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The threshold of parallel whistler-mode instability

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Abstract. Based on a weakly relativistic consideration of the condition of marginal stability of parallel whistler-mode waves, it is pointed out that these waves tend to be stabilized by a reduction of electron density. As a result the range of whistler-mode instability for a given value of anisotropy of electron distribution function can differ up to a factor of two for the range of electron densities typical for magnetospheric conditions. This effect, not noticed so far, needs to be taken into account when interpreting observations of whistler-mode waves in the magnetosphere.

Introduction

Since the publication of Kennel and Petschek's (1966) pioneering paper it has generally been believed that parallel whistler-mode waves propagating through a hot anisotropic but nonrelativistic plasma are unstable when:

$$A_e > A_{e0} \equiv \frac{Y}{Y-1} \qquad (1)$$

where A_e is the anisotropy of the electron distribution function (ratio of the electron perpendicular temperature $T_\perp$ to the electron parallel temperature $T_\parallel$), $Y=\Omega/\omega$, Ω is the modulus of electron gyrofrequency, ω is the wave frequency.

Condition (1) seems to be confirmed by simultaneous observations of natural whistler-mode radio-emissions in the inner magnetosphere (Burton, 1976; Solomon *et al.*, 1988) and in the magnetosheath (Thorne and Tsurutani, 1981; Tsurutani *et al.*, 1982), and has been widely used in the interpretation of whistler-mode wave phenomena in different regions of the magnetosphere (e.g. Hayashi *et al.*, 1968; Haugstad, 1976). Nonrelativistic approximation seemed to be justified by the condition:

$$w_\parallel \ll c \qquad (2)$$

(where $w_\parallel$ is the parallel thermal velocity of the electrons, c is the velocity of light), valid in most regions of the magnetosphere.

At the same time in many theoretical papers (see e.g. Jacquinot and Leloup, 1971; Sazhin, 1987a, b, 1989; Sazhin and Temme, 1990) it has been pointed out that the inequality (2) is the condition of weakly relativistic rather than nonrelativistic plasma. In particular, it has been shown that relativistic corrections to the whistler-mode refractive index N can be of the same order or even exceed the corresponding thermal corrections, and also relativistic corrections influence the threshold of whistler-mode instability determined by condition (1). However, the analyses of this threshold presented by Jacquinot and Leloup (1971) and Sazhin and Temme (1990) have not answered the question whether the relativistic effects on whistler-mode instability are in fact important in the magnetospheric conditions. The numerical curves presented by Jacquinot and Leloup (1971) refer to extreme, rather than typical, values of the magnetospheric parameters, while the analysis of Sazhin and Temme (1990) refers to the extreme case when:

$$\frac{2(Y-1)}{N^2} \ll 1. \qquad (3)$$

The condition (3) is satisfied for rather large values of N, unless ω is close to Ω. In this case the relativistic correction to A_{e0} appeared to be of the order of:

$$\delta = \frac{1}{2N^2} \ll 1 \qquad (4)$$

which seems to be not very significant in magnetospheric conditions, unless a very refined analysis of the instability is required.

In this paper we present an alternative analysis of the threshold of whistler-mode instability in a weakly relativistic plasma [inequality (2) is valid] without imposing any specific restrictions on the value of N, but assuming that:

$$|\Delta A_e| = |A_e - 1| \ll 1. \qquad (5)$$

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Inequality (5) seems to be satisfied in some regions of the magnetosphere and the magnetosheath (e.g. Tsurutani *et al.*, 1982; Cornilleau-Wehrlin *et al.*, 1985; Bahnsen *et al.*, 1985; Solomon *et al.*, 1988), although in other regions higher values of A_e were observed (e.g. Cornilleau-Wehrlin *et al.*, 1985; Solomon *et al.*, 1988). At this stage our theory cannot be applied to the case of highly anisotropic plasma. Another major restriction of our theory is that it does not take into account the possible dependence of A_e on $Y^{-1}=\omega/\Omega$ (see the discussion of this problem by Cornilleau-Wehrlin *et al.*, 1985; and Solomon *et al.*, 1989). The values of A_e, which we will eventually calculate, will refer to Y^{-1} corresponding to the frequency of marginal stability of whistler-mode waves.

The theoretical background of our analysis is discussed below, where we derive the weakly relativistic version of the condition for marginal stability of whistler-mode waves. Later, the latter condition is analysed, and we discuss its relevance to the magnetosphere.

Theory

Following Sazhin and Temme (1990), we assume an electron distribution function in the form:

$$f(p_\perp, p_\parallel)=(\pi^{3/2}\, p_{0\perp}^2\, p_{0\parallel})^{-1} \exp\left(-\frac{p_\perp^2}{p_{0\perp}^2}-\frac{p_\parallel^2}{p_{0\parallel}^2}\right) \tag{6}$$

where $p_{0\perp(\parallel)}$ is the electron thermal momentum in the direction perpendicular (parallel) to the magnetic field, $p_\perp$ and $p_\parallel$ are the electron momenta in the corresponding directions.

We restrict our analysis to considering wave frequencies well above proton gyrofrequency, so that the protons and other heavy ions can be considered as a neutralizing background. Finally, we assume the plasma to be weakly relativistic, so that (2) is valid. In view of all these assumptions and neglecting the contribution of higher order terms, we can write the dispersion equation for parallel whistler-mode waves as (Sazhin and Temme, 1990):

$$N^2=1-\frac{2X}{r}\left[\mathcal{F}_{1/2,2}-\frac{d\mathcal{F}_{3/2,2}}{dz}(A_e-1)\,N^2\right] \tag{7}$$

where

$$\mathcal{F}_{q,p}\equiv\mathcal{F}_{q,p}(z,a,b)=-i\int_0^\infty e^{izt-\frac{at^2}{1-it}}(1-it)^{-q}(1-ibt)^{-p}\,dt \tag{8}$$

is the generalized Shkarofsky function,

$$z=\frac{2(1-Y)}{r};\quad a=\frac{N^2}{r};\quad r=\frac{p_{0\parallel}^2}{m_e^2c^2};$$

$$b=A_e;\quad X=\Pi_0^2/\omega^2;\quad Y=\Omega_0/\omega;\quad A_e=p_{0\perp}^2/p_{0\parallel}^2$$

is the same in (1); Π_0 and Ω_0 are the electron plasma frequency and electron gyrofrequency at rest (in the non-relativistic limit, Y in this definition is the same as in (1)); and m_e is the electron mass at rest.

In view of our condition (5), Eq. (7) can be considerably simplified and written as (Sazhin 1989):

$$D=N^2-1+\frac{2X}{r}\left[\mathcal{F}_{5/2}+\Delta A_e(2-N^2)\,\mathcal{F}'_{7/2}\right]=0 \tag{9}$$

where:

$$\mathcal{F}_q\equiv\mathcal{F}_q(z,a)=-i\int_0^\infty e^{izt-\frac{at^2}{1-it}}(1-it)^{-q}\,dt \tag{10}$$

is the Shkarofsky function (Shkarofsky, 1966; Airoldi and Orefice, 1982; Krivenski and Orefice, 1983; Robinson, 1986, 1987):

$$\mathcal{F}'_q\equiv d\mathcal{F}_q/dz=\mathcal{F}_q-\mathcal{F}_{q-1}. \tag{11}$$

In general, Eq. (9) can be analysed by numerical methods only. Its analytical solution is possible only in some particular limiting cases (see e.g. Sazhin, 1987a, 1989; Sazhin and Temme, 1990). Here, we consider one more particular solution of (9) corresponding to marginally stable waves. This solution follows from the equation (Hasegawa, 1975):

$$\Im D=0 \tag{12}$$

in which the arguments of the Shkarofsky functions are assumed to be real. In view of (9) and (11), Eq. (12) can be written as:

$$\Im\mathcal{F}_{5/2}+\Delta A_e(2-N^2)(\Im\mathcal{F}_{7/2}-\Im\mathcal{F}_{5/2})=0. \tag{13}$$

In view of the fact that the argument z of the Shkarofsky function is negative for whistler-mode waves, we obtain (Robinson, 1986):

$$\Im\mathcal{F}_q=-\pi e^{z-2a}[(a-z)/a]^{(q-1)/2}\,I_{q-1}[2a^{1/2}(a-z)^{1/2}] \tag{14}$$

$q=5/2;\ 7/2$; I_ν are the modified Bessel functions; a and $a-z$ lie on the principal branch of the Riemann surface.

Substituting (14) into (13) we have:

$$\Delta A_e=\frac{1}{\left[1-\left(\dfrac{a-z}{a}\right)^{1/2} I_{5/2}/I_{3/2}\right](2-N^2)} \tag{15}$$

where the argument of the Bessel functions is the same as in (14).

When deriving (15) we took into account that

$$I_{3/2}=\sqrt{\frac{2z}{\pi}}\left[\frac{\cosh z}{z}-\frac{\sinh z}{z^2}\right]\neq 0.$$

Equation (15) can be further simplified if we remember that we consider a weakly relativistic plasma for which $r\ll 1$, $a\gg 1$ and $-z\gg 1$. In this case the argument of the Bessel functions is large and we can write their asymptotics as ($\tilde{z}=2a^{1/2}(a-z)^{1/2}$):

$$I_\nu(\tilde{z})\sim\frac{\exp(\tilde{z})}{\sqrt{2\pi\tilde{z}}}\sum_{k=0}^{\infty}\frac{(-1)^k}{(2\tilde{z})^k}\frac{\Gamma(\nu+k+\frac{1}{2})}{k!\,\Gamma(\nu-k+\frac{1}{2})} \tag{16}$$

or, more specifically:

$$I_{5/2}(\tilde{z})\sim\frac{\exp(\tilde{z})}{\sqrt{2\pi\tilde{z}}}\left(1-\frac{3}{\tilde{z}}+\dots\right) \tag{17}$$

$$I_{3/2}(\tilde{z})\sim\frac{\exp(\tilde{z})}{\sqrt{2\pi\tilde{z}}}\left(1-\frac{1}{\tilde{z}}+\dots\right). \tag{18}$$

In view of (17)–(18) and remembering that $|2\sqrt{a(a-z)}| \gg 1$ we can simplify (15) to:

$$\Delta A_e = \frac{1}{\left[1-\left(\frac{a-z}{a}\right)^{1/2}+\frac{1}{a}\right](2-N^2)}. \tag{19}$$

In most cases of whistler-mode propagation in the magnetosphere where $r \ll 1$ the term $1/a$ is very small and can be neglected. Hence, Eq. (19) can be further simplified to:

$$\Delta A_e = \frac{1}{\left[1-\sqrt{\frac{N^2-2+2Y}{N^2}}\right](2-N^2)}. \tag{20}$$

For the case when $N^2 \gg 1$ we obtain from (20):

$$\Delta A_e = \frac{1}{Y-1} + \frac{1}{2N^2}\left(\frac{Y+3}{Y-1}\right). \tag{21}$$

In the limit $N^2 \to \infty$ condition (21) reduces to:

$$\Delta A_e = \frac{1}{Y-1}. \tag{22}$$

This is the nonrelativistic threshold of whistler-mode instability similar to that expressed by inequality (1). In view of our assumption (5) we should restrict our analysis to $Y \gg 1$. Hence, (20) reduces to:

$$\Delta A_e = \frac{1}{Y-1} + \frac{1}{2N^2}. \tag{23}$$

This expression is similar to that obtained by Sazhin and Temme (1990) based on a different approach to the problem. This coincidence of the results can justify both approaches. Also, as was indicated by Sazhin and Temme (1990), Eq. (23) is essentially consistent with the results of Jacquinot and Leloup (1971).

Analysis

When analysing the condition (20) of marginal whistler-mode stability, we assume for simplicity that the plasma temperature is so low that the thermal and relativistic corrections to N^2 can be neglected, so that we can write:

$$N^2 = 1 + \frac{\nu Y^2}{Y-1} \tag{24}$$

where: $\nu = \Pi_0^2/\Omega_0^2$.

Also, Eq. (24) could be obtained if we assume that plasma consists of two species of electrons: hot, which are responsible for wave amplification or damping; and cold which are responsible for wave propagation. This assumption has been widely used for the analysis of whistler-mode waves in the magnetosphere (see e.g. Etcheto *et al.*, 1973).

The inclusion of thermal and relativistic corrections to N^2 typical for magnetospheric conditions ($T_e \lesssim 1$ keV) would slightly change the value of ΔA_e defined by (20), but not the conclusions of the paper. These corrections

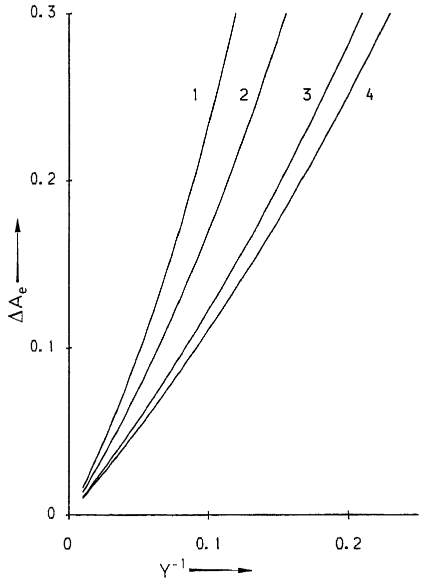

Fig. 1. The plots of ΔA_e, corresponding to marginal whistler-mode instability, versus $Y^{-1} = \omega/\Omega_0$ for $\nu = \Pi_0^2/\Omega_0^2 = 0.5$ (curve *1*), $\nu = 1$ (curve *2*), $\nu = 5$ (curve *3*), and $\nu = 100$ (curve *4*). The region of wave instability is to the left of the corresponding curves

have been considered in detail by Sazhin (1987a, b, 1989) and Sazhin and Temme (1990).

In Fig. 1 we show the plots ΔA_e versus Y^{-1} based on Eq. (19) with N^2 determined by Eq. (24). As one can see from this figure, the value of ΔA_e dramatically increases with decreasing electron plasma density. In particular, the decrease of ν from 100 to 0.5 results in almost a doubling of ΔA_e for a given frequency. This means that in a rarefied plasma ($\nu = 0.5$) the threshold value of ΔA_e at which an instability can develop at a given frequency is about twice as large, when compared with that predicted by the condition (1), valid for infinitely large values of N^2. In other words, in a rarefied plasma ($\nu = 0.5$) with a given value of ΔA_e, the instability develops below about half the frequency predicted by (1). As soon as the range of ν from 0.5 or less to about 100 or more becomes applicable under magnetospheric conditions (see Curtis, 1978; Tsurutani *et al.*, 1982), the dependence of ΔA_e on ν, as shown in Fig. 1, cannot be neglected when interpreting whistler-mode wave observations in the magnetosphere and, in particular, chorus emissions observed in the outer magnetosphere where $\nu < 1$ (see Anderson and Maeda, 1977; Curtis, 1978). Nonrelativistic analysis of this phenomenon similar to that of Curtis (1978) predicts a considerably wider frequency band of these emissions, when compared with the results which would follow from weakly relativistic analysis (cf. curves 1 and 4 in Fig. 1).

Note that although the curves 1, 2 and 3 in Fig. 1 follow from relativistic consideration of the problem of marginal stability of the waves, they do not depend on electron energy. Hence, they remain valid for arbitrarily low energy plasma, although in this case the increment of whistler-mode instability or decrement of their damping is very small as well. In other words, the effect of relativistic stabilization of whistler-mode instability considered, for example by Jacquinot and Leloup (1971) and Sazhin and Temme (1990), turns into its stabilization due to finite electron density. To our knowledge this effect has not been noticed before.

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