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A RELATIVISTIC THEORY OF THE R WAVE CUT-OFF

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Abstract—R wave propagation at frequencies near the cut-off frequency is studied in detail based on a weakly relativistic approximation. It is pointed out that the deviation of the relativistic cut-off frequency from the corresponding value of the cut-off frequency predicted by the theory of wave propagation in a cold plasma is particularly noticeable in a rarefied plasma. Our theory predicts that the R wave in the frequency range under consideration can propagate without collisionless damping or amplification. The obtained results are applicable to the analysis of low frequency cut-off in the dynamic spectra of natural magnetospheric radio emissions at frequencies above the electron plasma frequency (non-thermal continuum).

1. INTRODUCTION

It is well known that, for certain values of wave frequencies and plasma parameters, the dispersion equation for wave propagation in a cold plasma has solutions corresponding to the wave refractive index N equal to zero (see e.g. Stix, 1962). Frequencies at which this happens are called cut-off frequencies. The analysis of wave propagation in the vicinity of these frequencies is particularly important for understanding different properties of wave propagation in plasmas, and, in particular, in the Earth's magnetosphere, and also for wave diagnostics of plasma parameters (see e.g. Gurnett and Shaw, 1973; Gurnett and Frank, 1974). Although the results of the latter diagnostics based on a cold plasma theory appeared to be in a qualitative agreement with the *in situ* measurements of these parameters, the more refined quantitative diagnostics require taking into account the effects of finite plasma temperature as well. One can easily see that the finite temperature effects do not influence wave propagation within the non-relativistic theory, but this influence appears, and can be quite significant, if we base our analysis on a more general weakly relativistic approximation. Some basic results of an approximate relativistic theory of plasma cut-offs have been discussed by Sazhin (1987), who pointed out that the relativistic effects can lead to a noticeable shift in the cut-off frequencies. In view of the importance of this shift for practical applications (in particular, for diagnostics of plasma parameters in the Earth's magnetosphere), we feel that further developments of the relativistic theory of plasma cut-offs would be justified.

The results of one of these developments are presented in this paper. In contrast to Sazhin (1987) we restrict ourselves to the analysis of only one type of wave, namely the so-called R wave propagating strictly parallel to the magnetic field. In the cold plasma limit this wave is described by the following dispersion equation (Stix, 1962):

$$N^2 = R, \tag{1}$$

where

$$R = 1 + \frac{\nu Y^2}{Y-1}, \tag{2}$$

$\nu = \Pi^2/\Omega^2$, $Y = \Omega/\omega \leqslant (\sqrt{1+4\nu}-1)/2\nu \equiv \Omega/\omega_{\mathrm{cf0}}$, Π is an electron plasma frequency, Ω is the electron gyrofrequency and ω is the wave frequency.

In a weakly relativistic approximation the dispersion equation for this wave can be written as

$$N^2 = 1 - \frac{2\nu Y^2}{r}\left[\mathscr{F}_{1/2,2} - \frac{\mathrm{d}\mathscr{F}_{3/2,2}}{\mathrm{d}z}(A_e - 1)N^2\right], \tag{3}$$

where

$$\mathscr{F}_{q,p} \equiv \mathscr{F}_{q,p}(z,a,b) = -\mathrm{i}\int_0^\infty \exp\left(\mathrm{i}zt - \frac{at^2}{1-\mathrm{i}t}\right)(1-\mathrm{i}t)^{-q}(1-\mathrm{i}bt)^{-p}\,\mathrm{d}t \tag{4}$$

is the generalized Shkarofsky function, $z =$

$2(1-Y)/r$, $a = N^2/r$, $r = p_{0\parallel}^2/(m_e^2c^2)$, $b \equiv A_e = p_{0\perp}^2/p_{0\parallel}^2$. When deriving equation (3) we assumed that the electron distribution function had the form:

$$f(p_\perp, p_\parallel) = (j!\pi^{3/2}p_{0\perp}^{2j+2}p_{0\parallel})^{-1}p_\perp^{2j} \times \exp\left(-\frac{p_\perp^2}{p_{0\perp}^2} - \frac{p_\parallel^2}{p_{0\parallel}^2}\right), \quad (5)$$

where $p_{0\perp(\parallel)}$ is the electron thermal momentum in the direction perpendicular (parallel) to the magnetic field, $p_\perp$ and $p_\parallel$ are the electron momenta in the corresponding directions, $j = 0, 1, 2, \ldots$.

We restricted our analysis to the case $j = 0$ (bi-Maxwellian plasma). The generalization to $j \neq 0$ would be straightforward (Tsai *et al.*, 1981; Sazhin, 1989). The protons and other heavy ions were considered as a neutralizing background. Note that when $\omega < \Omega$ equations (1) and (3) describe the propagation of whistler-mode waves (see Sazhin and Temme, 1990).

In Section 2 we describe the R wave propagation near the cut-off frequency in an isotropic plasma [$b \equiv A_e = 1$ in equation (3)]. In Section 3 we generalize the results of Section 2 to the case of an anisotropic plasma. The main results of the paper are summarized in Section 4.

2. PROPAGATION IN AN ISOTROPIC PLASMA

In the case of an isotropic plasma ($A_e = 1$) equation (3) can be simplified to:

$$N^2 = 1 - \frac{2\nu Y^2}{r}\mathscr{F}_{5/2}, \quad (6)$$

where

$$\mathscr{F}_{5/2} \equiv \mathscr{F}_{5/2}(z) = -\mathrm{i}\int_0^\infty \exp\left(\mathrm{i}zt - \frac{at^2}{1-\mathrm{i}t}\right) \times (1-\mathrm{i}t)^{-5/2}\,\mathrm{d}t = \mathscr{F}_{1/2,2}(z, a, b = 1) \quad (7)$$

is the Shkarofsky function with the index $\tilde{q} = 5/2$.

Restricting our analysis to waves at frequencies near the cut-off frequency (ω_{cf}) we can assume that $N^2 \ll 1$ and $a \ll 1$, and write the following series expansion [cf. equation (10) in Robinson, 1986]:

$$\mathscr{F}_{5/2} = \mathrm{e}^{-a}\sum_{j=0}^\infty \frac{a^j}{j!}F_{(5/2)+j}(z-a), \quad (8)$$

where

$$F_{\tilde{q}}(z) = \mathscr{F}_{\tilde{q}}(z, a = 0) \quad (9)$$

is the Dnestrovskii function (see Robinson, 1986). This function can be expressed in terms of the incomplete gamma function

$$\Gamma(\alpha, z) = \int_z^\infty \mathrm{e}^{-t}t^{\alpha-1}\,\mathrm{d}t \quad (10)$$

and written as (Robinson, 1986)

$$F_{\tilde{q}}(z-a) = z^{\tilde{q}-1}\mathrm{e}^z\,\Gamma(1-\tilde{q}, z-a). \quad (11)$$

For $|z-a| \to \infty$ and $|\arg z| < 3\pi/2$ (which is justified for the waves under consideration) $\Gamma(\alpha, z)$ can be asymptotically expanded (see Abramovitz and Stegun, 1964, p. 263):

$$\Gamma(1-\tilde{q}, z-a) \sim z^{-\tilde{q}}\mathrm{e}^{-z}\left[1 - \frac{\tilde{q}}{z-a} + \frac{\tilde{q}(\tilde{q}+1)}{(z-a)^2} + \cdots\right]. \quad (12)$$

Having substituted (12) into (11) we obtain

$$F_{\tilde{q}}(z-a) \sim \sum_{j=0}^\infty \frac{(-1)^j(z-a)^{-1-j}\Gamma(\tilde{q}+j)}{\Gamma(\tilde{q})}, \quad (13)$$

where $\Gamma(z)$ is the Euler gamma function.

Having substituted (13) into (8) and (8) into (6) we obtain the final equation describing the propagation of the R wave at frequencies close to the cut-off frequency, ω_{cf}, in an isotropic plasma:

$$N^2 = 1 - \frac{2\nu Y^2}{r}\mathrm{e}^{-a}\sum_{j=0}^\infty \frac{a^j}{j!} \times \sum_{\tilde{j}=0}^\infty \frac{(-1)^{\tilde{j}}(z-a)^{-1-\tilde{j}}\Gamma(\frac{5}{2}+j+\tilde{j})}{\Gamma(\frac{5}{2}+j)}. \quad (14)$$

Let us now consider some limiting cases of equation (14). We begin with deriving the equation for the cut-off frequency ω_{cf}. In order to do this we should put $N^2 = a = 0$ in equation (14) and reduce it to:

$$\frac{2\nu Y^2}{r}\sum_{\tilde{j}=0}^\infty \frac{(-1)^{\tilde{j}}z^{-1-\tilde{j}}\Gamma(\frac{5}{2}+\tilde{j})}{\Gamma(\frac{5}{2})} = 1. \quad (15)$$

Considering the contribution of only the zeroth order term in equation (15) we can simplify this equation to:

$$\frac{\nu Y^2}{1-Y} = 1. \quad (16)$$

This is exactly the equation for the cut-off frequency in a cold plasma ($R = 0$) [see equation (2)].

Considering the contribution of two lowest order terms in equation (15), we reduce this equation to:

$$\frac{\nu Y^2}{1-Y} - \frac{5\nu Y^2 r}{4(1-Y)^2} = 1. \quad (17)$$

This equation is compatible with equation (3.11) by Sazhin (1987) taken for an isotropic plasma ($A_e = 1$).

The third term in expansion (15) allows us to generalize equation (17) to:

$$\frac{\nu Y^2}{1-Y} - \frac{5\nu Y^2 r}{4(1-Y)^2} + \frac{35\nu Y^2 r^2}{16(1-Y)^3} = 1. \quad (18)$$

Equation (18) [or its simplified versions (16) and (17)] can be used for different purposes. For example, for the calculation of $\omega \equiv \omega_{cf}$ when Π and Ω are known, or for the calculation of Ω when ω_{cf} and Π are known, or for the calculation of Π when ω_{cf} and Ω are known. In particular, assuming that r is small, but keeping the second order terms we can rewrite equation (18) as:

$$\nu = \frac{1-Y_{cf}}{Y_{cf}^2}\left[1 + \frac{5r}{4(1-Y_{cf})} - \frac{5r^2}{8(1-Y_{cf})^2}\right], \quad (19)$$

where $Y_{cf} = \Omega/\omega_{cf}$.

As follows from equation (19), the relativistic corrections, given by the terms containing r in equation (19), are generally small when r is small and Y_{cf} is not close to unity. However, if Y_{cf} is close to unity, which is the case for the R wave cut-off in a rarefied plasma ($\nu \leqslant 1$), these relativistic corrections can be significant even for small r (cf. similar conclusions for whistler-mode waves by Sazhin and Temme, 1990, 1991a,b).

Now we return to equation (14) and consider the contribution of finite although small a. Keeping terms up to the second order with respect to a and z^{-1} we rewrite this equation as:

$$N^2 = R + \nu Y^2\left[\frac{5r}{4(1-Y)^2} - \frac{35r^2}{16(1-Y)^3}\right]. \quad (20)$$

Restricting ourselves to considering the contribution of the terms up to the first order only we can simplify equation (20) to

$$N^2 = R + \frac{5\nu Y^2 r}{4(1-Y)^2}. \quad (21)$$

At the frequency corresponding to the cut-off in a cold plasma, ω_{cf0}, when $R = 0$, we have:

$$N^2(\omega = \omega_{cf0}) = \frac{5\nu Y^2 r}{4(1-Y)^2}. \quad (22)$$

This equation is compatible with equation (3.8) given by Sazhin (1987) and taken in the limit $A_e = 1$.

The main restriction of the equations derived in this section is that they are valid for an isotropic plasma only. In the next section we generalize the results to the case of an anisotropic plasma.

3. PROPAGATION IN AN ANISOTROPIC PLASMA

In the case of wave propagation in an anisotropic plasma we should base our analysis on the general equation (3).

For small a we can write the following expansion for $\mathscr{F}_{q,p}$ (see Temme *et al.*, in press):

$$\mathscr{F}_{q,p} = e^{-a} \sum_{j=0}^{\infty} \frac{a^j}{j!} \phi_{j+q,p} \quad (23)$$

with

$$\phi_{q,p} = \int_0^{\infty} \frac{e^{-(z-a)s}}{(1+s)^q(1+bs)^p}\,ds. \quad (24)$$

The asymptotic expansion of equation (24) for large $z-a$ can be obtained either by expanding the denominator of the integrand for small s or by integrating equation (24) by parts. In both cases we obtain the following first three terms of this expansion:

$$\phi_{q,p} \sim \frac{1}{z-a} - \frac{q+bp}{(z-a)^2} + \frac{q(q+1)+2bpq+b^2p(p+1)}{(z-a)^3} + \cdots. \quad (25)$$

For an isotropic plasma ($b = 1$) with $q = 1/2$ and $p = 2$, expansion (25) reduces to the first three terms in expansion (13). We were unable to write an asymptotic expansion of $\mathscr{F}_{q,p}$ in a general form similar to (13) for the case of an anisotropic plasma.

Having substituted (25) into (23) and keeping only terms up to the third order in a or z^{-1} we can simplify equation (23) to

$$\mathscr{F}_{q,p} \sim \frac{1}{z} - \frac{q+bp}{z^2} + \frac{q(q+1)+2bpq+b^2p(p+1)}{z^3} + \cdots. \quad (26)$$

Having substituted (26) into (3) and neglecting the higher order terms, we obtain:

$$N^2 = 1 - \frac{\nu Y^2}{1-Y}\left[1 - \frac{1+4A_e}{2z} + \frac{3+8A_e+24A_e^2}{4z^2} + \frac{(A_e-1)N^2}{z}\right]. \quad (27)$$

Assuming $N^2 = 0$ in equation (27) we can reduce it to an equation determining the cut-off frequency, ω_{cf}, in an anisotropic plasma:

$$\frac{\nu Y^2}{1-Y}\left[1 - \frac{(1+4A_e)r}{4(1-Y)} + \frac{(3+8A_e+24A_e^2)r^2}{16(1-Y)^2}\right] = 1. \quad (28)$$

At $A_e = 1$ equation (28) reduces to equation (18). The analysis of equation (28) can be performed in a way similar to that which was discussed in Section 2 for an isotropic plasma.

Equation (27) can be explicitly resolved with respect to N^2. Assuming $|R|$ to be small and neglecting the higher order terms we obtain

$$N^2 = R + \frac{\nu Y^2 r(1+4A_e)}{4(1-Y)^2} - \frac{R(A_e-1)\nu Y^2 r}{2(1-Y)^2} - \frac{r^2 \nu Y^2}{16(1-Y)^3}[(3+8A_e+24A_e) + 2\nu Y^2(-1-3A_e+4A_e^2)^{-1}]. \quad (29)$$

Neglecting the second order terms we can further simplify equation (29) to

$$N^2 = R + \frac{\nu Y^2 r(1+4A_e)}{4(1-Y)^2}. \quad (30)$$

At the cut-off frequency in a cold plasma, when $R = 0$, equation (30) gives an expression for N^2 at $\omega = \omega_{\mathrm{cf0}}$ which exactly coincides with that derived by Sazhin (1987) [see his equation (3.8)]. For isotropic plasma, expressions (29) and (30) reduce to (20) and (21), respectively.

Note that equations (3.8) and (3.11) in Sazhin (1987) can be considered as particular cases of equation (30) corresponding to $R = 0$ and $N^2 = 0$, respectively. Equation (29) can be used either for more refined analysis of the R wave propagation in the vicinity of the cut-off frequency, or for the rigorous justification of the range of applicability of equation (30). As one can see from equation (29) the neglect of second order terms with respect to r is not always justified, particularly for Y close to 1 and/or large A_e.

Note that in the case $z \geqslant a$ the function $\phi_{q,p}$ [see equation (24)] is real for real z and a. Hence, based on equation (23), we can see that $\mathscr{F}_{q,p}$ is real as well, which means that the waves described by equation (3) can propagate without growth or damping for these values of parameters. Remembering that for the R wave and frequencies near the cut-off frequency z is large and positive, while a is small, we can see that the condition $z \geqslant a$ for these waves is surely satisfied and we can expect that the interaction of these waves with charged particles will not lead to the change of their amplitude. In the case of an isotropic plasma we have $\mathscr{I}\mathscr{F}_{\bar{q}} = 0$ when $z \geqslant a$, as also follows from equation (43) of Robinson (1986). The non-relativistic theory of R wave propagation in a hot plasma would predict the absence of growth or damping of the waves only when $\omega = \omega_{\mathrm{cf0}}$. At $\omega > \omega_{\mathrm{cf0}}$ this theory would predict slight damping or amplification of the waves depending on the value of A_e.

In the Earth's magnetosphere the R wave cut-off was observed as a lower frequency cut-off in the dynamic spectrum of the natural electromagnetic radiation at frequencies above the electron plasma frequency (non-thermal continuum) (Gurnett and Shaw, 1973; Gurnett, 1975). Based on the theory of wave propagation in a cold plasma this cut-off was used to estimate the magnetospheric magnetic field $|\mathbf{B}_0|$ bringing results in qualitative agreement with the *in situ* measurements of this field. However, we strongly believe that before this method of diagnostics of $|\mathbf{B}_0|$ or electron density could be recommended for practical applications, the relativistic effects on R wave propagation near the cut-off frequency need to be taken into account.

4. CONCLUSIONS

An expression for the R wave refractive index $\mathbf{N}$ at frequencies near the cut-off frequency, where $|\mathbf{N}|$ is small, is presented in the form of a Taylor series with respect to N^2 and an asymptotic series with respect to the squared electron thermal velocity. This expression allowed us to obtain a rather accurate expression for the cut-off frequency, generalizing the expression obtained earlier by Sazhin (1987). It is pointed out that the relativistic corrections to the corresponding expression for the R wave cut-off frequency in a cold plasma are particularly important in the case of wave propagation in a rarefied plasma (the electron plasma frequency is less than the electron gyrofrequency). Relativistic theory predicts the propagation of these waves without collisionless damping or amplification.

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