c}$$

$$\varphi_{(8.12)}(x,t) = \varphi^0_{(8.12)}(x) \equiv \delta_{2\bar{t}} u(x,t), \quad t = 2\tau, \quad (x,t) \in S^{[2]}.$$

First we estimate

$$\varphi^0_{(8.12)}(x) - \varphi^0_{(8.11)}(x) = \delta_{2\bar{t}} u(x,t) - \delta_{2\bar{t}} z(x,t), \quad t = 2\tau.$$

For this purpose we expand the function $u(x,t)$ into a Taylor series in terms of t:

$$u(x,t) = a^{(1)}(x) t + a^{(2)}(x) t^2 + v_2(x,t) \equiv u^{[2]}(x,t) + v_2(x,t), \quad (x,t) \in \overline{G} \tag{8.13}$$

where the coefficients $a^{(1)}(x)$, $a^{(2)}(x)$ should be determined. Taking $u(x,t)$ in the form (8.13) and introducing it into into equation (1.1a) we arrive at the system

$$\begin{aligned}-p(x,0)a^{(1)}(x) &= f(x,0)\\ -2p(x,0)a^{(2)}(x) &+ \varepsilon a\frac{\partial^2}{\partial x^2}a^{(1)}(x)+b\frac{\partial}{\partial x}a^{(1)}(x)\\ &- \left(c(x,0)+\frac{\partial}{\partial t}p(x,0)\right)a^{(1)}(x)=\frac{\partial}{\partial t}f(x,0)\end{aligned}$$

from which the functions $a^{(1)}(x)$, $a^{(2)}(x)$ can be found successively. The function $v_2(x,t)$ is the solution of the boundary value problem

$$\begin{aligned}L_{(1.1)}v_2(x,t) &= f_{(8.14)}(x,t)\equiv f(x,t)-L_{(1.1)}u^{[2]}(x,t), \quad (x,t)\in G \quad (8.14)\\ v_2(x,t) &= \varphi_{(8.14)}(x,t)\equiv \varphi(x,t)-u^{[2]}(x,t), \quad (x,t)\in S.\end{aligned}$$

Estimating $f_{(8.14)}(x,t)$ and $\varphi_{(8.14)}(x,t)$ and using the maximum principle, we derive the estimate

$$|v_2(x,t)|\leqslant Mt^3, \quad (x,t)\in\overline{G}. \tag{8.15}$$

Further we have to construct the function $z(x,t)$ in the form

$$\begin{aligned}z(x,t) &= (b_0^{(1)}(x)+b_1^{(1)}(x)\tau)t+b_0^{(2)}(x)t^2+v_2^h(x,t)\equiv z^{[2]}(x,t)+v_2^h(x,t)\\ &\quad (x,t)\in\overline{G}_h\end{aligned}$$

i.e. as an expansion in terms of τ and t. Inserting $z(x,t)$ into the equation (5.3), we arrive at the equations

$$\begin{aligned}-p(x,0)b_0^{(1)}(x) &= f(x,0)\\ -2p(x,0)b_0^{(2)}(x) &+ \varepsilon a\frac{\partial^2}{\partial x^2}b_0^{(1)}(x)+b\frac{\partial}{\partial x}b_0^{(1)}(x)\\ &- \left(c(x,0)+\frac{\partial}{\partial t}p(x,0)\right)b_0^{(1)}(x)=\frac{\partial}{\partial t}f(x,0)\\ b_0^{(2)}(x)+b_1^{(1)}(x) &= 0.\end{aligned}$$

Thus, we have

$$z^{[2]}(x,t)=u^{[2]}(x,t)+b_1^{(1)}(x)\tau t, \quad (x,t)\in\overline{G}_h. \tag{8.16}$$

The function $v_2^h(x,t)$ is the solution of the discrete boundary value problem

$$\begin{aligned}\Lambda_{(5.3)}v_2^h(x,t) &= f_{(8.17)}(x,t)\equiv f(x,t)-\Lambda_{(5.3)}z^{[2]}(x,t), \quad (x,t)\in G_h \quad (8.17)\\ v_2^h(x,t) &= \varphi_{(8.17)}(x,t)\equiv\varphi(x,t)-z^{[2]}(x,t), \quad (x,t)\in S_h.\end{aligned}$$

Taking into account the estimates of the functions $f_{(8.17)}(x,t)$ and $\varphi_{(8.17)}(x,t)$, we derive the estimate

$$\left|v_2^h(x,t)\right|\leqslant M\left(N^{-1}\ln N+t\right)t^2, \quad (x,t)\in\overline{G}_h. \tag{8.18}$$

By virtue of relations (8.15), (8.16), (8.18) the following inequality is valid:

$$\begin{aligned}&\left|\varphi^0_{(8.12)}(x)-\varphi^0_{(8.11)}(x)\right|=|\delta_{2\bar t}u(x,t)-\delta_{2\bar t}z(x,t)|\leqslant M\left(N^{-1}\ln N+\tau\right) \quad (8.19)\\ &\qquad (x,t)\in\overline{G}_h, \quad t=2\tau.\end{aligned}$$

We continue the proof by estimating $\delta_{2\bar t}u(x,t)-\delta_{2\bar t}z(x,t)$ for $t>2\tau$. Note that the functions $\delta_{2\bar t}u(x,t)$ and $\delta_{2\bar t}z(x,t)$ are the solutions of the differential and difference equations obtained from equations (1.1) and (5.3), respectively, by applying the operator $\delta_{2\bar t}$. Moreover, the difference equation for $\delta_{2\bar t}z(x,t)$ approximates the differential equation for $\delta_{2\bar t}u(x,t)$ ε-uniformly. On the boundary S_h, for $x=0$ or $x=1$ we have $\delta_{2\bar t}u(x,t)=\delta_{2\bar t}z(x,t)$. Taking into account the estimate (8.19), we find

$$|\delta_{2\bar t}u(x,t)-\delta_{2\bar t}z(x,t)|\leqslant M\left(N^{-1}\ln N+\tau\right], \quad (x,t)\in\overline{G}_h, \quad t\geqslant 2\tau. \tag{8.20}$$

Thus, we arrive at the estimates

$$\begin{aligned}\left|\delta_{\bar t}u(x,t)-\delta_{\bar t}z^{(1)}(x,t)\right| &\leqslant M\left(N^{-1}\ln N+\tau\right), \quad (x,t)\in\overline{G}_h, \quad t\geqslant\tau\\ \left|\delta_{2\bar t}u(x,t)-\delta_{2\bar t}z^{(1)}(x,t)\right| &\leqslant M\left(N^{-1}\ln N+\tau\right), \quad (x,t)\in\overline{G}_h, \quad t\geqslant 2\tau \quad (8.21)\\ \left|u(x,t)-z^{(2)}(x,t)\right| &\leqslant M\left(N^{-1}\ln N+\tau^2\right), \quad (x,t)\in\overline{G}_h.\end{aligned}$$

This completes the proof.

Now we make two remarks which are the direct corollary of Theorem 5.1.

Remark 8.1. We obtained the estimate (8.20) for $z^{(k)}(x,t)$, $k=1$. In exactly the same way we derive the same bound for $k=2$ and thus we obtain

$$\left|\delta_{2\bar t}u(x,t)-\delta_{2\bar t}z^{(k)}(x,t)\right|\leqslant M\left(N^{-1}\ln N+\tau^k\right), \quad (x,t)\in\overline{G}_h, \quad t\geqslant 2\tau, \quad k\leqslant 2. \tag{8.22}$$

Remark 8.2. Making use of (8.22), in the same way as we derived the estimate (8.21) we can obtain

$$\left|\delta_{3\bar{t}}u(x,t) - \delta_{3\bar{t}}z^{(1)}(x,t)\right| \leqslant M\left(N^{-2}\ln N + \tau\right), \quad (x,t) \in \overline{G}_h, \quad t \geqslant 3\tau. \quad (8.23)$$

Now we will dwell briefly on the difference between the proof of (8.21) and the proof of (8.23). To estimate the difference between $\delta_{3\bar{t}}u(x,t)$ and $\delta_{3\bar{t}}z(x,t)$ for $t = 3\tau$ we represent the function $u(x,t)$ [with the condition (5.9)] in the form

$$u(x,t) = a^{(2)}(x)t^2 + a^{(3)}(x)t^3 + v_3(x,t) \equiv u^{[3]}(x,t) + v_3(x,t), \quad (x,t) \in \overline{G}$$

and the function $z(x,t)$ in the form

$$\begin{aligned} z(x,t) &= u^{[3]}(x,t) + (b_1^{(1)}(x)\tau + b_2^{(1)}(x)\tau^2)t + b_1^{(2)}(x)\tau t^2 + v_3^h(x,t) \\ &\equiv z^{[3]}(x,t) + v_3^h(x,t), \quad (x,t) \in \overline{G}_h. \end{aligned}$$

The coefficients of these series are found using equations (1.1) and (5.3), respectively. For the coefficients we have the system

$$-2p(x,0)a^{(2)}(x) = \frac{\partial}{\partial t}f(x,0)$$

$$-3p(x,0)a^{(3)}(x) + \varepsilon a\frac{\partial^2}{\partial x^2}a^{(2)}(x) + b\frac{\partial}{\partial x}a^{(2)}(x) - \left(c(x,0) + 2\frac{\partial}{\partial t}p(x,0)\right)a^{(2)}(x)$$

$$= \frac{1}{2}\frac{\partial^2}{\partial t^2}f(x,0)$$

$$-b_1^{(1)}(x) + a^{(2)}(x) = 0$$

$$-2p(x,0)b_1^{(2)}(x) + \frac{\partial}{\partial t}p(x,0)a^{(2)}(x) + 3p(x,0)a^{(3)}(x)$$

$$-\left(\frac{\partial}{\partial t}p(x,0) + c(x,0)\right)b_1^{(1)}(x) + \varepsilon a\frac{\partial^2}{\partial x^2}b_1^{(1)}(x) + b\frac{\partial}{\partial x}b_1^{(1)}(x) = 0$$

$$-b_2^{(1)}(x) - a^{(3)}(x) + b_1^{(2)}(x) = 0.$$

The unknown functions $a^{(2)}$, $a^{(3)}$, $b_1^{(1)}$, $b_1^{(2)}$, $b_2^{(1)}$ can be found successively. For the functions $v_3(x,t)$ and $v_3^h(x,t)$ the following estimates are derived:

$$\begin{aligned} |v_3(x,t)| &\leqslant Mt^4, \quad (x,t) \in \overline{G} \\ |v_3^h(x,t)| &\leqslant M\left(N^{-1}\ln N + t\right)t^3, \quad (x,t) \in \overline{G}_h. \end{aligned}$$

From these inequalities and the expression for $z^{[3]}(x,t)$ it follows that (8.23) holds ε-uniformly for $t = 3\tau$. The remainder part of the proof of the estimate (8.23) is similar with small variations to the proof of the estimate (8.21).

8.3. The proof of Theorem 5.2

Notice that if the following relations hold for the functions $z^{(1)}(x,t)$, $z^{(2)}(x,t)$

$$\left|\delta_{3\bar{t}}u(x,t) - \delta_{3\bar{t}}z^{(1)}(x,t)\right| \leqslant M\left(N^{-1}\ln N + \tau\right), \quad (x,t) \in G_h, \quad t \geqslant 3\tau \quad (8.24)$$

$$\left|\delta_{2\bar{t}}u(x,t) - \delta_{2\bar{t}}z^{(2)}(x,t)\right| \leqslant M\left(N^{-1}\ln N + \tau^2\right), \quad (x,t) \in G_h, \quad t \geqslant 2\tau$$

then for the difference $u(x,t) - z^{(3)}(x,t) \equiv \omega^{(3)}(x,t)$ we obtain

$$\left|\Lambda_{(5.3)}\omega^{(3)}(x,t)\right| \leqslant M\left(N^{-1}\ln N + \tau^3\right), \quad (x,t) \in G_h, \quad \omega^{(3)}(x,t) = 0, \quad (x,t) \in S_h.$$

Hence we have

$$\left|u(x,t) - z^{(3)}(x,t)\right| \leqslant M\left(N^{-1}\ln N + \tau^3\right), \quad (x,t) \in \overline{G}_h.$$

Thus, for the proof of the theorem it is sufficient to show the validity of inequalities (8.24). These inequalities follow from (8.22), (8.23). This completes the proof of Theorem 5.2.

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On high-order compact schemes in the finite element method

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Abstract — In this paper we propose the new family of the spaces of grid functions, which is the basis for the construction of high-order schemes in the finite element method. We show the completeness of this family in the space $H^1(\Omega)$ and study the error of approximate solutions in grid norms. The second order of accuracy on a nonuniform grid and the fourth order on a uniform grid are obtained.

In the theory of difference methods one of the remarkable approximations is the Mikeladze nine-point scheme for the Poisson equation. It has the error $O(h^4)$ on a uniform rectangular grid and even $O(h^6)$ on a square grid and under additional smoothness conditions on the right-hand side and the solution of the equation as well [6]. This scheme has initiated numerous studies on high-order difference methods and has given rise to the so-called compact difference schemes, including those for differential equations in more general form (see [2] and the bibliography therein). It was also shown in [3] that compact fourth-order accurate schemes can be obtained in the framework of the finite volume method which uses the approximation of parametrized conservation laws.

The question arises of whether a compact fourth-order scheme can be obtained in the framework of projection approaches. In other words, to what basis in the finite element method does the Mikeladze scheme correspond? As far as we know, the work [4] is the first work on this subject. It describes the sufficiently versatile method of constructing basis functions which are based on piecewise linear and bilinear fillings and are the basis for constructing the system of the Galerkin method coinciding, in particular, with the Mikeladze scheme. The obtained space of grid functions approximates the initial function space in a weak sense (the family completeness is not available). The necessary and sufficient conditions for the classes of basis functions which ensure the third and fourth local orders of the approximation of projection difference schemes were obtained in [9]. The use of these conditions for the choice of concrete grid spaces is rather nontrivial. Moreover, the error of grid solutions was not studied in [9].

In the present work for the Laplace equation with the Dirichlet boundary conditions we propose an alternative to the approach used in [4]. Complementing the

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