Uniform Asymptotic Expansions of Integrals

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The purpose of the paper is to give an account of several aspects of uniform asymptotic expansions of integrals. We give examples of standard forms, the role of critical points and methods to construct the expansions.

1. Introduction

Asymptotic expansions of integrals is an important topic of classical analysis. Many results are available for the well-known higher transcendental functions of mathematical physics and probability theory, and for integrals occurring as solutions of physical problems.

Here we are concerned with uniform expansions of integrals of the type

$$I(z) = \int_C f(t) e^{-z\phi(t,\alpha)} dt \tag{1.1}$$

where C is a contour in the complex t-plane and z is a large parameter. Note that the value of $I(z)$ depends on the parameter α. We suppose that for certain values of α the asymptotic behaviour of $I(z)$ will change.

For obtaining uniform expansions the following major steps can be distinguished:

- (i) trace the points on C or near C that significantly contribute to $I(z)$;
- (ii) transform the integral into a standard form;
- (iii) construct a formal uniform expansion;
- (iv) investigate the asymptotic properties of the expansion;
- (v) construct error bounds;
- (vi) extend the results to wider domains of the parameters.

The first three are most frequently the only possibilities to investigate in practical problems. In applications this formal approach is usually accepted. Often the contributions in the expansion have a physical meaning and then just the form of the expansion is the ultimate requirement. In a systematic study of uniform asymptotic expansions the remaining steps should be incorporated. Also, in numerical applications efficient error bounds are particularly important and in this area point (v) cannot be forgotten.

The above points are not the only problems to be investigated. Several problems arising in physics (for instance in optics and in scattering theory) yield integrals which are generalizations of Airy-type integrals. Then the approximants are higher transcendental functions which fall outside the classical ones. The computational problems for these generalizations are not easy to solve.

In this paper we discuss several aspects of the steps enumerated above. We give definitions of asymptotic expansions, we consider critical points and various methods and techniques to construct the coefficients and, for some cases, error bounds. Several unsolved problems are mentioned.

A standard reference work for asymptotic expansions is OLVER [11], also for special functions; see also OLVER [12] for uniform expansions for special functions. WONG [27] gives a survey with recent results on error bounds for asymptotic expansions of integrals.

This paper is concerned with the classical aspects of asymptotic analysis. Recently new investigations of integrals have been initiated by MASLOV and HÖRMANDER, see DUISTERMAAT [6]. Uniformity problems are cast into the theory of unfoldings of singularities. This approach falls outside the scope of the present publication. An introduction to Maslov's work can be found in POSTON, STEWART [13].

2. DEFINITIONS OF ASYMPTOTIC EXPANSIONS

We use the terminology of generalized asymptotic expansions. First we introduce the concept of asymptotic scale:

a sequence of functions $\{\phi_n(x)\}$ is called an *asymptotic sequence* or *scale* when $\phi_{n+1}(x)=o[\phi_n(x)]$ as $x\to\infty$.

Then we have the definition:

the formal series $\Sigma_{n=0}^{\infty} f_n(x)$ is said to be an *asymptotic expansion* of $f(x)$ *with respect to the scale* $\{\phi_n\}$ if

$$f(x)-\sum_{n=0}^{N} f_n(x) = o[\phi_N(x)] \quad as \quad x\to\infty, \quad N=0,1,\ldots; \tag{2.1}$$

in this case we write

$$f(x) \sim \sum_{n=0}^{\infty} f_n(x); \quad \{\phi_n(x)\} \quad as \quad x\to\infty.$$

In uniform expansions it is required that the 'o' sign holds uniformly (with respect to $\alpha\in A$, say). This general set up is extensively described in ERDÉLYI, WYMAN [7].

When $f_n=\phi_n$ we have a Poincaré type asymptotic expansion; when $f_n=\phi_n=x^{-n}$ we obtain the definition of Poincaré and Stieltjes, who both introduced the definition of this kind in 1886.

Observe that in (2.1) no requirements are put on $\{f_n\}$: it need not be an asymptotic scale. Rather useless expansions may arise (from an asymptotical point of view) when it is not. Also, we can take the scale too rough to measure the error in (2.1).

EXAMPLE 2.1. Take $f_n(x)=(x+n)^{-2}$, and $\phi_n(x)=\log^{-n}x$, $x>1$, $n=0,1,2,\ldots$. Then we have

$$\sum_{n=N+1}^{\infty}(x+n)^{-2} = \mathcal{O}(x^{-1})=o[\phi_m(x)] \quad as \quad x\to\infty$$

for all N,m. So we can write

$$f(x) \sim \sum_{n=0}^{\infty}(x+n)^{-2}; \ \{\log^{-n}x\} \quad as \quad x\to\infty$$

where for f we can take the convergent sum, which represents $d^2 ln\Gamma(x)/dx^2$ (Γ is the Euler gamma function).

Some expansions are provided with a 'thin' scale in which successive terms become more and more indistinguishable. The following example is in WIMP [24], a survey on uniform scale functions and asymptotic expansion of integrals.

EXAMPLE 2.2. The coefficients a_n of the expansion $\Gamma(1-t)=\Sigma_{n=0}^{\infty}a_n t^n$, $|t|<1$, satisfy the expansion

$$a_n \sim \sum_{k=0}^{\infty}\frac{(-1)^k}{k!(k+1)^{n+1}}\ ; \ \{\phi_k(n)\} \quad as \quad n\to\infty,$$

where $\phi_k(n)=(k+1)^{-n}$. The series converges rather fast. However, the scale satisfies $\phi_{k-1}(n)/\phi_k(n)= (1+1/k)^{-n}$, which indeed is $o(1)$ as $n\to\infty$, $k\geqslant 1$, k fixed. But as k increases this ratio tends to unity (n fixed).

For some functions we need a *compound asymptotic expansion.* That is we have a decomposition

$$\begin{aligned} f(x) &= A_1(x)f_1(x)+ \ldots +A_k(x)f_k(x) \\ f_k(x) &\sim \sum_{j=0}^{\infty}f_{jk}(x); \quad \{\phi_{jk}\} \quad as \quad x\to\infty, \end{aligned} \tag{2.2}$$

where, for each k, $\{\phi_{jk}(x)\}$ is an asymptotic scale. In complicated problems the f_k are not known a priori.

It may be rather difficult to investigate whether an expansion is uniform with respect to a parameter α. A non-uniformity may be recognized when in (2.2) $A_k(x)$, $f_{jk}(x)$ or $\phi_{jk}(x)$ are singular at certain values of the uniformity parameter α, whereas $f(x)$ remains regular for these values.

EXAMPLE 2.3. Consider the exponential polynomial in the form

$$e_n(x) = e^{-x}\sum_{s=0}^{n}\frac{x^s}{s!}, \quad x>0, \quad n=0,1,2,\ldots .$$

We have $\lim_{x\to\infty}e_n(x)=0$, $\lim_{n\to\infty}e_n(x)=1$; so the first limit cannot be uniformly valid when n grows with x. Asymptotic expansions for large n, which are uniform with respect to unrestricted real values of x can be given in terms of error functions. Any approximation in terms of elementary functions breaks down when x passes the value n, which is not a singularity for $e_n(x)$. The function $e_n(x)$ is related to the incomplete gamma functions and to the Poisson distribution. For more information we refer to WONG [25], TRICOMI [21], and [19].

3. CRITICAL POINTS

There is a systematic approach to obtain the asymptotic expansion of (1.1). We have to look for certain distinguished points whose immediate neighbourhoods determine completely the asymptotic behaviour of the integral. Such points are called *critical points* by VAN DER CORPUT [5]. Possible candidates are:
- the end points of the contour;
- singular points of the integrand;
- stationary or saddle points of ϕ (i.e., where $\partial\phi/\partial z$ vanishes).

The contribution of a single critical point to the asymptotic value of $I(z)$ is known for a great variety of critical points. We mention some key words in this respect: Watson's lemma, the method of Laplace, the method of steepest descent, the method of saddle points, the principle of stationary phase and the method of Darboux. We give a formulation of one of the most important tools.

LEMMA (WATSON). *Consider the Laplace integral*

$$I(z) = \int_0^\infty e^{-zt} f(t) dt. \tag{3.1}$$

Assume that
(i) *f is locally integrable on* $[0,\infty)$*;*
(ii) $f(t) \sim \Sigma_{s=0}^{\infty} a_s t^{(s+\lambda-\mu)/\mu}$ *as* $t \to 0^+$, μ, λ *fixed,* $\mu>0$, $\mathrm{Re}\lambda>0$*;*
(iii) *the abscissa of convergence of (3.1) is not* $+\infty$.

Then,

$$I(z) \sim \sum_{s=0}^{\infty} \Gamma(\frac{s+\lambda}{\mu}) a_s z^{-(s+\lambda)/\mu} \tag{3.2}$$

as $z \to \infty$ *in the sector* $|\arg z| \leq \frac{1}{2}\pi - \delta (< \frac{1}{2}\pi)$ *where* $z^{(s+\lambda)/\mu}$ *has its principal value.*

PROOF. See OLVER [11, p. 113]. □

Oberve that (3.2) is obtained by substituting (ii) into (3.1) and by interchanging the order of summation and integration. In (ii) λ and μ are fixed. When

$\lambda = \mathcal{O}(z)$ (or larger) the expansion (3.2) has no meaning. A modification of the lemma is needed then to give a uniform expansion, see [20].

When in (1.1) α ranges over a domain A the critical points may be variable. For certain values in A two or more critical points may coalesce. Usually, the form of the expansion changes and it is unlikely that the sum of the contributions of each critical point will be uniformly valid. For instance, coefficients of the several expansions may become singular when α takes these distinguished values.

The systematic approach of van der Corput to add several contributions from the critical points was an important step to take away part of the mystery of asymptotics. In uniform problems it is also important to systematize. We can single out the following possibilities for (1.1):

- singularity coincides with stationary point;
- end-point of contour coincides with stationary point;
- two stationary points coincide.

In VAN DER WAERDEN [23], CHESTER, FRIEDMAN, URSELL [4] and BLEISTEIN [1] important contributions are given for these cases.

By introducing several auxiliary parameters much more situations can occur. Some of them correspond with important physical applications or with problems for the well-known special functions of mathematical physics. A survey is given by OLVER [12].

The approximants in uniform expansions are usually more complicated than the elementary functions used in earlier days. Now we use error functions, Airy functions, Bessel functions, parabolic cylinder functions, etc. The computational problem has been solved for most of these functions, and now they are accepted as approximants.

In classifying relevant cases of coalescing critical points it is instructive to look at approaches via the WKB or Liouville-Green methods for differential equations. Most functions from mathematical physics can be investigated in both directions: they have an integral representation and they satisfy a differential equation. See again [12] for more details on this point.

4. EXAMPLES OF STANDARD FORMS

In the table we give standard forms of integrals for which well-known special functions are used as approximants. We give the critical points, the coalescence of which causes uniformity problems, and references to the literature.

	Standard form	Approximant	Critical points	References
(4.1)	$\int_{-\infty}^{\infty} e^{-zt^2} \frac{f(t)}{t-i\alpha} dt$	Error function	$t=0,\ t=i\alpha$	[23]
(4.2)	$\int_{-\infty}^{\infty} e^{-zt^2} \frac{f(t)}{(t-i\alpha)^{\mu}} dt$	Parabolic cylinder function	$t=0,\ t=i\alpha$	[1]
(4.3)	$\int_{-\infty}^{\alpha} e^{-zt^2} f(t)dt$	Error function	$t=0,\ t=\alpha$	[19]
(4.4)	$\int_{0}^{\infty} t^{\beta-1} e^{-z(\frac{1}{2}t^2-\alpha t)} f(t)dt$	Parabolic cylinder function	$t=0,\ t=\alpha$	[1], [8] [14], [25]
(4.5)	$\int_{c} e^{z(\frac{1}{3}t^3-\alpha t)} f(t)dt$	Airy function	$t=\pm\sqrt{\alpha}$	[4], [10]
(4.6)	$\int_{0}^{\infty} t^{\alpha-1} e^{-zt} f(t)dt$	Gamma function	$t=0,\ t=\alpha/z$	[20]
(4.7)	$\int_{\alpha}^{\infty} t^{\beta-1} e^{-zt} f(t)dt$	Incomplete gamma function	$t=0,\ t=\alpha$	[9], [15] [17], [28]
(4.8)	$\int_{0}^{\infty} t^{\beta-1} e^{-z(t+\alpha/t)} f(t)dt$	Bessel function	$t=0,\ t=\pm\sqrt{\alpha}$	[18]
(4.9)	$\int_{\alpha}^{\infty} f(\sqrt{t^2-\alpha^2})\sin zt\,dt$	Bessel function	$t=\pm\alpha$	[22], [26]
(4.10)	$\int_{0}^{\infty} \frac{\sin z(t-\alpha)}{t-\alpha} f(t)dt$	Sine integral	$t=0,\ t=\alpha$	[29]

REMARKS

1. Functions f are supposed to be regular in neighbourhoods of the critical points.
2. The integrals reduce to their approximants when $f=1$, except in (4.9) where it occurs for $f(t)=t^{\beta}$.
3. Quite different integrals may have the same approximants.
4. Different intervals of integration are investigated too.
5. In (4.5), (4.8) two saddle points coalesce with each other when $\alpha=0$; both cases are different, however. In (4.8) we have an additional critical point at $t=0$ (end point and singularity).

6. In all cases elementary approximants can be used for fixed values of the uniformity parameter α.
7. Several of the examples need further investigations with respect to the construction of error bounds and the determination of maximal regions of validity.

5. Transformation to standard forms

Once the critical points are located and the asymptotic phenomena are recognized, a next step may be a transformation to one of the standard forms. To obtain an optimal representation, such a transformation may be rather complicated. As a consequence, it may cause serious problems for the construction of error bounds and for the computation of the coefficients. In this section we consider two examples. The first one (on incomplete gamma functions) is relatively simple; the second one is more difficult to investigate due to the role of the uniformity parameter.

5.1 Incomplete gamma functions

These are defined by

$$P(a,x) = \frac{1}{\Gamma(a)}\int_0^x t^{a-1}e^{-t}dt, \quad Q(a,x) = \frac{1}{\Gamma(a)}\int_x^\infty t^{a-1}e^{-t}dt. \qquad (5.1)$$

We consider positive values of x and a. The function $e_n(x)$ of Example 2.3 is a special case: $e_n(x) = Q(n+1,x)$. We are interested in the asymptotic expansion which is valid for $a\to\infty$ and $x\in[0,\infty)$ (uniformly). The function $t^a e^{-t}$ attains its maximal value at $t=a$. When x and a are nearly equal this point is close to the end points of the intervals of integration in (5.1). Hence, we recognize (4.3). We rewrite $P(a,x)$ in the form

$$P(a,x) = \frac{e^{-a}a^a}{\Gamma(a)}\int_0^{x/a} e^{-a(t-1-\ln t)}dt.$$

The transformation into the standard form is defined by the mapping $\zeta:\mathbb{R}^+\to\mathbb{R}$, given by

$$\tfrac{1}{2}\zeta^2(t) = t-1-\ln t, \quad \operatorname{sign}\zeta(t) = \operatorname{sign}(t-1). \qquad (5.2)$$

The result is

$$P(a,x) = \frac{e^{-a}a^a}{\Gamma(a)}\int_{-\infty}^{\eta} e^{-\frac{1}{2}a\zeta^2} f(\zeta)d\zeta,$$

$$(5.3)$$

$$Q(a,x) = \frac{e^{-a}a^a}{\Gamma(a)}\int_{\eta}^{\infty} e^{-\frac{1}{2}a\zeta^2} f(\zeta)d\zeta,$$

where $\eta = \zeta(\lambda)$, $\lambda = x/a$, $f(\zeta) = t^{-1}dt/d\zeta = \zeta/(t-1)$, $f(0) = 1$.

By tracing the (complex) singularities of the mapping in (5.2) we can infer that f is analytic in the strip $|\mathrm{Im}\zeta|<\sqrt{2\pi}$. To give the main steps in this analysis we observe that ζ is analytic at $t=1$, but not at $t_n=\exp(2\pi in)$, $n=\pm1, \pm2,\ldots$. (To obtain a sufficiently large ζ-domain we have to consider more than the principal sheet of the Riemann surface of the logarithm in (5.2)). Corresponding ζ-points follow from (5.2): $\frac{1}{2}\zeta_n^2=-2\pi in$; the points with $n=\pm1$ are nearest to the real line, $|\mathrm{Im}\zeta_{\pm1}|=\sqrt{2\pi}$.

This information is useful for estimating coefficients and remainders in the asymptotic expansions of (5.3). A first approximation to the functions in (5.3) is obtained by replacing $f(t)$ by $f(0)=1$. Then the integrals can be written in terms of the normal probability functions or error functions. In [19] the complete expansion is given, which is uniformly valid with respect to $\eta\in\mathbb{R}$, (or $x\in[0,\infty)$).

5.2 Anger function of large order

A second example is from OLVER [11, p. 352]. The integral is

$$A_{-\nu}(a\nu) = \int_0^\infty e^{-\nu(a\sinh t-t)}dt, \quad a>0, \quad \nu>0. \tag{5.4}$$

$A_\nu(z)$ is a so-called Anger function, which is related to Bessel functions; ν is the large parameter, a is restricted to $(0,1]$, where $a=1$ is a critical value. Write $a=1/\cosh\alpha$. Saddle points in (5.4) are zeros of $d[a\sinh t-t] = \cosh t/\cosh\alpha-1$. When $a\in(0,1)$ two real saddle points are $\pm\alpha$, which coalesce with each other when $a\to1$.

A transformation to the standard form (4.5) is obtained by using the mapping $\zeta:\mathbb{R}\to\mathbb{R}$ that is defined by

$$\sinh t/\cosh\alpha-t = \tfrac{1}{3}\zeta^3-\eta\zeta. \tag{5.5}$$

To make $\zeta(t)$ regular at $t=\pm\alpha$ the only possible choice for η is

$$\tfrac{2}{3}\eta^{\frac{3}{2}} = \alpha-\tanh\alpha. \tag{5.6}$$

So we obtain

$$A_{-\nu}(a\nu) = \int_0^\infty e^{-\nu(\frac{1}{3}\zeta^3-\eta\zeta)}f(\zeta)d\zeta, \tag{5.7}$$

where $f(\zeta)=dt/d\zeta$. The singularities of f arise from complex singular points of $\zeta(t)$. These arise from complex solutions of the equation $\cosh t=\cosh\alpha$, i.e., $t_k^{\pm}=\pm\alpha+2k\pi i$, $k=\pm1,\pm2,\ldots$. Corresponding $\zeta_k^{\pm}$ values follow from (5.5). For small values of a (i.e., large α and η) we have $\zeta_k^{\pm}\sim\sqrt{\eta}+\eta^{-1/4}\sqrt{2\pi ik}$. So, the singularities of $f(\zeta)$ are rather close to the saddle point $\zeta=\sqrt{\eta}$, when η is large.

As OLVER [11] shows this distance is not too small for obtaining an expansion for (5.7) that is uniformly valid with respect to $a \in [0,1]$, or $\eta \in [0,\infty)$. In Section 6 some more details about the expansion will be given.

REMARK. In both examples (5.3) and (5.7) the asymptotic nature of the expansion follows from the singularities of $f(\zeta)$, where f is considered as an analytic function of the complex variable ζ. This approach is natural for the special functions considered here. In a more general approach, where it may be assumed that f belongs to a function class C^k, the method of proof is quite different, of course.

6. THE CONSTRUCTION OF THE FORMAL EXPANSION

Several methods are available to obtain various kinds of asymptotic expansions. Roughly speaking we have the following three possibilities:

1. Expansions at the critical points;
2. Integration by parts;
3. Residue methods.

The third method is well known in the theory of Laplace and Mellin transformations for obtaining back transforms; see BLEISTEIN, HANDELSMAN [3] for a lot of information on the use of Mellin transforms in asymptotics. It will not be considered here. A new method based on a combination of 1. and 2. is discussed in Section 7.

The expansion at $t=0^+$ in Watson's lemma (Section 3) is an example of 1. To obtain uniform expansions integration by parts should not be done in a straightforward way. We now demonstrate a method of BLEISTEIN [1] that is very useful in various types of integrals.

6.1.

We consider (4.8) and we write $\mu^2=\alpha$. Saddle points occur at $t=\pm\mu$, where μ is supposed to be positive. The first step is the representation

$$f(t) = a_0+b_0t+(t-\mu^2/t)g(t) \tag{6.1}$$

where a_0,b_0 follow from substitution of $t=\pm\mu$. We have

$$a_0 = \frac{1}{2}[f(\mu)+f(-\mu)],\quad b_0=\frac{1}{2\mu}[f(\mu)-f(-\mu)].$$

Denoting (4.8) by $I(z)$ we obtain upon inserting (6.1) into (4.8)

$$I(z) = a_0\Phi_0+b_0\Phi_1+I_1(z) \tag{6.2}$$

where Φ_0, Φ_1 are modified Bessel functions

$$\Phi_j = 2(\mu/\sqrt{z})^{\beta+j}K_{\beta+j}(2\mu\sqrt{z}),\quad j=0,1.$$

An integration by parts gives

$$I_1(z) = \int_0^\infty t^{\beta-1}e^{-z(t+\mu^2/t)}(t-\mu^2/t)g(t)dt$$

$$= -\frac{1}{z}\int_0^\infty t^\beta g(t)de^{-z(t+\mu^2/t)} = \frac{1}{z}\int_0^\infty t^{\beta-1}e^{-z(t+\mu^2/t)}f_1(t)dt,$$

with $f_1(t)=t^{1-\beta}\frac{d}{dt}[t^\beta g(t)]= \beta g(t)+tg'(t)$. We see that $zI_1(z)$ is of the same form as $I(z)$. The above procedure can now be applied to $zI_1(z)$ and we obtain for

$$I(z) = \int_0^\infty t^{\beta-1}e^{-z(t+\mu^2/t)}f(t)dt, \tag{6.3}$$

the formal expansion

$$I(z) \sim \Phi_0\sum_{s=0}^\infty \frac{a_s}{z^s}+\Phi_1\sum_{s=0}^\infty \frac{b_s}{z^s}, \quad \text{as} \quad z\to\infty, \tag{6.4}$$

where we define inductively $f_0(t)=f(t)$, $g_0(t)=g(t)$ and for $s=1,2,...,$

$$f_s(t) = t^{1-\beta}\frac{d}{dt}[t^\beta g_{s-1}(t)]=a_s+b_st+(t-\mu^2/t)g_s(t),$$

$$a_s = \frac{1}{2}[f_s(\mu)+f_s(-\mu)], \quad b_s=\frac{1}{2\mu}[f_s(\mu)-f_s(-\mu)].$$

6.2.

Next we show that it is rather easy to obtain an expansion in which β acts as a second uniformity parameter. Then we exploit fully the fact that the Bessel functions in Φ_j are functions of two variables. The form of the new expansion is exactly as in (6.4), with the same Φ_j, but with different coefficients.

We write $\beta=2\nu z$, $\nu\in\mathbb{R}$. The saddle points $t_\pm$ are now zeros of $d[t+\mu^2/t-2\nu\ln t]/dt$, which gives $t_\pm=\nu\pm(\nu^2+\mu^2)^{1/2}$. The modification of (6.1) is

$$f(t) = c_0+d_0t+(t-2\nu-\mu^2/t)h_0(t)$$

and we obtain for (6.3) the formal expansion

$$J(z) \sim \Phi_0\sum_{s=0}^\infty \frac{c_s}{z^s}+\Phi_1\sum_{s=0}^\infty \frac{d_s}{z^s}, \quad \text{as} \quad z\to\infty. \tag{6.5}$$

Now the coefficients follow from

$$\tilde{f}_0(t) = f(t), \quad \tilde{f}_s(t)=t\frac{d}{dt}h_{s-1}(t)=c_s+d_st+(t-2\nu-\mu^2/t)h_s(t),$$

$$c_s = \frac{t_+\tilde{f}_s(t_-)-t_-\tilde{f}_s(t_+)}{t_+-t_-}, \quad d_s=\frac{\tilde{f}_s(t_+)-\tilde{f}_s(t_-)}{t_+-t_-}.$$

6.3.

When f of (6.3) is analytic, say in the strip $|\mathrm{Im}\,t|<a$, then each f_s, $\tilde{f}_s$ and hence all coefficients are well defined for all $\nu\in\mathbb{R}$, $\mu\geq 0$. Extension to complex values of ν, μ and z is possible when more is known about f.

The expansions (6.4), (6.5) might be applied to the confluent hypergeometric function

$$U(a,b,z) = \frac{1}{\Gamma(a)}\int_0^\infty t^{a-1}(1+t)^{b-a-1}e^{-zt}dt, \tag{6.6}$$

where $\mathrm{Re}\,a>0$, $b\in\mathbb{C}$, $\mathrm{Re}\,z>0$.

In [18] (6.6) was transformed into (6.3) and an expansion was obtained by expanding f at the critical point $t=0$. The range of the parameters was rather limited but we obtained a manageable error bound, which was very useful in a numerical algorithm for (6.6).

Using the procedure for (6.5) we expect to be able to construct for (6.6) an expansion for $a\to\infty$, that is uniformly valid with respect to $z\in[0,\infty)$, $b\in(-\infty,\nu a)$, $\nu<1$, ν fixed. Further research is needed, however, to transform (6.6) into (6.3), to investigate the asymptotic nature of (6.4), and to construct error bounds.

6.4.

In OLVER [11] a uniform expansion of (5.7) is obtained by expanding $f(\zeta)$ at the critical point $\zeta=\sqrt{\eta}$. By writing $f(\zeta)=\Sigma q_s(\zeta-\sqrt{\eta})^s$, the expansion

$$A_{-\nu}(\nu a) \sim \sum_{s=0}^{\infty} q_s\frac{\pi Qi_s(\nu^{\frac{2}{3}}\eta)}{\nu^{(s+1)/3}}, \quad \nu\to\infty \tag{6.7}$$

follows. It is shown to be uniform with respect to $a\in[0,1]$, or $\eta\in[0,\infty)$ (see (5.6)). Here $Qi_s(y)$ is related to Airy functions,

$$Qi_s(y) = \frac{1}{\pi}\int_0^\infty e^{-\frac{1}{3}t^3+yt}(t-\sqrt{y})^s dt, \quad s=0,1,....$$

An integration by parts procedure for (5.7) is used by WONG [27]; the result is supplied with an error bound.

6.5.

The standard form (4.6) is investigated in [20] in both directions: integration by parts and expansion of f at the critical point $t=\alpha/z$. The asymptotic nature of the expansions is discussed and error bounds are given. The integration by parts procedure gives for

$$F_\lambda(z) = \frac{1}{\Gamma(\lambda)}\int_0^\infty t^{\lambda-1}e^{-zt}f(t)dt \tag{6.8}$$

the expansion

$$F_\lambda(z) \sim z^{-\lambda}\sum_{s=0}^{\infty} f_s(\mu)z^{-s}, \quad z\to\infty \tag{6.9}$$

where

$$f_0(t) = f(t), \; f_{s+1}(t)=t\frac{d}{dt}\frac{f_s(t)-f_s(\mu)}{t-\mu}, \quad s=0,1,2,... \tag{6.10}$$

with $\mu=\lambda/z$. In [20] it is shown that $\{f_s(\mu)z^{-s}\}$ is an asymptotic scale and that (6.9) is a Poincaré type expansion that is uniform with respect to $\mu\in[0,\infty)$. The main condition on f is that its singularities are not too close to $t=\mu$: let R_μ denote the radius of convergence of the Taylor expansion of f at $t=\mu$, then we require $R_\mu^{-1}=\mathcal{O}[(1+\mu)^{-\kappa}]$, $\mu\geqslant 0$ ($\kappa\geqslant\frac{1}{2}$, κ fixed).

7. A NEW CLASS OF POLYNOMIALS

In the previous sections three different types of expansions of the integrand function f are used for obtaining an asymptotic expansion:

(i) in (3.1) an expansion at the fixed critical point $t=0$;
(ii) for (6.7) an expansion at the movable saddle point $\zeta=\sqrt{\eta}$;
(iii) (6.4) and (6.5) are expansions that in fact are based on a two-point interpolation process for f.

The computation of the coefficients in the asymptotic expansion and the construction of error bounds becomes progressively more difficult in the above cases. Especially this is true when f is defined in terms of implicitly defined relations due to transformations to standard forms. When f is analytic in a neighbourhood of the critical points, in the first two cases representations of the coefficients are available in terms of Cauchy integrals. In general, such a representation is missing in the third case.

Therefore, a new approach to construct the coefficients of a uniform expansion is worth to mention. In this section we describe a recent method of SONI and SLEEMAN [16], where a set of polynomials is introduced to expand the function f. An interesting by-product of the method is a Cauchy-type integral for the coefficients that generalizes the representation for the Taylor expansion. We return to (6.8) to demonstrate the method (in [16] it is given for (4.4), but it has much wider applications).

Consider the formal expansion

$$g(t) = \sum_{s=0}^{\infty}\alpha_s P_s(t) \tag{7.1}$$

where g is defined by $f(t)=f(\mu)+(t-\mu)g(t)$ and where it is assumed that $\{P_s\}$ satisfies the following conditions:

(i) $P_0(t)=1, \; P_1(t)=t.$
(ii) $P_s(0)=0, \; s=1,2,....$
(iii) $tP'_s(t)=(t-\mu)P_{s-1}(t), \; s=2,3,....$

Then there is a unique polynomial solution $\{P_s\}$ satisfying the above three conditions; the coefficients $\{\alpha_s\}$ in (7.1) can be computed recursively and they appear as coefficients in the expansion (6.9).

SKETCH OF THE PROOF. From the recursion (iii) it follows that the first polynomials are

$$P_0(1) = 1,\ P_1(t)=t,\ P_2(t)=\frac{1}{2}t^2-\mu t,\ P_3(t)=\frac{1}{6}t^3-\frac{3}{4}\mu t^2+\mu^2 t.$$

The remaining P_s follow from (ii) and (iii) by writing

$$P_s(t) = \int_0^t (\tau-\mu)\tau^{-1}P_{s-1}(\tau)d\tau.$$

From (ii) we infer $\alpha_0 = g(0)$. Formal differentiation of (7.1) gives (with (iii))

$$tg'(t) = \alpha_1 t+\alpha_2(t-\mu)P_1(t)+\alpha_3(t-\mu)P_2(t)+...,$$

from which we obtain $\alpha_1 = g'(\mu)$. Next we write

$$g_1(t) := t\frac{g'(t)-g'(\mu)}{t-\mu} = \alpha_2 t+\alpha_3 P_2(t)+\alpha_4 P_3(t)+....$$

Applying again the operator $t\frac{d}{dt}$ we get $\alpha_2 = g'_1(\mu)$. In this way all coefficients α_s can be computed. To show that they turn up in (6.9) we insert (7.1) into (6.8) and we obtain (the term $s=0$ gives no contribution)

$$F_\lambda(z) \sim z^{-\lambda}f(\mu)+\sum_{s=1}^{\infty}\alpha_s\psi_s \tag{7.2}$$

$$\psi_s = \frac{1}{\Gamma(\lambda)}\int_0^\infty t^{\lambda-1}e^{-zt}(t-\mu)P_s(t)dt = \frac{1}{z}\psi_{s-1} \quad (s=2,3,...),$$

where we used (iii). Hence it follows that $\psi_s = \lambda/z^{\lambda+s+1}$, and that (7.2) can be written as

$$F_\lambda(z) \sim z^{-\lambda}[f(\mu)+\alpha_1\mu/z+\alpha_2\mu/z^2+...].$$

This gives the relation (using the unicity property of asymptotic expansions) $\alpha_s = f_s(\mu)/\mu$, $s=1,2,....$ □

The above method generalizes Watson's lemma: the expansion at $t=0$ in (3.1) is now replaced by the expansion (7.1). The polynomials P_s reduce to $t^s/s!$ when $\mu\to 0$. Hence in that event (7.1) is the Maclaurin expansion of $g(t)$, when g is analytic. It also may give a new approach for obtaining error bounds; some ideas are worked out in [16]. Furthermore, it gives an explicit representation of $f_s(\mu)$ or α_s. This result is not in [16].

First we compute the coefficients $Q_s(z)$ in the expansion

$$\frac{1}{z-t} = \sum_{s=0}^{\infty} Q_s(z)P_s(t). \tag{7.3}$$

Using (i), (ii), (iii) above we obtain

$$Q_0(z) = 1/z, \quad Q_{s+1}(z) = -\frac{d}{dz}[zQ_s(z)/(z-\mu)].$$

For analytic functions g we have

$$g(t) = \frac{1}{2\pi i}\int \frac{g(z)}{z-t}dz = \frac{1}{2\pi i}\int g(z)\sum_{s=0}^{\infty} Q_s(z)P_s(t)dz.$$

Hence, formally, we obtain for α_s of (7.1)

$$\alpha_s = \frac{1}{2\pi i}\int g(z)Q_s(z)dz = \frac{1}{2\pi i}\int \frac{f(z)-f(\mu)}{z-\mu}Q_s(z)dz.$$

Since $f(\mu)$ gives no contribution we arrive at

$$f_s(\mu) = \frac{\mu}{2\pi i}\int \frac{f(z)Q_s(z)}{z-\mu}dz, \quad s=1,2,.... \tag{7.4}$$

The contour encircles $z=\mu$ in positive direction and no singularities of f.

EXAMPLE. Take $f(z)=1/(z+1)$. The residue at $z=-1$ gives at once

$$f_s(\mu) = \frac{\mu}{\mu+1}Q_s(-1), \quad s=1,2,....$$

8. Some remarks on error bounds

Special functions of mathematical physics are frequently treated as examples to demonstrate the methods of asymptotical analysis. Functions of hypergeometric type satisfy a differential equation and they have integral representations. Error bounds for the remainders in the expansions of special functions are derived most frequently from a differential equation. In Olver's work, see [11], general methods are derived to obtain strict and realistic error bounds. For a survey on error bounds for expansions of integrals we refer to WONG [27], where also a chapter on uniform expansions is included. Wong's conclusion is that the error theory for uniform expansion is still in its infancy. We agree with him that it is important to develop the theory. In many applications there is no choice between integrals and differential equations.

In a recent paper URSELL [22] demonstrates how the maximum-modulus theorem for analytic functions can be used to bound the error term. In [20] error bounds are given for (6.9). We now review the method of that paper.

The remainder in (6.9) is defined by

$$F_\lambda(z) = z^{-\lambda}[\sum_{s=0}^{n-1} f_s(\mu)z^{-s} + z^{-n}E_n(z,\lambda)], \tag{8.1}$$

$$E_n(z,\lambda) = \frac{z^\lambda}{\Gamma(\lambda)}\int_0^\infty t^{\lambda-1}e^{-zt}f_n(t)dt, \tag{8.2}$$

where f_n is the iterated function given in (6.10). From the conditions on f it follows that

$$|f_n(t)| \leq M(\mu)(1+t)^{p-n}, \quad t \geq 0, \tag{8.3}$$

where $M(\mu)$ is bounded for fixed finite values of $\mu=\lambda/z, \mu\geq 0$; p is a fixed real number. So we obtain for E_n the bound (we consider real positive values of λ, z)

$$|E_n(z,\lambda)| \leq \frac{M(\mu)z^{\lambda}}{\Gamma(\lambda)} \int_0^{\infty} t^{\lambda-1} e^{-zt}(1+t)^{p-n} dt. \tag{8.4}$$

When $p-n \leq 0$ the integral is easily estimated and we obtain $|E_n(z,\lambda)| \leq M(\mu)$; when $p-n>0$ we have to accept the integral in the bound (although it is a well-known special function, see (6.6)). Another point is that (8.3) may be sharp for t-values far from the interesting point $t=\mu$. In that case, the right-hand side of (8.4) may grossly overestimate $|E_n|$. To obtain a more manageable and more realistic bound we define real numbers σ_n such that

$$|f_n(t)| \leq M|f_n(\mu)|[(t/\mu)^{-\mu} e^{t-\mu}]^{\sigma_n}, \quad t>0, \tag{8.5}$$

where M is a fixed constant exceeding unity; $f_n(\mu)$ is supposed to be non-vanishing. Now the estimate is sharp at $t=\mu$ and the bound is expressed in terms of $f_n(\mu)$, which is part of the asymptotic scale. For E_n we obtain

$$|E_n(z,\lambda)| \leq M|f_n(\mu)|R_n, \tag{8.6}$$

$$R_n = (1-\sigma_n/z)^{-\lambda}[(\lambda-\mu\sigma_n)/e]^{\mu\sigma_n}\Gamma(\lambda-\mu\sigma_n)/\Gamma(\lambda).$$

When $z-\sigma_n$ and $\lambda=\mu z$ are large, R_n is close to unity, which follows from the Stirling approximation of the gamma functions. Observe that σ_n does not depend on z, when we consider μ as an independent uniformity parameter. When σ_n is a bounded function of μ on $[0, \infty)$, (8.6) gives a sufficient condition to prove that (6.9) is a uniform expansion with respect to the scale $\{f_s(\mu)z^{-s}\}$ (when $f_s(\mu)$ happens to be zero for some μ, s the scale has to be modified). Of course the bound is useful when σ_n is not too large.

The best value of σ_n in (8.5) is given by

$$\sigma_n = \sup_{t>0} \frac{\ln|f_n(t)/[Mf_n(\mu)]|}{t-\mu-\mu\ln(t/\mu)}.$$

It should be remarked that, in general, it is rather difficult to compute σ_n, especially when f is obtained from a transformation to standard form.

The above method modifies a method of Olver [11] for Laplace integrals of the form (3.1) (non-uniform case).

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