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ON REPRESENTATIONS OF THE SYMMETRIC GROUPS, NILPOTENT MATRICES, SYSTEMS, VECTORBUNDLES AND SCHUBERT CELLS

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ON REPRESENTATIONS OF THE SYMMETRIC GROUPS, NILPOTENT MATRICES, SYSTEMS, VECTORBUNDLES AND SCHUBERT CELLS.

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Introduction. Let κ be a partition of n, $\kappa_1 = (\kappa_1,\dots,\kappa_m)$, $\kappa_1 \geq \dots \geq \kappa_m \geq 0$, $\Sigma\kappa_i = n$. We identify partitions $(\kappa_1,\dots,\kappa_m)$ and $(\kappa_1,\dots,\kappa_m,0,\dots,0)$. One defines a partial order on the set of all partitions as follows

$$(1.1) \quad (\kappa_1,\dots,\kappa_m) > (\kappa_1',\dots,\kappa_m') \Leftrightarrow \sum_{i=1}^{r} \kappa_i \leq \sum_{i=1}^{r} \kappa_i', \; r = 1, \dots, m$$

Thus for example $(2,2,1) > (3,2)$. If $\kappa > \kappa'$ we say that κ specializes to κ' or that κ is more general than κ'. The reverse order has been called the dominance order. It occurs naturally in several seemingly rather unrelated parts of pure and applied mathematics. Some of these occurrences can be labelled by the words and phrases

- (i) Snapper conjecture (on the representations of symmetric groups)
- (ii) Gale Ryser theorem (on existence of (0,1)-matrices)
- (iii) Muirheads inequality (a symmetric mean inequality)
- (iv) Gerstenhaber-Hesselink theorem (on orbit closure properties of SL_n acting on nilpotent matrices)
- (v) Kronecker indices (on the orbit closure, or degeneration, properties of linear control systems acted on by the socalled feedback group)
- (vi) Double stochastic matrices (when is a partition "an average" of another partition)
- (vii) Shatz's theorem (on degeneration of vectorbundles (over the Riemann sphere))

These will be described in more detail in section 2 below.
In addition the same ordening plays a considerable role in theoretical chemistry in the theory of chiral molecules, i.e. molecules that are optically active [11,16,18].

Certain of these manifestations of this specialization order are known to be intimatedly related. Thus (i), (ii), (iii) and (vi) are very much related [2,13] and so are (v) and (vii) [15]. This talk is a report of work done jointly with Clyde Martin of Case Western Reserve Univ, which shows that all these manisfestations of this order are intimately related and that their common meeting ground, so to speak, seems to be the ordering defined by closure relations of the Schubert-Cells (with respect to a standard basis) of a Grassmann manifold. i.e. a Schubert-cell SC(λ) is more general than SC(λ'); in symbols: SC(λ) > SC(λ'), iff $\overline{SC(\lambda)} \supset SC(\lambda')$. This order in turn is much related to the Bernstein-Gelfand-Gelfand ordening on the Weyl group S_n. It is in fact the quotient ordering induced by the canonical map of the manifold of all flags in $\mathbb{R}^{n+m}$ to the Grassmann manifold of n-planes in (n+m)-space. Full details will appear elsewhere [8].

2. SEVERAL MANIFESTATIONS OF THE SPECIALIZATION ORDER

2.1. The Snapper conjecture. Let $\kappa = (\kappa_1,\dots,\kappa_m)$ be a partition of n. Let S_κ be the corresponding Young subgroup $S_\kappa = S_{\kappa_1} \times\dots\times S_{\kappa_m}$, where S_{κ_i} is seen as the subgroup of S_n acting on the letters $\kappa_1+\dots+\kappa_{i-1}+1, \dots, \kappa_1+\dots+\kappa_i$. (If $\kappa_m = 0$ the factor S_{κ_m} is deleted). Take the trivial representation of S_κ and induce this up to S_n. Let $\rho(\kappa)$ denote the resulting representation. It is of dimension $n!/\kappa_1!\dots\kappa_m!$ and it can be easily described as follows. Take m symbols $a_1, \dots, a_m$ and consider all associative (but noncommutative) words $\varepsilon_1 \dots \varepsilon_n$ of length n in the symbols $a_1, \dots, a_m$ such that a_i occurs precisely κ_i times. Let $W(\kappa_1,\dots,\kappa_m) = W(\kappa)$ denote this set. Then S_n acts on $W(\kappa)$ by $\sigma(\varepsilon_1 \dots \varepsilon_n) = \varepsilon_{\sigma(1)}\varepsilon_{\sigma(2)} \dots \varepsilon_{\sigma(n)}$. Let $V(\kappa)$ be the vectorspace with the elements of $W(\kappa)$ as basis vectors. Extending the action of S_n linearly this gives a representation of S_n and this is the representation $\rho(\kappa)$.

Now the irreducible representations of S_n are also labelled by partitions. Let $[\kappa]$ be the irreducible representation belonging to the partition κ. Snapper [21] proved that $[\kappa]$ occurs in $\rho(\kappa')$ only if $\kappa < \kappa'$ and conjectured the reverse implication. Liebler and Vitale [14] proved that $\kappa < \kappa' \Rightarrow \rho(\kappa)$ is a direct summand of $\rho(\kappa')$ which of course implies that $\kappa < \kappa' \Rightarrow [\kappa]$ occurs in $\rho(\kappa')$. Another proof of this implication (via a different generalization) was given by Lam [13].

2.2. The Gale-Ryser theorem ([5,19]). Let μ and ν be two partitions of n. Then there is a matrix consisting of zero's and one's whose columns sum to μ and whose rows sum to ν iff $\nu > \mu^*$. Here μ^* is the dual partition of μ defined by $\mu_i^* = \# \{j \mid \mu_j \geq i\}$. (If S is a set then # S stands for the number of elements in that set.) For example $(2,2,1)^* = (3,2)$.

2.3. Doubly stochastic matrices. A matrix $M = (m_{ij})$ is called doubly stochastic if $m_{ij} \geq 0$ for all i,j and if all the columns and all the rows add up to 1. Let μ and ν be two partitions of n. One says that μ is an average of ν if there is a doubly stochastic matrix M such that $\mu = M\nu$. Then there is the theorem that μ is an average of ν iff $\mu > \nu$ (in the specialization order).

2.4. Muirhead's inequality. One of the best-known inequalities is $(x_1\dots x_n)^{1/n} \leq n^{-1}(x_1+\dots+x_n)$. A far-reaching generalization due to Muirhead [22] goes as follows. Given a vector $p = (p_1,\dots,p_n)$, $p_i \geq 0$ one defines a symmetrical mean (of the non-negative variables $x_1,\dots,x_n$) by the formula

(2.5) $$[p](x) = (n!)^{-1} \sum_{\sigma} x_1^{p_{\sigma(1)}} \cdots x_n^{p_{\sigma(n)}}$$

where the sum runs over all permutations $\sigma \in S_n$. Then one has Muirhead's inequality which states that $[p](x) \leq [q](x)$ for all non-negative values of the variables $x_1, \ldots, x_n$ iff p is an average of q, so that in case p and q are partitions of n this happens iff $p > q$. The geometric mean - arithmetic mean inequality thus arises from the specialization relation $(1,\ldots,1) > (n,0,\ldots,0)$.

2.6. Completely reachable systems. Let $L_{m,n}$ denote the space of all pairs of real matrices (A,B) of sizes $n \times n$ and $n \times m$ respectively. To such a pair (A,B) [illegible] associates a control system given by the differential equations

(2.7) $$\dot{x} = Ax + Bu, \; x \in \mathbb{R}^n, \; u \in \mathbb{R}^m$$

where the u's are the inputs or controls. The pair (A,B), or equivalently, the system (2.7), is said to be completely reachable if the reachability matrix $R(A,B) = (B \vdots AB \vdots \ldots \vdots A^nB)$ consisting of the $(n+1)(n \times m)$-blocks A^iB, $i = 0, \ldots, n$ has maximal rank n. In system theoretic terms this is equivalent to the property that for any two points $x, x' \in \mathbb{R}^n$ one can steer $x(t)$ to x' in finite time starting from $x(0) = x$ by means of suitable control functions $u(t)$.
Let $L^{cr}_{m,n}$ denote the space of all completely reachable pairs of matrices (A,B). The Lie-group F of all block lower diagonal matrices $\begin{pmatrix} S & 0 \\ K & T \end{pmatrix}$, $S \in GL_n(\mathbb{R})$, $T \in GL_m(\mathbb{R})$, K an $m \times n$ matrix, acts on $L^{cr}_{m,n}$ by according to the formula

(2.8) $$(A,B)^g = (SAS^{-1}+SBTS^{-1}K, SBT), \; g = \begin{pmatrix} S & 0 \\ K & T \end{pmatrix}$$

The 'generating transformations' $(A,B) \mapsto (SAS^{-1}, SB)$ (base change in state space), [illegible]B) $\mapsto$ (A,BT) (base change in input space) and $(A,B) \mapsto (A+BK,B)$ (state space feedback), occur naturally in design problems (of control loops) in electrical engineering. It is now a theorem of Kalman [10] that the orbits of F acting on $L^{cr}_{m,n}$ correspond bijectively with partitions of n. The partition belonging to $(A,B) \in L^{cr}_{m,n}$ is found as follows. Let d_j be the dimension of the subspace of $\mathbb{R}^n$ spanned by the vectors A^ib_r, $r = 1, \ldots, m$, $i \leq j$ where b_r is the r-th column of B. Let $e_j = d_j - d_{j-1}$, $d_{-1} = 0$ then the partition corresponding to (A,B) is the dual partition of $(e_0, e_1, e_2, \ldots, e_n)$, i.e. $\kappa(A,B) = (e_0, e_1, \ldots, e_n)^*$. The numbers $\kappa_1 \geq \ldots \geq \kappa_m$ making up $\kappa(A,B)$ are called the Kronecker indices of (A,B). (Because the problem of classifying pairs (A,B) up to feedback equivalence, i.e. up to the action of F, is a subproblem of the problem of classifying pencils of matrices studied by Kronecker: to (A,B) one associates the pencil $(A-sI \vdots B)$).

Let Θ_κ be the orbit of F acting on $L^{cr}_{m,n}$ labelled by κ. Then a second theorem, noted by a fair number of people independently of each other (Kalman, Hazewinkel, Byrnes, Martin,... , but never yet published, states that $\Theta_\kappa \supset \Theta_{\kappa'} \Longleftrightarrow \kappa > \kappa$. In control theoretic terms this theorem says something about degeneration of systems or system failure.

2.9. Vectorbundles over the Riemann sphere. Let E be a holomorphic vectorbundle over the Riemann sphere $S^2 = \mathbb{P}^1(\mathbb{C})$. Then according to Grothendieck [4] E splits as a direct sum of line bundles

$$E \simeq L(\kappa_1) \oplus \dots \oplus L(\kappa_m) \tag{2.10}$$

where L(i) is the unique (up to isomorphism) line bundle over $\mathbb{P}^1(\mathbb{C})$ of degree i, $L(i) = L(1)^{\otimes i}$, $i \in \mathbb{Z}$, where L(1) is the canonical very ample line bundle of $\mathbb{P}^1(\mathbb{C})$. Thus each holomorphic vectorbundle E over $\mathbb{P}^1(\mathbb{C})$ defines an m-tuple of integers $\kappa(E)$ (in decreasing order). The bundle E is called positive if $\kappa_i(E) \geq 0$ for all i = 1, ..., m. Concerning these positive bundles there is now the following degeneration result of Shatz [20]. Let E_t be a holomorphic family of m-dimensional vectorbundles over $\mathbb{P}^1(\mathbb{C})$. Then for all small enough t, $\kappa(E_t) > \kappa(E_0)$. And inversely if $\kappa > \kappa'$ then there is a homorphic family E_t such that $\kappa(E_t) = \kappa$ for t small $t \neq 0$ and $\kappa(E_0) = \kappa'$.

2.11. Orbits of nilpotent matrices. Let N_n be the space of all n × n complex nilpotent matrices. Consider $SL_n(\mathbb{C})$ or $GL_n(\mathbb{C})$ acting on N_n by similarity, i.e. $A^S = SAS^{-1}$ ($A \in N_n$, $S \in GL_n(\mathbb{C})$). By the Jordan normal form theorem the orbits of this action are labelled by partitions of n. Let $O(\kappa)$ be the orbit consisting of all nilpotent matrices similar to the one consisting of the Jordan blocks $J(\kappa_i)$, i = 1, ..., m where $J(\kappa_i)$ is the $\kappa_i \times \kappa_i$ matrix with 1's just above the diagonal and zero's everywhere else. Then the Gerstenhaber - Hesselink theorem says that $O(\kappa) \supset O(\kappa')$ iff $\kappa < \kappa'$. (Note the reversion of the order with respect to the result orbits described in 2.6 above.)

2.12. A schematic overview of the various relations between all these manifestations of the specialization order can be found in section 5 below.

3. GRASSMANN MANIFOLDS AND CLASSIFYING VECTOR BUNDLES

Before outlining how the various manifestations of the specialization order are connected to each other we need to define Grassmann manifolds, the classifying vectorbundle over them and their Schubert cell decomposition (in section 4 below).

3.1. Grassmann manifolds. Fix two numbers m,n ∈ N. Then the Grassmann manifold $G_n(\mathbb{C}^{n+m})$ consists of all n-dimensional subspaces of $\mathbb{C}^{n+m}$. Thus for example $G_1(\mathbb{C}^{m+1})$ is the m dimensional complex projective space $\mathbb{P}^m(\mathbb{C})$. Let $\mathbb{C}^{n\times(n+m)}_{reg}$ be the space of all complex $n\times(n+m)$ matrices of rank n. Let $GL_n(\mathbb{C})$ act on this space by multiplication on the left. Then the quotient space $\mathbb{C}^{n\times(n+m)}/GL_n(\mathbb{C})$ is $G_n(\mathbb{C}^{n+m})$. The identification is done by associating to $M \in \mathbb{C}^{n\times(n+m)}_{reg}$ the subspace of $\mathbb{C}^{n+m}$ generated by the rows of M.

$G_n(\mathbb{C}^{n+m})$ inherits a natural holomorphic manifold structure from $\mathbb{C}^{n\times(n+m)}$. For a detailed description of $G_n(\mathbb{C}^{n+m})$ cf. e.g. [17].

3.2. The classifying bundle. We define a holomorphic vector bundle ξ_m over $G_n(\mathbb{C}^{n+m})$ as follows. For each x let the fibre over x, $\xi_m(x)$, be the quotient space $\mathbb{C}^{n+m}/x$. More precisely define the bundle η_n over $G_n(\mathbb{C}^{n+m})$ by

(3.3) $$\eta_n = \{(x,v) \in G_n(\mathbb{C}^{n+m}) \times \mathbb{C}^{n+m} \mid v \in x\}$$

with the obvious projection $(x,v) \mapsto x$. Then ξ_m is the quotient bundle of the trivial vector bundle $G_n(\mathbb{C}^{n+m}) \times \mathbb{C}^{n+m}$ by η_n. Both ξ_m and η_n can be used as universal or classifying bundles (cf. [17] for η_n as a universal bundle). Let E be an m-dimensional vector bundle over a complex analytic manifold Π. Let $\Gamma(E) = \Gamma(E,\Pi)$ be the space of all holomorphic sections of E, i.e. the space of all holomorphic maps $s : \Pi \to E$ such that $p \circ s = \mathrm{id}$, where $p : E \to \Pi$ is the bundle projection. The universality, or classifying, property of ξ_m in the setting of complex analytic manifolds now takes the following form. Suppose $V \subset \Gamma(E)$ is an (n+m)-dimensional subspace such that for each $x \in \Pi$ the vectors $s(x)$, $s \in V$ span $E(x)$, the fibre of E over x. Now identy $V \simeq \mathbb{C}^{n+m}$ and associate to $x \in \Pi$ the point of $G_n(\mathbb{C}^{n+m})$ represented by $\mathrm{Ker}(V \to E(x))$. This gives a holomorphic map $\psi_E : \Pi \to G_n(\mathbb{C}^{n+m})$ such that the pullback of ξ_m by means of ψ_E is isomorphic to E, $\psi_E^{!}\xi_m \simeq E$. It is universality properties such as this one which account for the importance of the bundles ξ_m and/or η_n in differential and algebraic topology [17], algebraic geometry and also system and control theory (cf. [24] for the last mentioned).

The bundle ξ_m has a number of obvious holomorphic sections, viz. the sections defined by $\varepsilon_i(x) = e_i \bmod x$ where e_i is the i-th standard basis vector of $\mathbb{C}^{n+m}$, $i = 1,\dots,m$. And, as a matter of fact, it is not difficult to show that $\Gamma(\xi_m, G_n(\mathbb{C}^{n+m}))$ is (n+m)-dimensional and that the $\varepsilon_1,\dots,\varepsilon_n$ from a basis for this space of holomorphic sections.

4. SCHUBERT CELLS

4.1. Schubert cells. Consider again the Grassmann manifold $G_n(\mathbb{C}^{m+n})$. Let $\underline{A} = (A_1,\dots,A_n)$ be a sequence of n-subspaces of $\mathbb{C}^{n+m}$ such that $0 \neq A_1 \subsetneq A_2 \subsetneq \dots \subsetneq A_n$. To each such sequence $\underline{A}$ we associate the closed subset

(4.2) $$SC(\underline{A}) = \{x \in G_n(\mathbb{C}^{m+n}) \mid \dim(x \cap A_i) \geq i\}$$

and call it the closed Schubert-cell of the sequence $\underline{A}$. In particular if $0 < \lambda_1 < \lambda_2 < \ldots < \lambda_n \leq n+m$ is a strictly increasing sequence of natural numbers $\leq n+m$ then we define (setting $\lambda = (\lambda_1,\ldots,\lambda_n)$)

$$SC(\lambda) = SC(\mathbb{C}^{\lambda_1}, \ldots, \mathbb{C}^{\lambda_n}) \tag{4.3}$$

where $\mathbb{C}^r$ is viewed as the subspace of all vectors in $\mathbb{C}^{n+m}$ whose last $n+m-r$ coordinates are zero.

4.4. Flag manifolds and the Bruhat decomposition. A flag in $\mathbb{C}^{n+m}$ is a sequence of subspaces $\underline{F} = F_1 \subset \ldots \subset F_{n+m} \subset \mathbb{C}^{n+m}$ such that $\dim F_i = i$. Let $F\ell(\mathbb{C}^{n+m})$ denote the analytic manifold of all flags in $\mathbb{C}^{n+m}$. There is a natural holomorphic mapping $F\ell(\mathbb{C}^{n+m}) \to G_n(\mathbb{C}^{n+m})$ given by asspciating to a flag $\underline{F}$ its n-th element F_n. The flag manifold can be seen as the space of all cosets Bg, $g \in GL_{n+m}(\mathbb{C})$ where B is the Borel subgroup of all lower triangular matrices in $GL_{n+m}(\mathbb{C})$. The mapping $GL_{n+m}(\mathbb{C}) \to F\ell(\mathbb{C}^{n+m})$ associates to a matrix g the flag $\underline{F}(g)$ whose i-th element is the subspace of $\mathbb{C}^{n+m}$ spanned by the first in row vectors of g.

Now view S_{n+m}, the symmetric group on $n+m$ letters as a subgroup of $GL_{n+m}(\mathbb{C})$ by letting it permute the basis vectors ($\sigma(\rho_i) = \rho_{\sigma(i)}$). Then in $GL_{n+m}(\mathbb{C})$ we have the socalled Bruhat decomposition

$$GL_{n+m}(\mathbb{C}) = \bigcup_{\sigma} B\sigma B \qquad \text{(disjoint union)} \tag{4.5}$$

Where σ runs through the Weyl group S_{n+m} of $GL_{n+m}(\mathbb{C})$. An analogous decomposition holds in a considerable more general setting (reductive groups, cf. [25], section 28).

4.6. The Bernstein-Gelfand-Gelfand order. The closure of a double coset $B\sigma B$ is necessarily a union of other double cosets (by continuity). This defines an ordering on the Weyl group S_{n+m} defined by

$$\sigma > \tau \Leftrightarrow \overline{B\sigma B} \supset B\tau B \tag{4.7}$$

This ordering plays a considerable role in the study of cohomology of flag spaces [1] and also in the theory of highest weight representations [27, 26].

Let H be the subgroup of S_{n+m} consisting of all block lower triangular matrices of the form $\begin{pmatrix} S_{11} & 0 \\ S_{21} & S_{22} \end{pmatrix}$, $S_{11} \in G_n(\mathbb{C})$, $S_{22} \in G_m(\mathbb{C})$, S_{21} and arbitrary $m \times n$ matrix. Then, using the remarks made in subsection 4.4 above, one sees that $G_n(\mathbb{C}^{n+m})$ is the coset space $\{Hg \mid g \in GL_{n+m}(\mathbb{C})\}$. Now let $\sigma \in S_{n+m}$ and let $\lambda_1 < \ldots < \lambda_n$ be the n natural numbers in increasing order determined by $\sigma(e_{\lambda_i}) \in \{e_1, \ldots, e_n\}$, $i = 1,\ldots,n$. Then one easily sees that the image of $B\sigma B$ under $GL_{n+m}(\mathbb{C}) \to G_n(\mathbb{C}^{n+m})$, i.e. the set of all spaces spanned by matrices of the form $h\sigma b$, $h \in H$, $b \in B$ is the open Schubert cell of all elements in $G_n(\mathbb{C}^{n+m})$ spanned by the rows of a matrix of the form

$$\begin{bmatrix} * & \dots & * & 0 & \dots & 0 & 0 & & \dots & & 0 \\ * & \dots & * & * & \dots & * & 0 & & \dots & & 0 \\ & & & & \vdots & & & & & & \\ * & \dots & * & * & \dots & * & * & \dots & * & 0 \dots & 0 \end{bmatrix}$$

column λ_1 column λ_2

where the last * in each row is nonzero. The closure of this open Schubert-cell is the Schubert-cell $SC(\lambda)$ defined in (4.3) above.

One easily checks that

(4.8) $\qquad SC(\mu) \subset SC(\lambda) \Leftrightarrow \mu_i \leq \lambda_i,\ i = 1, \dots, n$

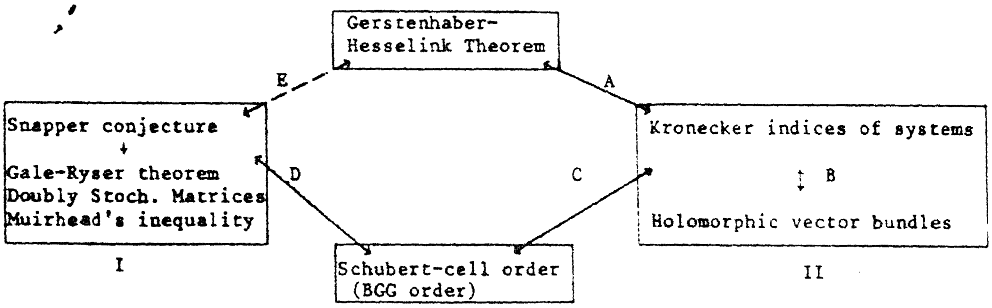
and this order on the Schubert cells $SC(\lambda)$, or the equivalent ordening on n-tuples of natural numbers, is therefore a quotient of the BGG order on the Weyl group S_{n+m}. It is the induced order on the set of cosets $(S_n \times S_m)\sigma$, $\sigma \in S_{n+m}$. (Obviously if $\tau \in S_n \times S_m$, then $\tau\sigma(e_{\lambda_i}) \in \{e_1, \dots, e_n\}$ if $\sigma(e_{\lambda_i}) \in \{e_1, \dots, e_n\}$).

(And inversely the Weyl order is determined by the associated orders of Schubert cells in the sense that $\sigma > \tau$ in S_n iff for all $k = 1, \dots, n-1$ we have for the associated Schubert cells in $G_k(\mathbb{C}^n)$ that $SC(\sigma) \subset SC(\tau)$; this is a rather efficient way of calculating the Weyl order).

5. INTERRELATIONS

Now that we have defined the concepts we need we can start to describe some interrelations between the various manifestations of the specialization order we discussed in section 2 above.

5.1. Overview of the various relations. A schematic overview of the various interconnections is given by the following diagram. In this diagram we have put together in boxes the manifestations which are more or less known to be intimatedly related and have explicitly indicated the new relations to be discussed in detail below.



5.2. On the various relations. The manifestations of the specialization order in box I are well known to be intimatedly related [2,5,11,13]. Very much related is also the Ruch - Schönhofer theorem [18] which states that $< \rho(\kappa), \bar{\rho}(\mu) > \; \geq 1$ iff $\kappa > \mu^*$ where $< , >$ denotes the usual innerproduct (which counts how many irreducible representations there are in common), and where $\bar{\rho}(\mu)$ is the representation of S_n obtained by inducing the alternating representation of S_μ. The link between this theorem and the Gale - Ryser theorem are given by Mackey's intertwining number theorem [29], §44 and Coleman's characterization [28] of double cosets of Young subgroups.

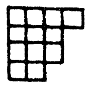
Relation A in the diagram is essentially established by giving two virtually identical proofs of the two theorems and these results can then be used to give natural continuous isomorphisms between feed-back orbits of systems and similarity orbits of nilpotent matrices. More details are in section 7 below. For connection B one associates to a system $\Sigma \in L^{cr}_{m,n}$ a vector bundle $E(\Sigma)$ of dimension m over $P^1(C)$. The construction used is a modification of the one in [15]. It has the advantage that one sees immediately that $\kappa(\Sigma) = \kappa(E(\Sigma))$. For connection C one uses the classifying morphism $\psi_E : P^1(C) \to G_m(C^{n+m})$ attached to a positive bundle E over $P^1(C)$ (cf. section 3.2 above). It turns out that the invariant of E can be recovered from ψ_E by considering the dimensions of the spaces $A_1, \ldots, A_n$ such that $Im\ \psi_E \subset SC(A)$. To establish a link between representations of S_{n+m} and Schubert cells we construct a family of representations of S_{n+m} parametrized by $G_m(C^{n+m})$, which can be used to give a deformation type proof of the Snapper conjecture (in the Liebler-Vitale form) cf. section 11 below. This is not the shortest proof but it certainly is a purely elementary proof which uses no representation theory at all [7]. Combining the links A, C, D gives of course a link from the Gerstenhaber - Hesselink theorem to the Snapper conjecture, albeit a tenuous one. However, there is also a very direct link due to Kraft [12] and this gives yet another proof of the Snapper conjecture.

One possible approach to the Snapper conjecture is of course via Young's rule (discussed below in section 8), which states that the irreducible representation $[\kappa]$ occurs in $\rho(\lambda)$ with a multiplicity equal to the number of semistandard κ-tableaux of type λ. Indeed it is easy to show that the existence of a semistandard κ-tableau of type λ implies that $\kappa < \lambda$. The inverse implication seems much more difficult to show directly. Yet this gives still another link between the Liebler - Vitale theorem (Snapper conjecture) and the Gerstenhaber - Hesselink theorem. Both can be seen as consequences of the statement that there exists a semistandard λ-tableaux of type μ iff $\lambda < \mu$, cf. section 7.6 below.

6. YOUNG'S RULE AND THE SPECIALIZATION ORDER

6.1. Young diagrams and semi-standard tableau. Let $\kappa = (\kappa_1, \ldots, \kappa_m)$ be a partition of n. As usual we picture κ as a Young diagram; that is an array of n boxes arranged in m rows with κ_i boxes in row i, as in the following example

(6.2) $\kappa = (4,3,3,2)$



Let $\lambda = (\lambda_1, \ldots, \lambda_s)$ be another partition of n. Then a semistandard κ-tableau of type λ is the Young diagram of κ with the boxes labelled by the integers $1, \ldots, s$ such that i occurs λ_i times, $i = 1, \ldots, s$ and such that the labels are nondecreasing in each row of the diagram and strictly increasing along each column. An example of a $(5,3,2)$-tableau of type $(4,2,2,2)$ is

(6.3)

1	1	1	1	4
2	2	3		
3	4			

We shall use $n(\kappa,\lambda)$ to denote the number of different semistandard κ-tableau of type λ.

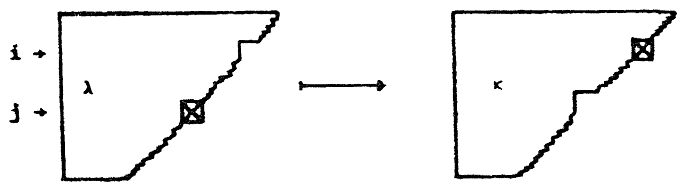
6.4. Young's rule. Let $[\rho]$ denote the irreducible representation associated to the partition ρ. Then Young's rule (cf. [30]) says that

6.5. Theorem. Let κ and λ be partitions of n. Then the number of times that the irreducible representation $[\lambda]$ occurs in the permutation representation $\rho(\kappa)$ is equal to the number of semistandard λ-tableaux of type κ.

6.6. The specialization order and semistandard tableaux. The implication $\kappa > \lambda \Rightarrow \rho(\lambda)$ is a direct summand of $\rho(\kappa)$ follows easily from this. First, however, state a lemma which is standard and seemingly unavoidable when dealing with the specialization order. Its proof is easy.

6.7. Lemma. Let $\lambda = (\lambda_1, \ldots, \lambda_m)$ and $\kappa = (\kappa_1, \ldots, \kappa_m)$ be two partitions of n and suppose that $\lambda > \kappa$ and $(\lambda > \mu > \kappa) \Rightarrow (\mu = \lambda$ or $\mu = \kappa)$ for all partitions μ. Then there are an i and a j, $i < j$ such that $\kappa_i = \lambda_i + 1$, $\lambda_i < \lambda_{i-1}$, $\kappa_j = \lambda_j - 1$, $\lambda_j > \lambda_{j+1}$, $\kappa_s = \lambda_s$, $s \neq i,j$

Pictorially the situation looks as follows



I.e. a box in row j which can be removed without upsetting # (row j) $\geq$ # (row j+1) (which means that we must have had $\lambda_j > \lambda_{j+1}$) is moved to a higher row i which is such that it can receive it without upsetting # (row i) $\leq$ # (row i-1) (which means that we must have had $\lambda_i < \lambda_{i-1}$).
Of course not all transformations of the type described above result in a pair λ,κ such that there is no μ strictly between λ and κ

6.8. Lemma. Let λ and κ be two partitions of n and suppose that there exists a semistandard λ-tableau of type κ. Then $\kappa > \lambda$.

Proof. In a semistandard λ-tableau of type κ all labels i must occur in the first i rows (because the labels in the columns must be strictly increasing). The number of labels j with $j \leq i$ is $\kappa_1+\ldots+\kappa_i$ and the number of places available in the first i rows is $\lambda_1+\ldots+\lambda_i$. Hence $\lambda_1+\ldots+\lambda_i \geq \kappa_1+\ldots+\kappa_i$ for all i so that $\lambda < \kappa$.

6.9. The implication [κ] occurs in $\rho(\lambda) \Rightarrow \kappa < \lambda$. Now suppose that [$\kappa$] occurs in $\rho(\lambda)$. Then there is semistandard κ-tableau of type λ by Youngs rule so that $\kappa < \lambda$ by lemma 6.8.

This implies of course that: ($\rho(\kappa)$ is a subrepresentation $\rho(\lambda)$) $\Rightarrow$ ($\kappa < \lambda$). Because there is obviously a semistandard κ-tableau of type κ (in fact precisely one).

7. NILPOTENT MATRICES AND SYSTEMS

As was remarked in section 5 above the connection A in the diagram above essentially consists of an almost identical proof of the two theorems. We start with a proof of the Gerstenhaber-Hesselink theorem. The first ingredient which we shall also need for the feedback orbits theorem is the following elementary remark on ranks of matrices.

7.1. Lemma. Let A(t) be a family of matrices depending polynomially on a complex or real parameter t. Suppose that rank $A(t) \leq$ rank $A(t_0)$ for all t. Then rank $A(t)$ = rank $A(t_0)$ for all but finitely many t. This follows immediately from the fact that a polynomial in t has only finitely many zeros.

7.2. Lemma. Let A be a nilpotent $n\times n$ matrix and let F be such that

$$F(\text{Ker } A^i) \subset \text{Ker } A^{i-1},\ i = 1,2,\ldots,n \tag{7.3}$$

Then $tA + (1-t)F$ is similar to A for all but finitely many t.

Proof. We show first that

$$\text{Ker}(tA + (1-t)F)^i \supset \text{Ker } A^i \tag{7.4}$$

for all t. Indeed from (7.3) with i=1 we see that F(Ker A) = 0 and it follows that $(tA + (1-t)F)(\mathrm{Ker}\ A) = 0$ which proves (7.4) for i=1. Assume with induction that (7.4) holds for all $i < s$. Then

$$(tA + (1-F))^s \mathrm{Ker}\ A^s = (tA + (1-t)F)^{s-1}(tA + (1-t)F)\mathrm{Ker}\ A^s$$
$$\subset (tA + (1-t)F)^{s-1}\mathrm{Ker}\ A^{s-1} = 0$$

because $A\ \mathrm{Ker}\ A^s \subset \mathrm{Ker}\ A^{s-1}$ and $F\ \mathrm{Ker}\ A^s \subset \mathrm{Ker}\ A^{s-1}$ by (7.3). This proves (7.4). Using 7.4 we know by (7.1) that for almost all t (take $t_0 = 1$)

$$\text{(7.5)} \qquad \mathrm{rank}(tA + (1-t)F)^i = \mathrm{rank}(A^i)$$

and because $tA + (1-t)F)$ and A are both nilpotent it follows that tA and $(1-t)F$ are similar for the all but finitely many t for which (7.5) holds.

Now let A be a nilpotent matrix. We say that A is of type $\kappa = (\kappa_1,\ldots,\kappa_m)$ if the Jordan normal form of A consists of m Jordan blocks of sizes $\kappa_i \times \kappa_i$, $i = 1,\ldots,m$. E.g. A is of type (4,2) iff its Jordan form is

$$\left[\begin{array}{cccc|cc} 0 & 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \\ \hline 0 & 0 & 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 & 0 & 0 \end{array}\right]$$

Consider Ker A, Ker A^2, ..., Ker A^n. Then A is of type κ iff $\dim(\mathrm{Ker}\ A^i) = \kappa_1^* + \ldots + \kappa_i^*$, $i = 1,\ldots,n$ where κ^* is the dual partition of κ. Thus in the example the kernel spaces Ker A^i are spanned by the basis vectors $\{e_1,e_5\}$, $\{e_1,e_2,e_5,e_6\}$, $\{e_1,e_2,e_3,e_5,e_6\}$, $\{e_1,e_2,e_3,e_4,e_5,e_6\}$.

7.6. <u>Semistandard tableaux and nilpotent matrices</u>. Let A be a nilpotent matrix of type κ. Let μ be another partition of n and suppose that there is a μ^*-tableau of type κ^*. Then there is nilpotent matrix F such that $F(\mathrm{Ker}\ A^i) \subset \mathrm{Ker}\ A^{i-1}$ for all i. This matrix F is constructed as follows. First choose a basis $e_1,\ldots,e_n$ of $\mathbb{R}^n$ such that the first $\kappa_1^* + \ldots + \kappa_i^*$ elements of this basis form a basis for Ker A^i, $i = 1,\ldots,n$. Now consider a semistandard μ^*-tableau T of type κ^*. Take the Young-diagram of μ^* and label the boxes of it by the basis vectors $e_1,\ldots,e_n$ in such a way that the boxes marked with i in the semistandard tableau T are filled with the basis vectors $e_{\kappa_1^*+\ldots+\kappa_{i-1}^*+1}, \ldots, e_{\kappa_1^*+\ldots+\kappa_i^*}$. This can be done because T is of type κ^* so that there are precisely κ_i^* boxes labelled i in T. Call this new μ^*-tableau T'. Now define F by $F(e_i) = e_{i'}$ if $e_{i'}$ is just above e_i in the μ^*-tableau T' and $F(e_j) = 0$ if e_j occurs in the first row of T'. Then obviously

$\dim \operatorname{Ker} F^i = \mu_1^* + \ldots + \mu_i^*$ so that F is of type μ and $F(\operatorname{Ker} A^i) \subset \operatorname{Ker} A^{i-1}$ because the μ^*-tableau T was semistandard which implies that the labels are strictly increasing along columns.

An example may illustrate things. Let $\kappa^* = (2,2,2)$, $\mu^* = (4,1,1)$. A μ^*-tableau of type κ^* is then

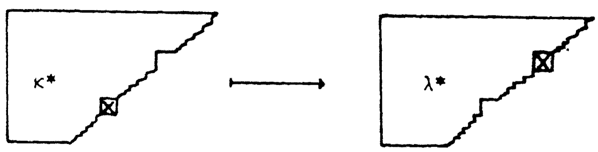
$$\begin{matrix} 1 & 1 & 2 & 3 \\ 2 \\ 3 \end{matrix}$$

Inserting $e_1,\ldots,e_6$ in such a way that e_1,e_2 are put into boxes marked with 1, e_3,e_4 in boxes marked with 2 and e_5,e_6 in boxes marked with 3 gives for example

$$\begin{matrix} e_1 & e_2 & e_3 & e_5 \\ e_4 \\ e_6 \end{matrix}$$

which yields an F defined by $F(e_6) = e_4$, $F(e_4) = e_1$, $F(e_1) = F(e_2) = F(e_3) = F(e_5) = 0$.

7.7. Proof of the Gerstenhaber-Hesselink theorem (Cf. 2.11 above). The implication $\Rightarrow$ is immediate. Indeed if $A_t \in O(\kappa)$ converges to $A_0 \in O(\lambda)$ as $t \to 0$ then $\operatorname{rank}(A_t^i) \geq \operatorname{rank}(A_0^i)$ for small t and all $i = 1,\ldots,n$. Hence $\dim(\operatorname{Ker} A_t^i) \leq \dim(\operatorname{Ker} A_0^i)$ for small t so that $\kappa_1 + \ldots + \kappa_i \leq \lambda_1 + \ldots + \lambda_i$ for all i, hence $\kappa^* > \lambda^*$ and $\kappa < \lambda$. To prove the opposite implication it suffices to show this in case that κ is obtained from λ by a transformation of the type described in lemma 6.7. (Because if $\overline{O(\kappa)} \supset O(\lambda)$ and $\overline{O(\lambda)} \supset O(\mu)$, then $\overline{O(\kappa)} \supset \overline{O(\lambda)}$, and hence $\overline{O(\kappa)} \supset O(\mu)$). Then λ^* is obtained from κ^* by a similar transformation. We recall the picture



Now put 1's in the first row of κ^*, 2's in the second row, etc. Transport the box ■ together with its label. The result is obviously a semistandard λ^*-tableau of type κ^*. Now let A be a nilpotent matrix of type κ. Then by the construction of 7.6 above there is an F of type λ such that $F \operatorname{Ker} A^i \subset \operatorname{Ker} A^{i-1}$ Then $tA + (1-t)F$ is similar to A for almost all t by lemma 7.2 so that there is a sequence of A's in $O(\kappa)$ converging to $F \in O(\lambda)$, proving that $O(\lambda) \subset \overline{O(\kappa)}$, which finishes the proof of the theorem.

Incidentally it is quite easy to describe F directly without ressorting to semistandard tableaux.

7.10. Kronecker indices of systems. Let $(A,B) \in L^{cr}_{m,n}$ be a completely reachable pair of matrices. Recall that this means that the matrix $R(A,B) = (B \mid AB \mid \ldots \mid A^nB)$ has rank n. Recall that the Kronecker indices $\kappa(A,B)$ of the pair (A,B) are defined as follows. Let for $i = 1,\ldots,n$

(7.11) $\quad V_i(A,B)$ = space spanned by the column vectors of A^jB, $j = 0,1,\ldots,i-1$

Let $d_i = \dim V_i(A,B)$, $e_i = d_i - d_{i-1}$, $d_0 = 0$. Then $e_i \leq e_{i-1}$, $i = 1,\ldots,n-1$, and $\kappa(A,B)$ is defined as the dual partition of n

(7.12) $$\kappa(A,B) = e(A,B)^*$$

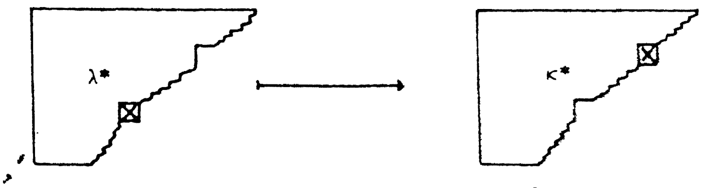
where $e(A,B) = (e_1,\ldots,e_n)$.

The orbits of the feedback group (cf. 2.6 above) acting on $L^{cr}_{m,n}$ are precisely the subsets of $L^{cr}_{m,n}$ with constant $\kappa(A,B)$. Let $U(\kappa)$ be this orbit. The "degeneration of systems theorem" now says

7.13. Theorem. $\overline{U(\kappa)} \supset U(\lambda) \Leftrightarrow \kappa > \lambda$

Here follows a proof which is virtually identical with the proof of the Gerstenhaber-Hesselink theorem given above. First if $(A_t,B_t) \to (A_0,B_0)$ as $t \to 0$, $(A_t,B_t) \in U(\kappa)$, $(A_0,B_0) \in U(\lambda)$, then $\text{rank}(A_t^{i-1}B_t \mid \ldots \mid A_tB_t \mid B_t) \geq \text{rank}(A_0^{i-1}B_0 \mid \ldots \mid A_0B_0 \mid B_0)$ for small t. Hence $\dim V_i(A_t,B_t) \geq \dim V_i(A_0,B_0)$ for small t. Hence $e(A_t,B_t) < e(A_0,B_0)$ for small t and $\kappa(A_t,B_t) > \kappa(A_0,B_0)$ for small t which proves the implication $\Rightarrow$.

To prove the inverse implication it suffices to prove this in the case λ is obtained from κ by a transformation as described in lemma 6.7 (exactly as in the case of the Gerstenhaber-Hesselink theorem). This means that κ^* is obtained from λ^* by a similar transformation:



Now let $(A,B) \in U(\kappa)$. Choose a basis $e_1,\ldots,e_n$ for $\mathbb{R}^n$ such that the first $\kappa_1^* + \ldots + \kappa_i^*$ elements of $e_1,\ldots,e_n$, form a basis for $V_i(A,B)$, $i = 1,\ldots,n$. Now write in the $e_1,\ldots,e_n$ in κ^* in the standard way and transport ⊠ backwards together with its label. E.g. if $\lambda^* = (4,3,2,2,1)$ and $\kappa^* = (4,4,2,1,1)$ then this would give

e_1	e_2	e_3	e_4	e_1	e_2	e_3	e_4
e_5	e_6	e_7		e_5	e_6	e_7	e_8
e_9	e_{10}			e_9	e_{10}		
e_{11}	e_8			e_{11}			
e_{12}				e_{12}			

Then the vectors in the first i rows of κ^* are a basis for $V_i(A,B)$. Now define a pair (F,G) in terms of λ^* as follows. G consists of the vectors in the first row of λ^* (plus a zero vector in case $\kappa_1^* > \lambda_1^*$), and F is defined by $F(e_i) = e_{i'}$ if $e_{i'}$ occurs just below e_i in λ^* and $F(e_i) = 0$ otherwise. Then (F,G) has following properties (all immediate)

(i) $(F,G) \in U(\lambda) \subset L^{cr}_{m,n}$

(ii) $V_i(F,G) \subset V_i(A,B)$

(iii) $FV_i(A,B) \subset V_{i+1}(A,B)$

(Of course (ii) follows from (iii) together with $V_1(F,G) \subset V_1(A,B)$). Now consider $A_t = tA + (1-t)F$, $B_t = tB + (1-t)G$. Then

(7.14) $\quad V_i(A_t,B_t) \subset V_i(A,B) \quad$ for all t

(7.15) $\quad V_i(A_t,B_t) = V_i(A,B) \quad$ for all but finitely many t

Indeed obviously $V_1(A_t,B_t) \subset V_1(A,B)$ because of (ii) above for i = 1. Now assume that (7.14) holds for all i < r. Then

$$\begin{aligned} V_r(A_t,B_t) &= (tA + (1-t)F)V_{r-1}(A_t,B_t) + V_{r-1}(A_t,B_t) \\ &\subset tAV_{r-1}(A,B) + (1-t)FV_{r-1}(A,B) + V_{r-1}(A,B) \\ &\subset V_r(A,B) + V_r(A,B) + V_{r-1}(A,B) = V_r(A,B) \end{aligned}$$

This proves (7.14) and (7.15) follows by means of lemma 7.1 (with $t_0 = 1$) because

$$\dim V_i(A_t,B_t) = \operatorname{rank}(A_t^{i-1}B_t \mid \ldots \mid B_t)$$

Now $(A_t,B_t) \to (F,G) \in U(\lambda)$ as $t \to 0$ and by (7.15) (and the theorem that the orbits under the feedback group are classified by the Kronecker indices) all but finitely many of the (A_t,B_t) are feedback equivalent to (A,B). Thus $U(\lambda) \ni (F,G) \in \overline{U(\kappa)}$ proving the theorem.

7.16. Remarks. The two proofs are very similar (up to duality in a certain sense). This can be given more precise form as follows. For a nilpotent matrix $N \in N_n$ let $\underline{s}(N) = \{(A,B) \in L^{cr}_{m,n} \mid N^iA^{i-1}B = 0,\ i = 1,\ldots,n\}$ and for

$(A,B) \in L^{cr}_{m,n}$ let $\underline{t}(A,B) = \{N \in N_n \mid N^i A^{i-1} B = 0,\ i = 1,\dots,n\}$. Then using the results above one shows that

$$\underline{t}\ \underline{s}(\overline{O(\kappa)}) = \overline{O(\kappa)},\quad \underline{s}\ \underline{t}(\overline{U(\kappa)}) = \overline{U(\kappa)}$$

so that $\underline{t}$ and $\underline{s}$ set up a bijective correspondence between the closures of orbits in the two cases and hence induce a bijective order preserving correspondence between the orbits themselves.

8. VECTORBUNDLES AND SYSTEMS

This section contains a modified version of the construction of Martin-Hermann [15] associating a vector bundle $E(\Sigma)$ over the Riemann sphere $\mathbb{P}^1(\mathbb{C})$ to every $\Sigma = (A,B) \in L^{cr}_{m,n}$. This version makes it almost trivial to see that $E(\Sigma)$ splits as a direct sum of line bundles $L(\kappa_i)$, $i = 1,\dots,m$ where $\kappa = (\kappa_1,\dots,\kappa_m)$ is the set of Kronecker indices of Σ.

The first thing needed is some more information on the universal bundle ξ_m.

8.1. On the universal bundle $\xi_m \to G_n(\mathbb{C}^{n+m})$. Let V be a complex $n+m$ dimensional vectorspace and $V^* = \mathrm{Hom}_{\mathbb{C}}(V,\mathbb{C})$ its dual vectorspace. Given $x \in G_n(\mathbb{C}^{n+m})$ define $x^* = \{v^* \in V^* \mid \langle v^*,v\rangle = 0 \text{ for all } v \in x\}$ where $\langle\ ,\ \rangle$ denotes the usual pairing $V^* \times V \to \mathbb{C}$. Then x^* is m-dimensional and $x \mapsto x^*$ defines a holomorphic isomorphism

$$d : G_n(V) \to G_m(V^*) \tag{8.2}$$

Now $v \in V/x$ defines a unique homomorphism $v^T : x^* \to \mathbb{C}$ as follows: $v^T(a) = \langle a,\hat{v}\rangle$ for all $a \in x^*$, where $\hat{v} \in V$ is any representant of v. This is well defined because $\langle a,b\rangle = 0$ for all $b \in x$ if $a \in x^*$. This defines an isomorphism between the pullback $(d^{-1})^! \xi_m$ and the dual of the subbundle η_m of $G_m(V^*)$ defined by

$$\eta_m = \{(x^*,w) \in G_m(V^*) \times V^* \mid w \in x^*\}$$

It follows that ξ_m is a subbundle of an $n+m$ dimensional trivial bundle $G_n(\mathbb{C}^{n+m}) \times \mathbb{C}^{n+m}$. Because $G_n(\mathbb{C}^{n+m})$ is projective (compact) all holomorphic maps $G_n(\mathbb{C}^{n+m}) \to \mathbb{C}$ are constant so that the space of holomorphic sections $\Gamma(G_n(\mathbb{C}^{n+m}) \times \mathbb{C}^{n+m}, G_n(\mathbb{C}^{n+m}))$ is of dimension $n+m$. As a subbundle of a trivial $(n+m)$-dimensional bundle ξ_m can therefore have at most $(n+m)$ linearly independent holomorphic sections. But we have already found $(n+m)$ linearly independent sections viz. the $\varepsilon_1,\dots,\varepsilon_{n+m}$ defined by $\varepsilon_i(x) = e_i \bmod x$ where e_i is the i-th standard basis vector of $\mathbb{C}^{n+m}$. Therefore

$$\dim \Gamma(\xi_m, G_n(\mathbb{C}^{n+m})) = n + m \tag{8.3}$$

Now let $A \in GL_{n+m}(\mathbb{C})$. Then A induces a holomorphic automorphism A_* of

$G_m(\mathbb{C}^{n+m})$ defined by $x \mapsto Ax$. Then of course there is an induced isomorphism $A^{-1} : \mathbb{C}^{n+m}/Ax \to \mathbb{C}^{n+m}/x$ which for varying x induces an isomorphism

$$A_*^{!}\xi_m \simeq \xi_m \ , \ A \in GL_{n+m}(\mathbb{C}) \tag{8.4}$$

8.5. <u>The line bundles</u> $L(i)$ <u>over</u> $\mathbb{P}^1(\mathbb{C})$. The Riemann sphere $\mathbb{P}^1(\mathbb{C}) = S^2$ can be obtained by gluing together two copies of $\mathbb{C}$ along the open subsets $\mathbb{C} \setminus \{0\}$ by the isomorphism

$$\mathbb{C} \setminus \{0\} \to \mathbb{C} \setminus \{0\}, s \mapsto t = s^{-1}$$

A line bundle over $\mathbb{P}^1(\mathbb{C})$ is then obtained by giving a holomorphic isomorphism $\mathbb{C} \setminus \{0\} \times \mathbb{C} \to \mathbb{C} \setminus \{0\} \times \mathbb{C}$ linear in the second variable compatible with the above isomorphism. Obviously the only possibilities are $(s,v) \mapsto (s^{-1}, s^i v)$ for $i \in \mathbb{Z}$. This gives us the following commutative diagram of identifications

$$\begin{array}{ccccccc}
\mathbb{C}\times\mathbb{C} & \supset & \mathbb{C}\setminus\{0\}\times\mathbb{C} & \xrightarrow{(s,v)\mapsto(s^{-1},s^i v)} & \mathbb{C}\setminus\{0\}\times\mathbb{C} & \subset & \mathbb{C}\times\mathbb{C} \\
s_1 \uparrow\downarrow & & \downarrow & & \downarrow & & \downarrow\uparrow s_2 \\
\mathbb{C} & \supset & \mathbb{C}\setminus\{0\} & \xrightarrow{s\mapsto s^{-1}=t} & \mathbb{C}\setminus\{0\} & \subset & \mathbb{C}
\end{array}$$

The corresponding holomorphic line bundle is denoted $L(-i)$. A section of $L(-i)$ consists of two holomorphic mappings s_1, s_2, of the form $s \mapsto (s, f(s))$, $t \mapsto (t, g(t))$ such that $s^i f(s) = g(s^{-1})$. It readily follows that $f(s)$ must be a polynomial of degree $\leq -i$. Thus

$$\dim\Gamma(L(i)) = 0 \text{ if } i < 0 \tag{8.6}$$

$$\dim\Gamma(L(i)) = i+1 \text{ if } i \geq 0 \tag{8.7}$$

8.8. <u>The (modified) Martin-Hermann vectorbundle of a system</u>. Let $\Sigma = (A,B)$ be a pair of real or complex matrices of sizes $n\times n$ and $n\times m$. Then (A,B) is completely reachable (cr) iff the $n\times(n+m)$ matrix $(sI - A \mid B)$ is of rank n for all complex values of s. So if $\Sigma = (A,B)$ is cr one can define a holomorphic map ψ_Σ by

$$\psi_\Sigma : \mathbb{P}^1(\mathbb{C}) \to G_n(\mathbb{C}^{n+m}), \ s \mapsto \mathrm{Row}(sI-A \mid B), \ \infty \mapsto \mathrm{Row}(I \mid 0) \tag{8.9}$$

where $\mathrm{Row}(M)$ for an $n\times(m+n)$ matrix M denotes the subspace of $\mathbb{C}^{n+m}$ generated by the rows of M. The vectorbundle $E(\Sigma)$ over $\mathbb{P}^1(\mathbb{C})$ is now defined by

$$E(\Sigma) = \psi_\Sigma^{!}\xi_m \tag{8.10}$$

8.11. <u>Proposition</u>. $E(\Sigma)$ depends only on the feedback orbit of Σ. Indeed one easily checks that $\Sigma = (A,B)$, $\Sigma' = (A',B') \in L_{m,n}^{cr}$ are feedback equivalent (cf. 2.6. above) iff there are constant invertible matrices P, Q such that $P(sI-A \mid B)Q = (sI-A' \mid B')$. Now $\mathrm{Row}(PM) = \mathrm{Row}(M)$ and postmultiplication with Q changes ψ_Σ to $Q_* \circ \psi_\Sigma$ and $E(\Sigma') = \psi^!_{\Sigma'}(\xi_m) = \psi^!_\Sigma(Q^!_*\xi_m) \simeq \psi^!_\Sigma(\xi_m) = E(\Sigma)$ by 8.4 above, proving the proposition.

Thus to determine $E(\Sigma)$ we can assume that $\Sigma = (A,B)$ is in Brunowsky canonical form which means that A, B take the form

$$A = \left[\begin{array}{c|c|c} \begin{matrix} 0 & 1 & & 0 \\ & \ddots & \ddots & \\ & & & 1 \\ 0 & & & 0 \end{matrix} & 0 & 0 \\ \hline 0 & \begin{matrix} 0 & 1 & & 0 \\ & \ddots & \ddots & \\ & & & 1 \\ 0 & & & 0 \end{matrix} & 0 \\ \hline 0 & 0 & \begin{matrix} 0 & 1 & & 0 \\ & \ddots & \ddots & \\ & & & 1 \\ 0 & & & 0 \end{matrix} \end{array}\right] \qquad B = \left[\begin{array}{ccc} 0 & 0 & 0 \\ \vdots & \vdots & \vdots \\ 0 & \cdot & \cdot \\ 1 & 0 & 0 \\ \hline 0 & 0 & 0 \\ \vdots & \vdots & \vdots \\ \cdot & 0 & \cdot \\ 0 & 1 & 0 \\ \hline 0 & 0 & 0 \\ \vdots & \vdots & \vdots \\ \cdot & \cdot & 0 \\ 0 & 0 & 1 \end{array}\right] \begin{array}{l} \Big\} \kappa_1 \\ \\ \Big\} \kappa_2 \\ \\ \Big\} \kappa_3 \end{array}$$

in case m=3, where $(\kappa_1, \kappa_2, \kappa_3) = \kappa(A,B)$ are the Kronecker indices of $\Sigma = (A,B)$. (The general case is evident from this example). The matrix $(sI - A \mid B)$ is now easily written down, and one observes that for all $s \neq 0$, ∞ [illegible] $e_1 \equiv e_2 \equiv \ldots \equiv e_\kappa \equiv e_{n+1}$ and $\mathrm{Row}(sI - A \mid B)$, [illegible] and $\psi_\Sigma(s)$, and for $s=0$, $e_2 \equiv \ldots \equiv e_{\kappa_1} \equiv e_{n+1} \equiv 0$ but $e_1 \not\equiv 0$ and for $s = \infty$, $e_1 \equiv \ldots \equiv e_{\kappa_1} \equiv 0$ and $e_{n+1} \not\equiv 0$. It follows that the vectors $e_1(\psi_\Sigma(s)), \ldots, e_{\kappa_1}(\psi_\Sigma(s)), e_{n+1}(\psi_\Sigma(s))$ span a one dimensional subspace of $\xi_m(\psi_\Sigma(s))$ for all s so that $E(\Sigma) = \psi^!_\Sigma \xi_m$ contains a line bundle L_1 which admits at least κ_1+1 linearly independent holomorphic sections viz. the $e_1, \ldots, e_{\kappa_1}, e_{n+1}$. Similar relations hold for $e_{\kappa_1+\ldots+\kappa_{i-1}+1}, \ldots, e_{\kappa_1+\ldots+\kappa_i}, e_{n+i}$ for all $i = 1, \ldots, m$ giving line subbundles L_i, $i = 1, \ldots, m$ which admit at least κ_i+1 holomorphic sections. This exhausts the e_j and because the $e_1(x), \ldots, e_{n+m}(x)$ span $\xi_m(x)$ for $x \in G_m(\mathbb{C}^{n+m})$ it follows that $E(\Sigma) = \oplus L_i$. As the pullback of a bundle [illegible] ξ_m, $E(\Sigma)$ itself is a subbundle of an $(n+m)$-dimensional trivial bundle. Because $E(\Sigma)$ is projective it follows (as before) that $E(\Sigma)$ has at most n+m linearly independent holomorphic sections. But L_i has at least κ_i+1 linearly independent sections, hence $\oplus L_i$ has at least $\Sigma(\kappa_i+1) = n+m$ linearly independent sections which proves that L_i has precisely κ_i+1 linearly independent sections and hence identifies L_i as the bundle $L(\kappa_i)$ described above in (8.5). We have reproved the theorem of Martin and Hermann [15]

8.12. <u>Theorem</u>. Keeping the notations introduced above in (8.10) and (8.5) we have $E(\Sigma) \simeq \bigoplus_{i=1}^{m} L(\kappa_i)$.

8.13. <u>The correspondence B</u>. (cf. the diagram in section 5 above). The mapping

$\Sigma \mapsto E(\Sigma)$ is obviously continuous. Thus the result $U(\kappa) \supset U(\lambda) \Leftrightarrow \kappa > \lambda$ can be deduced from Shatz's theorem (cf. 2.9). Inversely Shatz's theorem for positive bundles over $\mathbf{P}^1(\mathbb{C})$ can be deduced from the result on feedback orbits because every positive bundle arises as an $E(\Sigma)$. By tensoring with a suitable $L(r)$, r high enough, the result is then extended to arbitrary bundles over $\mathbf{P}^1(\mathbb{C})$.

9. VECTOR BUNDLES, SYSTEMS AND SCHUBERT CELLS

9.1. Partitions and Schubert-cells. Let κ be a partition of n. To κ we associate the following increasing sequence of n numbers $\tau(\kappa)$.

$$(9.2)\quad \tau(\kappa) = (\underbrace{2,3,\dots,\kappa_1+1}_{\kappa_1},\underbrace{\kappa_1+3,\dots,\kappa_1+\kappa_2+2}_{\kappa_2},\dots,\underbrace{\kappa_1+\dots+\kappa_{m-1}+m+1,\dots,\kappa_1+\dots+\kappa_m+m}_{\kappa_m})$$

Let $\tau_j(\kappa)$, $j = 1,\dots,n$ be the j-th element of this sequence. It is an easy exercise to check that

$$(9.3)\qquad \kappa > \lambda \Leftrightarrow \tau_i(\kappa) \geq \tau_i(\lambda) \text{ for all } i = 1,\dots,n$$

Thus the specialization order is a suborder of the inclusion ordening between closed Schubert cells, because $SC(\tau) \supset SC(\tau') \Leftrightarrow \tau_i \geq \tau_i'$, $i = 1,\dots,n$. And in turn as we saw above in section 4 the Schubert-cell order is a quotient of the BGG order on the Weyl group S_{n+m}.

9.4. Systems and Schubert cells. Let $(A,B) \in L^{cr}_{m,n}$ be a system and as in section 8.8 consider the associated holomorphic morphism $\psi_\Sigma : \mathbf{P}^1(\mathbb{C}) \to G_n(\mathbb{C}^{n+m})$. Suppose that (A,B) are in Brunovsky canonial form. Then simple inspection of the matrix $(sT-A \mid B)$ (cf. the example below proposition 8.11) shows that $\mathrm{Im}\ \psi_\Sigma \subset SC(\tau(\kappa))$, where $\kappa = \kappa(A,B)$. Now let (A,B) be any system in $L^{cr}_{m,n}$. Then it is feedback equivalent to a Brunovsky canonical one so that $(sT-A \mid B) = P(sT-A_0 \mid B_0)Q$ for certain constant invertible matrices P,Q where (A_0,B_0) is a canonical pair. Premultiplication with P does not change ψ_Σ and postmultiplication with Q induces an automorphism of $G_n(\mathbb{C}^{n+m})$ which takes the "standard basis" Schubert-cell $SC(\tau(\kappa))$ into another Schubert-cell of the same dimension type. Thus we have shown.

9.5. Theorem. Let $\Sigma \in L^{cr}_{m,n}$, $\kappa = \kappa(\Sigma)$ and let $\psi_\Sigma : \mathbf{P}^1(\mathbb{C}) \to G_n(\mathbb{C}^{n+m})$ be the Martin-Hermann morphism of Σ. Then there is a Schubert-cell $SC(\underline{A})$, $\underline{A} = (A_1,\dots,A_n)$ such that $\mathrm{Im}\ \psi_\Sigma \subset SC(\underline{A})$ and $\dim A_i = \tau_i(\kappa)$, where $\tau_i(\kappa)$ is defined by (9.2).

The converse is also true but considerably more difficult to prove, cf.[8]:

9.6. Theorem. With the notations of theorem 9.5 let the Schubert-cell $SC(\underline{B})$, $\underline{B} = (B_1,\dots,B_n)$, in $G_n(\mathbb{C}^{n+m})$ be such that $\mathrm{Im}\ \psi_\Sigma \subset SC(\underline{B})$. Then $\dim B_i \geq \tau_i(\kappa)$.

9.7. Vectorbundles and Schubert cells. Because every positive vectorbundle over $\mathbb{P}^1(\mathbb{C})$ arises as the bundle $E(\Sigma)$ of some system Σ one has the obvious analogues of theorems 9.5 and 9.6 for positive bundles over $\mathbb{P}^1(\mathbb{C})$. Here the morphism ψ_Σ must of course be replaced by the classifying morphism (cf. section 3.2 above) of a positive vector bundle E, and $n+m$ and m are determined respectively as $\dim \Gamma(E, \mathbb{P}^1(\mathbb{C}))$ and $\dim E$.

10. A FAMILY OF REPRESENTATIONS OF S_{n+m} PARAMETRIZED BY $G_n(\mathbb{C}^{n+m})$

10.1. <u>Construction of the family</u>. Let M be the regular representation of S_{n+m}. That is M has a basis e_σ, $\sigma \in S_{n+m}$ and $\tau \in S_{n+m}$ acts by $\tau(e_\sigma) = e_{\tau\sigma}$. Now consider the universal bundle ξ_m over $G_n(\mathbb{C}^{n+m})$ and the $(n+m)$ holomorphic sections $\varepsilon_1, \ldots \varepsilon_{n+m}$ defined by $\varepsilon_i(x) = e_i \bmod x \in \mathbb{C}^{n+m}/x$. Take the $(m+n)$-fold tensor product of ξ_m and define a family of homomorphisms parametrized by $G_n(\mathbb{C}^{n+m})$ by

(10.2)
$$\pi_x : M \to \xi_m(x)^{\otimes(n+m)}$$
$$e_{\sigma^{-1}} \mapsto \varepsilon_{\sigma(1)}(x) \otimes \ldots \otimes \varepsilon_{\sigma(n+m)}(x)$$

(More precisely (10.2) defines a homomorphism of vectorbundles $G_n(\mathbb{C}^{n+m}) \times M \to \xi_m^{\otimes(n+m)}$).

The group S_{n+m} acts on $\xi_m(x)^{\otimes(n+m)}$ by permuting the factors and it is a simple exercise to see that π_x is equivariant with respect to this action, i.e. that $\pi_x(\tau v) = \tau\pi_x(v)$ for all $v \in M$, $\tau \in S_{n+m}$ (Here the product $\tau\sigma \in S_{n+m}$ is interpreted as first the automorphism σ of $\{1,\ldots,n+m\}$ and then the automorphism τ).

Thus $\operatorname{Im} \pi_x = \pi(x)$ is a representation of S_{n+m} for all x giving us a family of representations parametrized by $G_n(\mathbb{C}^{n+m})$. Fixing a point $x_0 \in G_n(\mathbb{C}^{n+m})$ and choosing m independent sections of ξ_m in a neighbourhood U of x_0, this gives families of homomorphisms of representations

(10.3)
$$M \xrightarrow{\pi'_x} (\mathbb{C}^m)^{\otimes(n+m)}, \quad x \in U \subset G_n(\mathbb{C}^{n+m})$$

such that $\operatorname{Im} \pi'_x \simeq \pi(x)$ for $x \in U$.

10.4. <u>Permutation representations and Schubert-cells (On connection D)</u>. Let $x \in G_n(\mathbb{C}^{n+m})$ be a subspace of $\mathbb{C}^{n+m}$ spanned by the rows of a matrix of the form (m=8, n=5)

$$\left[\begin{array}{cccc|ccc|c} * & * & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & * & * & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & * & * & 0 & 0 & 0 & 0 \\ \hline 0 & 0 & 0 & 0 & * & * & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & * & * & 0 \end{array}\right]$$

where all the *'s are nonzero. Then obviously the representation $\pi(x)$ of S_8 is isomorphic to $\rho(\tilde{\kappa})$ with $\tilde{\kappa} = (4,3,1)$. Note that x is in the standard Schubert-cell $SC(\tau(\kappa))$, with $\kappa = (3,2,0)$. This holds in general and it is not difficult to extend this to

10.5. Proposition. Let κ be an m-part partition of n, $\tilde{\kappa} = (\kappa_1+1,\ldots,\kappa_m+1)$. Then for almost all $x \in SC(\tau(\kappa))$, the representation $\pi(x)$ of S_{n+m} contains the representation $\rho(\tilde{\kappa})$ and for some $x \in SC(\tau(\kappa))$ $\pi(x) \simeq \rho(\tilde{\kappa})$.

Conjecturally the reverse holds also. That is if for almost all x in a standard Schulbert-cell $SC(\lambda)$ we have that $\pi(x)$ contains $\rho(\tilde{\kappa})$ then $\lambda_i \geq \tau_i(\kappa)$, $i = 1,\ldots,n$. And I am even inclined to believe that if $x \in SC(\lambda)$ and $\pi(x)$ contains (or is equal to) $\rho(\tilde{\kappa})$ then $\lambda_i \geq \tau_i(\kappa)$.

Note also that the matrices (10.5) are precisely the type of matrices $(sI-A|B)$ for a system $\Sigma = (A,B)$ in feedback canonical form ($s \neq 0, \infty$), suggesting that there is a natural representation of S_{n+m} attached to Σ awaiting interpretation.

11. DEFORMATIONS OF REPRESENTATION HOMOMORPHISMS AND SUBREPRESENTATIONS

11.1. On proving Snapper-type results. Suppose we have given a continuous family of homomorphisms of finite dimensional representations over $\mathbb{C}$ of a finite group G

$$\pi_t : M \to V \qquad (11.2)$$

Suppose that $\operatorname{Im} \pi_t \simeq \rho$ for $t \neq 0$ (and small) and $\operatorname{Im} \pi_0 \simeq \rho_0$. Then the representation ρ_0 is a direct summand of ρ. This is seen as follows. Because the category of finite dimensional representations of G is semisimple there is a homomorphism of representations $\phi_0 : \operatorname{Im} \pi_0 \to M$ such that $\pi_0 \circ \phi_0 = \mathrm{id}$. Then $_t \circ \phi_0 : \operatorname{Im} \pi_0 \to \operatorname{Im} \pi_t$ is still injective for small t (by the continuity of $_t$) which gives us ρ_0 as a subrepresentation and hence a direct summand of ρ.

It is almost equally easy to construct a surjective homomorphism $\operatorname{m} \pi_t \to \operatorname{Im} \pi_0$ (which is more or less what we shall do below in 11.3 in (a ketch of) a proof of the Liebler-Vitale theorem that $\kappa < \lambda \Rightarrow \rho(\kappa)$ is a direct ummand of $\rho(\lambda)$).

11.2. The inverse result. Inversely if ρ_0 is a subrepresentation of ρ then here is a family of representations (11.2) such that $\operatorname{Im} \pi_t \simeq \rho$ for $t \neq 0$ and $\operatorname{Im} \pi_0 \simeq \rho_0$, and if ρ is generated (as a $\mathbb{C}[G]$-module) by one element one can take for M in (11.2) the regular representation. Indeed if ρ_0 is a subrepresentation of ρ then $\rho = \rho_0 \oplus \rho_1$. Let $\pi : M \to \rho = \rho_0 \oplus \rho_1$ be a surjective map of representations Let π_0, π_1 be the two components of π. Let $s = (s_0, s_1)$ be a section of π. Then $\pi_0 s_0 = \mathrm{id}$, $\pi_1 s_1 = \mathrm{id}$, $\pi_0 s_1 = 0$, $\pi_1 s_0 = 0$ and it follows that $\pi(t)$

consisting of the components π_0 and $t\pi_1$ is still surjective. Hence Im $\pi(t) = \rho$ and Im $\pi(0) = \rho_0$.

11.3. On a proof of the Liebler-Vitale theorem. It is quite conceivable that the grand family constructed in section 10 above contains all subfamilies needed to prove the Liebler-Vitale theorem by means of the deformation argument of section 11.1 above. So far, however, we have not found them. A somewhat more complicated argument immediately suggested by the structure of the family of representations constructed in section 10 above does give a proof. It is perhaps best illustrated by means of an example.

Consider an $x \in G_3(\mathbb{C}^5)$ spanned by the rows of a matrix of the form

$$\begin{bmatrix} 1 & -1 & 0 & 0 & 0 \\ 0 & 1 & -1 & 0 & 0 \\ z & 0 & 0 & -1 & t \end{bmatrix}$$

Let $f_1,\dots,f_5$ be the images of the standard basis vectors $e_1,\dots,e_5$ in $\mathbb{C}^5/x$. Then $f_1 = f_2$, $f_2 = f_3$, $f_4 = zf_1 + tf_5$ so that f_1 and f_5 are a basis for $\mathbb{C}^5/x$ for all values of z and t. Let $(1) \in S_5$ be the identity representation. The image of $e_{(1)} \in M$ in $(\mathbb{C}^5/x)^{\otimes 5}$ is by the definition (10.2) equal to

$$f_1 \otimes f_2 \otimes f_3 \otimes f_4 \otimes f_5 = zf_{11115} + tf_{11155} \quad . \tag{11.4}$$

where f_{11115} is short for $f_1 \otimes f_1 \otimes f_1 \otimes f_1 \otimes f_5$ and similarly for other 5-tuples of indices. Symmetrizing the element (11.4) with respect to the permutation (45) gives us

$$z(f_{11115} + f_{11151}) + 2tf_{11155} \tag{11.5}$$

Let V_1 be the subrepresentation of Im π_x (generated by the element (11.5). (The representation Im π_x is the subrepresentation of $(\mathbb{C}^5/x)^{\otimes 5}$ generated by (11.4)). Now (11.5) is invariant under the Young subgroup $S_3 \times S_2$. Hence dim $V_1 \le 5!/3!2!$. On the other hand if $t \neq 0$ then setting $z = 0$ in (11.5) (which corresponds to the surjective map mentioned just above 11.2 associated to a family of representations) obviously maps V_1 onto the vector space with a basis all symbols $f_{\cdot\cdot\cdot\cdot\cdot}$ with three of the indices equal to 1 and 2 equal to 5. This is $\rho(3,2)$ of dimension $5!/3!2!$ so that $V_1 \simeq \rho(3,2)$ if $t \neq 0$. Now for $z \neq$ set $t = 0$ in (11.4) to obtain a homomorphism of representations

$$\text{Im}\ \pi_x \to \rho(4,1)$$

It is now not hard to prove that (cf. [7] for a detailed proof)

11.6. Proposition. The composed homomorphism of representations $\rho(3,2) \simeq V_1 \subset \text{Im}\ \pi_x \to \rho(4,1)$ is surjective

This then proves that $\rho(4,1)$ is a direct summand of $\rho(3,2)$. The argumen generalizes without difficulty for partitions $\kappa > \lambda$ such that λ is obtained from κ by a transformation of the type described in 6.7 above.

This is by no means the easiest way to prove the Liebler-Vitale theorem. It is perfectly easy to describe the morphism $\rho(\kappa) \to \rho(\lambda)$ directly and then the general analogue of proposition 11.6 yields the Liebler-Vitale result. This proof uses no representation theory at all (except the definition of the permutation representations $\rho(\kappa)$); cf. [7] for details.

References

1. I.N. Bernstein, I.M. Gelfand, S.I. Gelfand, Schubert cells and cohomology of the spaces G/P, Russian Math. Surveys 28, 3 (1973), 1-26.

2. T. Brylawski, The lattice of integer partitions, Discrete Math. 6 (1973),201-219.

3. M. Gerstenhaber, On dominance and varieties of commuting matrices, Ann.of Math. 73 (1961), 324-348.

4. A. Grothendieck, Sur la classification des fibrés holomorphes sur la sphère de Riemann, Amer.J.Math. 79 (1957), 121-138.

5. L. Harper, G.-C. Rota, Matching theory: an introduction, In: P. Ney (ed.), Advances in Probability Vol. 1, Marcel Dekker, 1971, 171-215.

7. M. Hazewinkel, T. Vorst, On the Snapper, Liebler-Vitale, Lam theorem on permutation representations of the symmetric groups, in preparation.

8. M. Hazewinkel, C. Martin, Representations of the symmetric group, the specialization order, systems and Grassmann manifolds, in preparation.

9. W. Hesselink, Singularities in the nilpotent scheme of a classical group, Trans. Amer.Math.Soc. 222 (1976), 1-32.

10. R.E. Kalman, Kronecker invariants and feedback (+ errata), In: L. Weiss (ed.) Ordinary differential equations, Acad.Pr., 1972, 459-471.

11. A. Kerber, The diagram lattice as structural principle in mathematics. In: P. Kramers, A. Rieckers (eds.), Group theoretical methods in physics (Tübingen, 1977), Lect.Notes in Physics 79, Springer, 1978, 53-71.

12. H.P. Kraft, Letter to M. Hazewinkel, June 2, 1980 (This material will be presented at the Torun conference in Sept. 1980).

13. T.Y. Lam, Young diagrams, Schur functions, the Gale Ryser theorem and a conjecture of Snapper, J. pure and appl.Algebra 10 (1977), 81-94.

14. R.A. Liebler, M.R. Vitale, Ordening the partition characters of the symmetric group, J. of Algebra 25, (1973), 487-489.

15. C. Martin, R. Hermann, Applications of algebraic geometry to system theory: the McMillan degree and Kronecker indices of transfer functions as topological an holomorphic system invariants, SIAM J. Control Opt. 16 (1978), 743-755.

16. A. Mead, E. Ruch, A. Schönhofer, Theory of chirality functions, generalized for molecules with chiral ligands, Theor.Chim.Acta 29 (1973), 269-304.

. Milner, J. Stasheff, Characteristic classes, Princeton Univ.Pr., 1974.

. Ruch, A. Schönhofer, Theorie der Chiralitätsfunktionen, Theor.Chim.Acta 19 (1970), 225-287.

. Ryser, Combinatorial mathematics, Carus Math. Monographs 14, Wiley, 1963.

.S. Shatz, The decomposition and specialization of algebraic families of vector bundles, Compositio Math. 35 (1977).

. Snapper, Group characters and nonnegative integral matrices, J.of Algebra 19 (1971), 520-535.

F. Muirhead, Some methods applicable to identities and inequalities of symmetric algebraic functions of n letters, Roc. Edinburgh Math.Soc. 21 (1903), 144-157.

I. Hazewinkel, A partial survey of the uses of algebraic geometry in systems and control theory, Proc. INDAM Vol. 24 (Severi centennial conference, Rome April 1979), Acad.Press, to appear.

I. Hazewinkel, (Fine) moduli spaces in linear system theory: what they are and what they are good for. In C. Martin, C. Byrnes (eds.), Algebraic and geometric methods in linear systems theory, Reidel Publ.Cy, to appear.

J.E. Humphreys, Linear algebraic group, Springer, 1975.

J.C. Jantzen, Moduln met einem höchsten Gewicht, Lect.Notes in Math. 750, Springer, 1979.

J.E. Humphreys, Highest weight modules for semi-simple Lie algebras, In: Proc. Workshop on present trends in representation theory, Carleton Univ., 1979, Lect.Notes Math., Springer, to appear.

A.J. Coleman, Induced representations with applications to S_n, Queen's papers in pure and applied math. 4, Queen's Univ., Kingston, Canada, 1966.

.W. Curtis, I. Reiner, Representation theory of finite group and associative algebras, Interscience, 1962.

G.D. James, The representation theory of the symmetric groups, Lect. Notes in Math. 682, Springer, 1978.