Algorithm 819: AIZ, BIZ: Two Fortran 77 Routines for the Computation of Complex Airy Functions

AMPARO GIL
U. Autónoma de Madrid
JAVIER SEGURA
U. Carlos III de Madrid
and
NICO M. TEMME
CWI

Two Fortran 77 routines for the evaluation of Airy functions of complex arguments $Ai(z)$, $Bi(z)$ and their first derivatives are presented. The routines are based on the use of Gaussian quadrature, Maclaurin series and asymptotic expansions. Comparison with a previous code by D. E. Amos [1986] is provided.

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1. INTRODUCTION

This algorithm computes the Airy functions $\mathrm{Ai}(z)$ and $\mathrm{Bi}(z)$ and their first derivatives in the complex plane. Airy functions are solutions of the differential equation

$$w'' - zw = 0. \tag{1}$$

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J. Segura acknowledges financial support from DAAD for his research stay at Kassel University, Mathematics Department.
Authors' addresses: A. Gil, Departamento de Mathemáticas, U. Autónoma de Madrid, 28049-Madrid, Spain; email: amparo.gil@uam.es; J. Segura, Departamento de Mathemáticas, U. Carlos III de Madrid, 28911-Leganés (Madrid), Spain; email: jsegura@math.uc3m.es; N. M. Temme, CWI, P.O. Box 94079, 1090 GB Amsterdam, The Netherlands; email: nico.temme@cwi.nl.

The program gives the option of computing scaled Airy functions in order both to enlarge the range of computation and to reduce accuracy problems for large $|z|$ [Gil et al. 2002].

Amos' [1986] code is a well-known package and includes an algorithm for the computation of complex Airy functions. The algorithms are based on the evaluation of modified Bessel functions for complex arguments, and the Airy functions are evaluated through relations between Airy and Bessel functions. Our goal is to provide a stand-alone algorithm, which is more convenient for the direct computation of Airy functions. In addition, as we will show, the combination of two complex modified Bessel functions produces, in certain regions, quite extensive errors, and we found that it is better to avoid combining the Bessel functions in this way.

The current algorithm is based on Maclaurin series for small $|z|$, Gauss-Laguerre quadrature for intermediate values and asymptotic expansions for large $|z|$.

The relative accuracy is better than 10^{-13} when scaled Airy functions are computed, with the natural exception of the vicinity of the real or complex zeros of Airy functions, where relative precision loses meaning and scaling does not help. The functions Ai and Bi (and their derivatives) have zeros on the negative real axis, while Bi and its derivative have zeros close to the anti-Stokes lines $\mathrm{ph}z = \pm\pi/3$.

The computation gradually loses accuracy for the modulus of unscaled Airy function as $|z|$ becomes large (for $|z| > 30$). The source of these errors is the evaluation of the dominant exponential factor $e^{-\frac{2}{3}z^{3/2}}$ for large complex arguments. A similar gradual loss of accuracy takes place for the phase of unscaled Airy functions. Additionally, the dominant exponential factor introduces an infinite number of curves where the real or imaginary parts of the unscaled functions cancel; over these curves, relative precision for the phase loses meaning. The code provides an absolute error smaller that 10^{-13} for $\min(|R(z)|, 1/|R(z)|)$ being the ratio between the real and the imaginary part of the function ($\mathrm{Ai}(z)$, $\mathrm{Ai}'(z)$, $\mathrm{Bi}(z)$ or $\mathrm{Bi}'(z)$), except very close to the zeros of the function.

It is observed that when these unavoidable cancellations take place, the codes AIZ and BIZ behave better than Amos' code. See Section 4.2 for further details.

2. METHOD OF COMPUTATION

We briefly describe the numerical methods considered for different regions in the complex plane. For a detailed discussion, we refer the reader to [Gil et al. 2002].

The three ingredients of the algorithm are:

(1) Maclaurin series ([Abramowitz and Stegun 1964], 10.4.2) for $|y| < 3$ and $-2.6 < x < 1.3$ ($z = x + iy$).

(2) Gauss-Laguerre quadrature [Gil et al. 2002] for $|z| < 15$ and where Maclaurin series are not used.

(3) Asymptotic expansions ([Abramowitz and Stegun 1964]. 10.4.59) for $|z| > 15$.

Except in the case of Maclaurin series, the domain of computation is the principal sector $|\mathrm{ph} z| \leq 2\pi/3$. For $2\pi/3 < |\mathrm{ph} z| \leq \pi$ the connection formula ([Abramowitz and Stegun 1964], 10.4.7) is considered:

$$\mathrm{Ai}(z) = -e^{-2\pi i/3}\mathrm{Ai}(e^{-2\pi i/3}z) - e^{2\pi i/3}\mathrm{Ai}(e^{2\pi i/3}z). \tag{2}$$

The conjugation property $\mathrm{Ai}(x+iy) = \overline{\mathrm{Ai}(x-iy)}$ is considered for negative $\Im z$.

The code also computes scaled Airy functions. The dominant term in the asymptotic behavior of $\mathrm{Ai}(z)$ for large $|z|$ is the quantity $e^{-\zeta}$, with $\zeta = \frac{2}{3}z^{\frac{3}{2}}$; this factor appears explicitly both in the integral representations that we consider and in the asymptotic expansions. The scaling of the Airy functions consists in eliminating this exponential behavior by considering the new functions:

$$\tilde{\mathrm{Ai}}(z) = e^{\zeta}\mathrm{Ai}(z), \quad \tilde{\mathrm{Ai}}'(z) = e^{\zeta}\mathrm{Ai}'(z). \tag{3}$$

The same scaling was considered by Amos [1986]. With this, not only is the range of computation extended, but errors are reduced by eliminating the exponential factor, which, as discussed in Gil et al. [2002], unavoidably introduces gradual loss of precision as $|z|$ grows. Also, in many physical problems, combinations (ratios, for example, see Jeffreys [1956]) of Airy functions are used, and scaled functions may then be very convenient.

When scaling is considered, the connection formula (2) has to be modified (see Gil et al. [2002]). The function $\tilde{\mathrm{Ai}}$ is analytic in the complex plane cut along the negative real axis. At the negative real axis, $\mathrm{ph} z = \pi$, we take the prescription of continuity of this function when approaching the negative real axis through positive values of $\Im z$.

2.1 Connection Formulas and Scaling

The computation of $\mathrm{Ai}(z)$ in $|\mathrm{ph} z| \leq 2\pi/3$ provides two numerically satisfactory solutions of the differential equation (1) in the whole complex plane, that is, two independent solutions such that both are not simultaneously dominant. This is so because $\mathrm{Ai}(z)$, $\mathrm{Ai}(e^{2\pi i/3}z)$ and $\mathrm{Ai}(e^{-2\pi i/3}z)$ are independent solutions of Eq. (1) and $\mathrm{Ai}(z)$ is dominant only in $\pi/3 < |\mathrm{ph} z| < \pi$. However, for completeness we extended the range of computation to the whole complex plane through the use of Eq. (2). Similarly, we also compute the solution $\mathrm{Bi}(z)$ through two connection formulas ([Abramowitz and Stegun 1964], 10.4.9, 10.4.6):

$$\begin{aligned} \mathrm{Bi}(z) &= i\mathrm{Ai}(z) + 2e^{-i\pi/6}\mathrm{Ai}(e^{-2\pi i/3}z), \\ \mathrm{Bi}(z) &= e^{i\pi/6}\mathrm{Ai}(e^{2\pi i/3}z) + e^{-i\pi/6}\mathrm{Ai}(e^{-2\pi i/3}z). \end{aligned} \tag{4}$$

We apply the first connection formula for $0 < |\mathrm{ph} z| \leq 2\pi/3$ and the second one for the rest of the complex plane. In this way, both relations are numerically satisfactory and the computation through (2) is avoided (therefore we avoid an extra evaluation).

The scaling of $\mathrm{Bi}(z)$ is not so obvious as that of $\mathrm{Ai}(z)$ because $\mathrm{Bi}(z)$ shows different dominant behavior depending on the sector in the complex plane: for large $|z|$ it behaves as e^{ζ} for $|\mathrm{ph} z| < \pi/3$ and as $e^{-\zeta}$ for $\pi/3 < |\mathrm{ph} z| < \pi$. The Amos solution is to rescale using a factor $e^{-|\Re(\zeta)|}$, which indeed enlarges the range of computation. We find this method has a major drawback: although scaling succeeds in enlarging the range of computation, it does not get rid of the whole dominant exponential contribution which introduces, as we will later show, a gradual decrease of accuracy as $|z|$ increases for any value of $|\mathrm{ph} z|$. In other words, the computable range includes values of z for which precision can be completely lost. We prefer to rescale the function in the following way:

$$\begin{aligned} \tilde{\mathrm{B}}\mathrm{i}(z) &= e^{-\zeta}\mathrm{Bi}(z), \quad \text{where } |\mathrm{ph} z| < \pi/3, \\ \tilde{\mathrm{B}}\mathrm{i}(z) &= e^{\zeta}\mathrm{Bi}(z), \quad \text{where } |\mathrm{ph} z| \geq \pi/3. \end{aligned} \tag{5}$$

We have now an additional discontinuity at $|\mathrm{ph} z| = \pi/3$. However, the main advantage is that errors are considerably reduced by scaling out the dominant exponential factors.

As happened with $\tilde{\mathrm{A}}\mathrm{i}$, the connection formulas must be modified for the computation of $\tilde{\mathrm{B}}\mathrm{i}$. Namely, we consider the following relations:

$$\begin{aligned} \tilde{\mathrm{B}}\mathrm{i}(z) &= ie^{-2\zeta}\tilde{\mathrm{A}}\mathrm{i}(z) + 2e^{-i\pi/6}\tilde{\mathrm{A}}\mathrm{i}(e^{-2\pi i/3}z) \text{ where } 0 \leq \mathrm{ph} z < \pi/3, \\ \tilde{\mathrm{B}}\mathrm{i}(z) &= i\tilde{\mathrm{A}}\mathrm{i}(z) + 2e^{-i\pi/6}e^{2\zeta}\tilde{\mathrm{A}}\mathrm{i}(e^{-2\pi i/3}z) \text{ where } \pi/3 \leq \mathrm{ph} z \leq 2\pi/3, \\ \tilde{\mathrm{B}}\mathrm{i}(z) &= e^{i\pi/6}\tilde{\mathrm{A}}\mathrm{i}(e^{2\pi i/3}z) + e^{-i\pi/6}e^{2\zeta}\tilde{\mathrm{A}}\mathrm{i}(e^{-2\pi i/3}z) \text{ where } 2\pi/3 < \mathrm{ph} z \leq \pi, \end{aligned} \tag{6}$$

and use complex conjugation when $\Im z < 0$.

2.2 Representations Near the Negative Real Axis and Near the Anti-Stokes Lines

The Airy functions $\mathrm{Ai}(z)$ and $\mathrm{Bi}(z)$ and their derivatives have zeros on the negative real axis [Abramowitz and Stegun 1964, pg 450]. Convenient representations in a region that contains the negative axis are

$$\begin{aligned} \mathrm{Ai}(-z) &= z^{-\frac{1}{4}}\pi^{-\frac{1}{2}}[\sin\chi\, P(z) - \cos\chi\, Q(z)], \\ \mathrm{Bi}(-z) &= z^{-\frac{1}{4}}\pi^{-\frac{1}{2}}[\cos\chi\, P(z) + \sin\chi\, Q(z)], \\ \mathrm{Ai}'(-z) &= -z^{\frac{1}{4}}\pi^{-\frac{1}{2}}[\cos\chi\, S(z) + \sin\chi\, T(z)], \\ \mathrm{Bi}'(-z) &= z^{\frac{1}{4}}\pi^{-\frac{1}{2}}[\sin\chi\, S(z) - \cos\chi\, T(z)], \end{aligned} \tag{7}$$

where

$$\chi = \zeta + \frac{1}{4}\pi, \quad \zeta = \frac{2}{3}z^{\frac{3}{2}}. \tag{8}$$

These representations can be used in the sector $|\mathrm{ph} z| < 2\pi/3$, $z \neq 0$, and are obtained by writing the asymptotic expansions [Abramowitz and Stegun 1964, 10.4.60, 10.4,62, 10.4.64, 10.4.67] as exact identities. The functions P, Q, S, T represent the slowly varying parts of the Airy functions for large $|z|$ and the user of the algorithms may be interested in these quantities which, generally speaking, can be computed with better accuracy than the Airy functions themselves.

Explicit representations for the functions P, Q, S, T follow from inverting the relations in (7). More interesting is a representation in terms of scaled Airy functions, and, by using the connection formulas (2) and (4), it is straightforward to verify that

$$\begin{aligned} P(z) &= z^{\frac{1}{4}}\pi^{\frac{1}{2}}[e^{-\pi i/12}\tilde{\mathrm{Ai}}(e^{-\pi i/3}z)+e^{\pi i/12}\tilde{\mathrm{Ai}}(e^{\pi i/3}z)],\\ Q(z) &= iz^{\frac{1}{4}}\pi^{\frac{1}{2}}[e^{-\pi i/12}\tilde{\mathrm{Ai}}(e^{-\pi i/3}z)-e^{\pi i/12}\tilde{\mathrm{Ai}}(e^{\pi i/3}z)],\\ S(z) &= -z^{-\frac{1}{4}}\pi^{\frac{1}{2}}[e^{\pi i/12}\tilde{\mathrm{Ai}}'(e^{-\pi i/3}z)+e^{-\pi i/12}\tilde{\mathrm{Ai}}'(e^{\pi i/3}z)],\\ T(z) &= -iz^{-\frac{1}{4}}\pi^{\frac{1}{2}}[e^{\pi i/12}\tilde{\mathrm{Ai}}'(e^{-\pi i/3}z)-e^{-\pi i/12}\tilde{\mathrm{Ai}}'(e^{\pi i/3}z)]. \end{aligned} \tag{9}$$

From the Wronskian relation for $\mathrm{Ai}(z)$ and $\mathrm{Bi}(z)$, that is,

$$\mathrm{Ai}(z)\mathrm{Bi}'(z)-\mathrm{Ai}'(z)\mathrm{Bi}(z)=1/\pi, \tag{10}$$

it follows that

$$P(z)S(z)+Q(z)T(z)=1. \tag{11}$$

Considering the relations in (7), we notice that some loss of accuracy is expected in the evaluation of $\mathrm{Ai}(-z)$ (and $\mathrm{Bi}(-z)$ and their derivatives) close to the negative real axis when $|z|$ becomes large. This is due to the evaluation of the sine of cosine functions for large arguments together with the cancellation between the P and Q (S and T) terms. The scaled functions, as described in the previous section, do not improve the accuracy near the negative real axis and similar problems take place (in fact, scaling on the negative real axis is of no practical use). In Section 4.2, we will describe these intrinsic numerical difficulties in more detail.

The functions $\mathrm{Bi}(z)$ and $\mathrm{Bi}'(z)$ have complex zeros near the anti-Stokes lines $\mathrm{ph}z=\pm\pi/3$, and in this area also difficulties arise. We have

$$\begin{aligned} \mathrm{Ai}(e^{\pi i/3}z) &= \tfrac{1}{2}e^{-\pi i/3}[\mathrm{Ai}(-z)+i\mathrm{Bi}(-z)],\\ \mathrm{Bi}(e^{\pi i/3}z) &= \tfrac{1}{2}e^{\pi i/6}[3\mathrm{Ai}(-z)-i\mathrm{Bi}(-z)], \end{aligned} \tag{12}$$

and similar formulas with i replaced by $-i$ throughout. The first relation follows from the first relation in (4), and the second one then follows from the second relation in (4). We have in terms of the auxiliary functions

$$\begin{aligned} \mathrm{Ai}(e^{\pi i/3}z) &= \frac{1}{2}z^{-\frac{1}{4}}\pi^{-\frac{1}{2}}e^{\pi i/6-i\chi}[P(z)+iQ(z)],\\ \mathrm{Bi}(e^{\pi i/3z}) &= \sqrt{2/\pi}z^{-\frac{1}{4}}e^{\pi i/6}[\sin\kappa\, P(z)-\cos\kappa\, Q(z)],\\ \mathrm{Ai}'(e^{\pi i/3}z) &= \frac{1}{2}z^{\frac{1}{4}}\pi^{-\frac{1}{2}}e^{-2\pi i/3-i\chi}[S(z)+iT(z)],\\ \mathrm{Bi}'(e^{\pi i/3}z) &= \sqrt{2/\pi}z^{\frac{1}{4}}[\cos\kappa\, S(z)+\sin\kappa\, T(z)], \end{aligned} \tag{13}$$

where $\kappa=\chi-\frac{1}{2}i\ln 2$. Compare the results for $\mathrm{Bi}'(e^{\pi i/3}z)$ and $\mathrm{Bi}'(e^{\pi i/3}z)$ with the asymptotic expansions in Abramowitz and Stegun [1964, 10.4.65 and 10.4.67].

We can expect that, unavoidably, the accuracy in the evaluation of $\mathrm{Bi}(z)$ and its derivative will decrease as larger values of $|z|$ are considered close to the anti-Stokes lines $\mathrm{ph}z=\pm\pi/3$ (see Section 4.2). Scaling does not avoid this loss of accuracy and, in fact, it is of little practical use over the anti-Stokes lines.

3. DESCRIPTION OF THE ROUTINES

We now describe the inputs and outputs of the main routines AIZ and BIZ. We also describe the dependencies between the subroutines in AIZ (and BIZ).

Both AIZ and BIZ call the function D1MACH to obtain the machine dependent constants (overflow and underflow numbers and the smallest relative spacing). The user should uncomment the corresponding data lines in D1MACH.

SUBROUTINE AIZ(IFUN,IFAC,X0,Y0,GAIR,GAII,IERRO)

- INPUT:
  - IFUN
    - IFUN=1, the code computes $\mathrm{Ai}(z)$.
    - IFUN=2, the code computes $\mathrm{Ai}'(z)$
  - IFAC:
    - IFAC=1, the code computes $\mathrm{Ai}(z)$ or $\mathrm{Ai}'(z)$
    - IFAC=2, the code computes normalized Airy functions.
  - X0: real part of the argument Z
  - Y0: imaginary part of the argument Z
- OUTPUT:
  - GAIR: real part of the Airy function (or derivative)
  - GAII: imaginary part of the Airy function (or derivative)
  - IERRO: control of over/underflow for unscaled Airy functions. If IERRO=0, the computation was successful. If IERRO=1, the Airy function overflows or underflows. Scaled functions do not overflow or underflow.

The routine AIZ uses complex conjugation and a connection formula in order to restrict the computations to the principal sector $0 \le \mathrm{ph} z < 2\pi/3$. The routine AIZ depends on the following subroutines (included in the code):

(1) SERAI (SERAID): implements the Maclaurin series for $\mathrm{Ai}(z)$ ($\mathrm{Ai}'(z)$).
(2) EXPAI (EXPAID): computes the asymptotic expansion for $\mathrm{Ai}(z)$ ($\mathrm{Ai}'(z)$) in the sector $0 \le \mathrm{ph} z \le 2\pi/3$
(3) AIRY1 (AIRY1D): computes $\mathrm{Ai}(z)$ ($\mathrm{Ai}'(z)$) by Gauss-Laguerre quadrature in the sector $0 \le \mathrm{ph} z \le \pi/2$
(4) AIRY2 (AIRY2D): computes $\mathrm{Ai}(z)$ ($\mathrm{Ai}'(z)$) by Gauss-Laguerre quadrature in the sector $\pi/2 < \mathrm{ph} z \le 2\pi/3$
(5) Auxiliary routines:
PHASE: computes the phase of z in $-\pi < \mathrm{ph} z \le \pi$
FG (called by SERAI), FGP (called by SERAID)

SUBROUTINE BIZ(IFUN,IFAC,X0,Y0,GBIR,GBII,IERRO)

- INPUT: Same as for AIZ
- OUTPUT:
  - GBIR: real part of the Airy function (or derivative)
  - GBII: imaginary part of the Airy function (or derivative)

- IERRO: control of over/underflow for unscaled Airy functions. If IERRO=0, the computation was successful. If IERRO=1, the Airy function overflows or underflows. Scaled functions do not overflow or underflow.

As previously described, the computation of $\mathrm{Bi}(z)$ is based on that of $\mathrm{Ai}(z)$ through connection formulas.

4. NUMERICAL VERIFICATION

Three independent numerical verifications of the code have been performed: Wronskian tests, comparison with trapezoidal quadrature method [Gil et al. 2002] and comparison with Amos' code. We conclude that the relative accuracy reachable for scaled Airy functions is better than 10^{-13} while the unscaled functions tend to lose precision gradually as $|z|$ increases. The accuracy is fixed by the selection of the number of points for the Gauss-Laguerre quadrature (40 points truncated to 25 function evaluations, see Gil et al. [2002]), which provides a good compromise between accuracy and efficiency.

As noted in the introduction, there are unavoidable exceptions where relative precision can be lost. At the real or complex zeros of Airy functions, both the real and imaginary parts cancel and relative precision loses meaning; near the zeros, both for scaled and unscaled functions, 10^{-13} is the absolute precision attainable for the modulus of the functions while their phase becomes meaningless.

When either the real or imaginary part of the function vanish (but not both simultaneously), the modulus does not suffer loss of relative accuracy, but the phase of the function does. For the case of unscaled Airy functions, these cancellations are mainly caused by the leading exponential factor $e^{-\frac{2}{3}z^{3/2}}$ (see Figure 2, left), which is removed in the case of scaled functions. For the case of unscaled Airy functions, the real or imaginary parts can vanish only near the real axis and near the zeros.

Of course, the same cancellations take place for Amos' code. As we will show, our codes behave better than Amos' in the regions where cancellations occur.

4.1 Wronskian Checks

We can test formulas 10.4.10-13 in Abramowitz and Stegun [1964] to check the accuracy of the algorithm. One should only perform such tests for satisfactory pairs of solutions, otherwise, if the two functions are dominant within one sector, the result will unavoidably suffer from strong cancellations.

We test the Wronskians for $\Im z \geq 0$ (as explained for $\Im z < 0$ we apply complex conjugation). This test is included in the package (WRTEST.F). The four Wronskian tests are not independent, and it is sufficient to test, for instance, 10.4.11 and 10.4.12, where the solutions involved constitute a satisfactory pair of numerical solutions, namely:

$$\begin{aligned} e^{2\pi i/3}\mathrm{Ai}(z)\mathrm{Ai}'(ze^{2\pi i/3}) - \mathrm{Ai}(z)'\mathrm{Ai}(ze^{2\pi i/3}) &= \frac{1}{2\pi}e^{-\pi i/6}\ ; \quad 0 \leq \mathrm{ph} z \leq \pi/3, \\ e^{-2\pi i/3}\mathrm{Ai}(z)\mathrm{Ai}'(ze^{-2\pi i/3}) - \mathrm{Ai}(z)'\mathrm{Ai}(ze^{-2\pi i/3}) &= \frac{1}{2\pi}e^{\pi i/6}\ ; \quad 0 \leq \mathrm{ph} z \leq \pi. \end{aligned} \tag{14}$$

For scaled functions $\tilde{\mathrm{Ai}}$, exactly the same relations hold. We only need to decide how to deal with the discontinuity cut at the negative real axis. If the normalized functions are defined to be continuous when approaching the negative real axis with positive real values of $\Im z$, then both equations hold in the same sectors (otherwise the cases $\mathrm{ph} z = \pi/3$ and $\mathrm{ph} z = \pi$ should be considered separately). We adopt the continuity convention described above.

The Wronskians between Bi and Ai are not independent of the above mentioned results. However, it is useful to also check such Wronskians particularly in the case of scaled functions $\tilde{\mathrm{Bi}}$, given the different scaling used for two different sectors. Two numerically satisfactory Wronskians are:

$$\begin{aligned} W[\mathrm{Ai}(z), \mathrm{Bi}(z)] &= 1/\pi \quad \text{for} \quad 0 \leq \mathrm{ph} z \leq \pi/3, \\ W[\mathrm{Bi}(z), \mathrm{Ai}(e^{-2\pi i/3} z)] &= \frac{e^{2\pi i/3}}{2\pi} \quad \text{for} \quad \pi/3 \leq \mathrm{ph}(z) \leq \pi, \end{aligned} \tag{15}$$

which, with the scaling prescriptions previously described, also apply when replacing Ai and Bi by $\tilde{\mathrm{Ai}}$ and $\tilde{\mathrm{Bi}}$, except for $\mathrm{ph} z = \pi/3$ in the first Wronskian.

All the Wronskian checks for scaled Airy functions $\tilde{\mathrm{Ai}}$ and $\tilde{\mathrm{Bi}}$ are consistent with a relative accuracy better than 10^{-13} . For unscaled Airy functions Ai and Bi, gradual loss of precision is observed for $|z| > 30$ (see Figure 2).

4.2 Comparison with Nonoscillating Integral Representations. Accuracy of the Codes

In Gil et al. [2002], nonoscillating integral representations, based on steepest descent contours of integration, were introduced. These integral representations are numerically stable for the computation of Airy functions in the complex plane. The trapezoidal rule is a good choice for the computation of the resulting integrals due to the steep decrease of the integrands at infinity.

Although the computation using the trapezoidal rule is less efficient than the methods actually used in the code AIZ and BIZ, the accuracy can be selected and then it is a convenient test-bench for the codes. In particular, the normalized functions $\tilde{\mathrm{Ai}}$ and $\tilde{\mathrm{Ai}}'$ can be checked against the non-oscillating integrals, computed with an accuracy better than 10^{-14}.

4.2.1 *Scaled Airy Functions.* It is observed that the relative accuracy when evaluating scaled Airy functions with the codes AIZ and BIZ is better than 10^{-13}. For the comparisons, we check the quantities

$$M(z) \equiv |\Re\tilde{\mathrm{Ai}}| + |\Im\tilde{\mathrm{Ai}}| \quad \text{and} \quad R(z) \equiv \Im\tilde{\mathrm{Ai}}/\Re\tilde{\mathrm{Ai}}. \tag{16}$$

between both codes.

As explained, there is an exception to this. Close to the curves $\Re\tilde{\mathrm{Ai}} = 0$, relative error for the evaluation of the real part loses meaning; of course, similar things happen with the imaginary part. And when both the real and imaginary parts tend to cancel (at the zeros of $\mathrm{Ai}(-x)$, $x > 0$), only the absolute error for the modulus of the function makes sense. Figure 1 shows these points of discrepancy near the negative real axis. The rings where the code fails to compute $R(z)$ with 10^{-13} relative accuracy correspond to the level curves $\Re\tilde{\mathrm{Ai}} = 0$, $\Im\tilde{\mathrm{Ai}} = 0$ which appear in couples and touch at the zeros of Ai in the negative real axis. As can

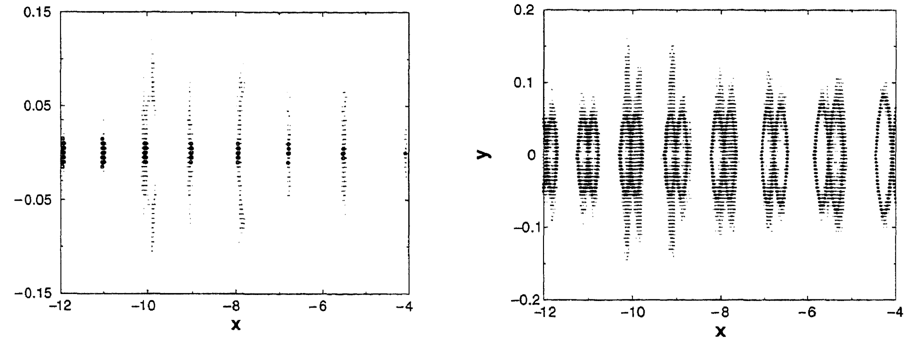

Fig. 1. Points where AIZ and ZAIRY (Amos' code) fail to compute the scaled Airy function with 10^{-13} relative accuracy. On the left, the points of discrepancy for $M(z)$ (Eq. (16)) are shown. The figure on the right is for $R(z)$. Black points correspond to AIZ and grey points to ZAIRY.

be seen in the same figure, this unavoidable error is under better control in our code than in Amos'.

In the case of scaled Airy functions $\tilde{\mathrm{Bi}}(z)$ the same phenomena appear near its negative real zeros and its complex zeros, which lie close to the rays $\mathrm{ph} z = \pm\pi/3$. The same happens for the derivatives.

The loss of accuracy near the zeros of Airy functions increases as $|z|$ becomes larger. In particular, we observe that for $|z| \leq 1000$, the absolute accuracy for the modulus near the zeros of $\tilde{\mathrm{Ai}}$ is better than 10^{-13} but as $|z|$ increases, the absolute accuracy worsens. This is as expected because the spacing between zeros is $\Delta x \simeq \pi/\sqrt{-x}$ for large $-x$. For $-x$ as large as 10^6, the absolute accuracy reachable for the modulus goes down to 10^{-8}. The same loss of accuracy for large $|z|$ takes places for $\tilde{\mathrm{Bi}}$ when $\mathrm{ph} z \simeq \pm\pi/3$, $\mathrm{ph} z \simeq \pm\pi$, where the zeros of this function lie.

4.2.2 *Unscaled Airy Functions.* Unscaled Airy functions overflow or underflow for values of $\frac{2}{3}|z|^{3/2}$ larger than log(OVER), OVER being the machine overflow number.

The relative error in the evaluation of the modulus of unscaled Airy functions tends to decrease as $|z|$ increases and becomes generally worse than 10^{-13} for $|z| > 30$. The main reason lies in the errors that the leading exponential term $e^{-2z^{3/2}/3}$ unavoidably introduces for large $|z|$, as discussed in Gil et al. [2002]. This source of error is common to Amos' codes and AIZ/BIZ and is also present when computing the Airy functions through steepest descent integrals. Figure 2 (left) illustrates such gradual loss of accuracy.

Furthermore, when considering unscaled functions, the phase of Airy functions suffers infinitely many cancellations, given the oscillatory nature of the dominant exponential factor in the complex plane. This factor introduces an infinite number of curves where the real or imaginary parts of the function vanish, apart from those close to the zeros of the function, see Figure 2 (right). Over these curves relative precision for the phase is meaningless. The code computes $min(|R(z)|, 1/|R(z)|)$ with absolute accuracy better than 10^{-13} for $|z| < 30$. Of course, at the zeros of the function, $R(z)$ becomes meaningless.

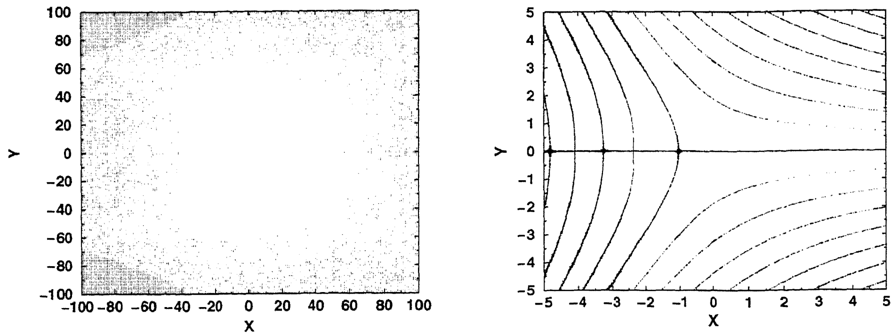

Fig. 2. Points where AIZ fails to compute $M(z)$ (left) and $R(z)$ (right) for the unscaled Airy function with 10^{-13} relative accuracy. AIZ is tested against steepest descent integrals. In the left figure, Ai overflows in the darker shaded regions (top and bottom left corners).

Figure 2 (right) illustrates the regions where a relative accuracy of 10^{-13} is not attainable for the computation of the phase (related to $R(z)$) of $\mathrm{Ai}(z)$. Over these curves the real or imaginary parts of the Airy function tend to cancel. As commented, the absolute error for $min(|R(z)|, 1/|R(z)|)$ is better than 10^{-13} over these curves. The comparison between Amos' and steepest descent integrals is similar, except that larger errors are observed near the negative real zeros and additional errors appear close to the line $\mathrm{ph}z = \pi/3$ and z real and positive (see next section and Figures 1, 3 and 4).

4.3 Comparison with Amos' Code

Amos' code computes Airy functions through its connection with Bessel functions of complex arguments. In particular, Eq. 10.4.14 in Abramowitz and Stegun [1964] is used:

$$\mathrm{Ai}(z) = \frac{1}{\pi}\sqrt{z/3}K_{1/3}(\zeta), \quad \zeta = \frac{2}{3}z^{3/2}, \tag{17}$$

and similar relations are applied for the rest of Airy functions. The code relies on the computation of Bessel functions for $\Re\zeta > 0$, that is, $|\mathrm{ph}z| \leq \pi/3$, and for the rest of the complex plane connection formulas of the type:

$$K_\nu(\zeta e^{\pm i\pi}) = e^{\mp i\pi\nu}K_\nu(z) \mp i\pi I_\nu(z). \tag{18}$$

are used.

However, the implementation of these connection formulas seems to produce some accuracy problems, particularly near the anti-Stokes lines (see also Section 2.2). Other accuracy problems appear, as we discuss next. We compare our code with Amos' code, focusing on the lines $\mathrm{ph}(z) = 0$, $\mathrm{ph}(z) = \pi/3$ and $\mathrm{ph}(z) = \pi$.

Figure 1 shows the discrepancies with Amos' code close to the negative real axis, both for $M(z)$ and $R(z)$. As previously mentioned, loss of relative precision is expected near the zeros of $\tilde{\mathrm{A}}\mathrm{i}(-x)$; however, it is apparent that our code is more stable in this region.

Figure 3, shows the discrepancies found with Amos' code on the positive real axis, clearly due to a failure in the evaluation of the imaginary part for

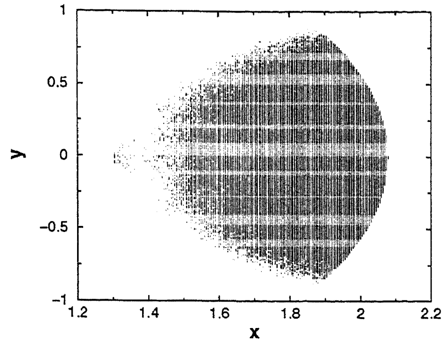

Fig. 3. Points near the positive real axis where ZAIRY (Amos' code) fails to compute $R(z)$ for the scaled Airy function with 10^{-13} relative accuracy. The results from AIZ match with those from steepest descent methods within this precision.

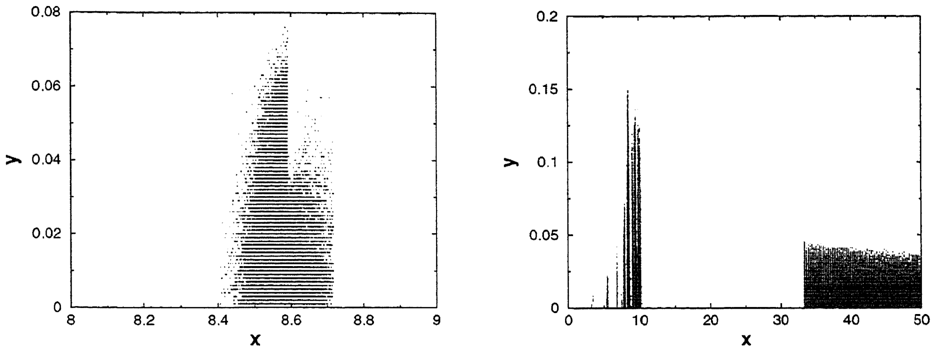

Fig. 4. Points close to the ray ph$z = \pi/3$ where ZAIRY (Amos' code) fails to compute $M(z)$ (left) and $R(z)$ (right) for the scaled Airy function with 10^{-13} relative accuracy. The match between AIZ and steepest descent integrals is perfect for this precision. The axes have been rotated by 60°.

moderate values of z close to this axis. Our code matches perfectly with the non-oscillating integral representation for the same demanded accuracy.

Finally, Figure 4 shows the discrepancies found close to the ray ph$z = \pi/3$ for Amos' code. Again AIZ shows no discrepancies with respect to steepest descent integrals.

4.4 Performance of the Code

The codes have been tested in different platforms (Sun OS, Linux-Debian, Windows), different computers (Sun station, PC Pentium II, Laptop Pentium II) and different compilers (g77 for Unix/Linux, g77 for MS-DOS, f90 for Sun OS, Compaq FORTRAN) with similar results.

When compared with Amos' code, it is observed that our codes run generally faster and that, in the rare situations where they are slower (less than a factor 2 slower) they are more accurate. Figure 5, shows the regions where AIZ is faster than ZAIRY.

For a Pentium II-300 MHz computer, with g77 running under Debian Linux 2.1, the typical running times are in the range $10\,\mu s$ to $100\,\mu s$ for the

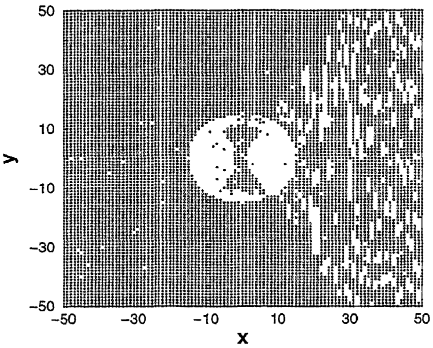

Fig. 5. Points in the complex plane where AIZ is faster than ZAIRY (Amos' code).

evaluation of one complex function for one value of z. When computing series or asymptotic expansions, the typical running times are in the range $10\,\mu s - 20\,\mu s$ while the Gauss-Laguerre integrals spend around $100\,\mu s$ for each z. Considering a Pentium II-500 MHz, with Compaq Fortran running under Windows 98, the typical range of CPU times becomes $5-60\,\mu s$; series and asymptotic expansions spend from $5\,\mu s$ to $9\,\mu s$ while Gauss-Laguerre quadrature typically needs $60\,\mu s$.

When using connection formulas, two functions have to be evaluated; in this case, the total running time will be twice as large.

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