ECONOMETRIC INSTITUTE

INTRODUCTION TO GEOMETRICAL METHODS FOR THE THEORY OF LINEAR SYSTEMS

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INTRODUCTION TO GEOMETRICAL METHODS FOR THE THEORY OF LINEAR SYSTEMS

C. Byrnes, M. Hazewinkel, C. Martin, and Y. Rouchaleau

In this joint totally tutorial chapter we try to discuss those definitions and results from the areas of mathematics which have already proved to be important for a number of problems in linear system theory.

Depending on his knowledge, mathematical expertise and interests, the reader can skip all or certain parts of this chapter 0. Apart from the joint section, the basic function of this chapter is to provide the reader of this volume with enough readily available background material so that he can understand those parts of the following chapters which build on this--for a mathematical system theorist perhaps not totally standard--basic material. The joint section is different in nature; it attempts to explain some of the ideas and problems which were (and are) prominent in classical algebraic geometry and to make clear that many of the problems now confronting us in linear system theory are similar in nature if not in detail. Thus we hope to transmit some intuition why one can indeed expect that the tools and philosophy of algebraic geometry will be fruitful in dealing with the formidable array of problems of contemporary mathematical system theory. This section can, of course, be skipped without endangering one's chances of understanding the remainder of this chapter and the following chapters.

The contents of this introductory chapter are:

C. I. Byrnes and C. F. Martin, Geometrical Methods for the Theory of Linear Systems, 1-84

1. SOME PROBLEMS OF CLASSICAL ALGEBRAIC GEOMETRY

The purpose of this section is to give insight into certain of the problems and achievements of 19th century algebraic geometry, in a historical perspective. It is our hope that this perspective, which for several reasons is limited, will go some of the distance towards explaining some natural interrelations between algebraic geometry and analysis, as well as a natural connection between algebraic geometry and linear system theory.

1.1 Plane algebraic curves

To begin, perhaps the most primitive objects of algebraic geometry are varieties, e.g., plane curves in $\mathbb{C}^2$ (say the variety defined by the equation $y = x^2$), and the most primitive relations are those of incidence, e.g., the intersection of varieties. To fix the ideas, let us consider the problem of describing all plane curves in $\mathbb{C}^2$ and the problem of describing their intersections. Since any two distinct irreducible (i.e., the polynomial $f(x,y)$, whose locus is the curve, is irreducible) curves intersect in finitely many points, the first problem of describing such an intersection is to compute the number of such points in terms of the two curves.

Now, whenever one speaks of a scheme for the description or classification of objects, such as plane curves, one has in mind a certain notion of equivalence. And, quite often, this involves the notions of transformation. For example, if $SL(2,\mathbb{C})$ is the group of 2×2 matrices with determinant 1, then $g \in SL(2,\mathbb{C})$ acts on $\mathbb{C}^2$ by linear change of variables and it has been known since the introduction of Cartesian coordinates that a linear change of coordinates leaves the degree of a curve invariant. That is, if $f(x,y)$ is homogeneous, then

$$f^g(x,y) = f\left(g^{-1}\begin{pmatrix}x\\y\end{pmatrix}\right) \qquad (1.1.1)$$

has the same degree as f. So, for homogeneous f, we may begin the classification scheme by fixing the degree. Now any f which is homogeneous of degree 1 is a linear functional, and these are well understood. If f is homogeneous of degree 2, then one can check that the discriminant

$$\Delta(f) = b^2 - 4ac,$$

where

$$f(x,y) = ax^2 + bxy + cy^2 ,$$

is invariant under $SL(2,\mathbb{C})$; i.e.,

$$\Delta(f) = \Delta(f^g), \qquad \text{for all } g \in SL(2,\mathbb{C}) . \qquad (1.1.2)$$

This explains, in part, why the discriminant is so important in analytic geometry, but there really is a lot more to the story. First of all, (1.1.2) asserts that the discriminant of f, $\Delta(f)$, is the same regardless of the choice of coordinates used to express f (provided we allow only volume preserving, orientation preserving changes of coordinates). But this is also true for Δ^2, $\Delta^2 + 3$, etc. In 1801, Gauss [2,4] proved an important result: *any polynomial in* a, b, c *which is invariant under* $SL(2,\mathbb{C})$ *is a polynomial in* Δ. That is, let V denote the 3-dimensional space of quadratic forms in 2 variables, let R denote the ring of polynomials on V (i.e., polynomials in a,b,c) and let $R^{SL(2,\mathbb{C})}$ denote the subring of invariant polynomials, i.e., the polynomials satisfying (1.1.2).

1.1.3 <u>Theorem</u> (Gauss). $R^{SL(2,\mathbb{C})} = \mathbb{C}[\Delta]$ *and, if* $\Delta(f_1) = \Delta(f_2) \neq 0$ *then* $f_1^g = f_2$ *for some* $g \in SL(2,\mathbb{C})$.

Thus, Gauss classifies homogeneous $f(x,y)$ of degree 2 by the table:

Quadratic Form	Complete Invariant
f s.t. $\Delta(f) \neq 0$	$\Delta(f)$
f s.t. $\Delta(f) = 0$	rank of $f = \begin{pmatrix} a & b/2 \\ b/2 & c \end{pmatrix}$

(1.1.4)

Clearly, the same kind of question is equally important for homogeneous forms of degree r, in $n \geq 2$ variables. In 1845, Cayley posed the general problem, in the same notation as above [2]:

1.1.5 <u>Cayley's Problem</u>: *Describe the algebra* $R^{SL(n,\mathbb{C})}$ *as explicitly as possible; e.g., is* $R^{SL(n,\mathbb{C})}$ *finitely generated by some invariants* $\Delta_1,\ldots,\Delta_\ell$?

Now, the case $n = 3$ is particularly relevant for our discussion of plane curves. For, one may always "homogenize" a polynomial, and this process allows one to express the number of points of intersection of 2 plane curves in a beautiful formula, due to Bézout. Returning to our example,

$$X = \{(x,y) : y = x^2\} ,$$

to homogenize $f(x,y) = y-x^2$ is to substitute x/z, y/z for x,y and then to clear denominators with the result being the homogeneous polynomial $\tilde{f}(x,y,z) = yz - x^2$ satisfying

$$\tilde{f}(x,y,1) = f(x,y) . \quad (1.1.6)$$

Geometrically, since $\tilde{f}(x,y,z)$ is homogeneous the locus of $\tilde{f}$ contains the line connecting any nonzero solution with the origin. Indeed, the intersection of $\tilde{f}(x,y,z) = 0$ with the plan $z = 1$

is given by the zeroes of f, as in (1.1.6), and the locus of $\tilde{f}$ contains all lines through this curve. However, there is more, the line $(0,y,0)$ also lies in the locus of $\tilde{f}$.

Next, if one considers the projective plane

$$\mathbb{P}^2 = \{\text{lines thru } 0 \text{ in } \mathbb{C}^3\}$$

then, by homogeneity, the locus of $\tilde{f}$ is a collection of points in $\mathbb{P}^2$--one for each of the points in $f(x,y) = 0$ and one more, the line $(0,y,0)$, which may be regarded as the point at ∞. To make this more precise, we give "homogeneous coordinates" to a point $P \in \mathbb{P}^2$; i.e., regarding P as a line in $\mathbb{C}^3$, choose some non-zero $(x,y,z) \in P$ noting that any other choice (x',y',z') is a non-zero multiple of (x,y,z). The equivalence class $[x,y,x]$ is called "homogeneous coordinates" for P and to check membership of $P_0 = [x_0,y_0,x_0]$ in the locus of a homogeneous $f(x,y,z)$ it is enough to evaluate $f(x_0,y_0,z_0)$.

As an example of these ideas in control theory, consider the transfer function

$$T(s) = \begin{pmatrix} \frac{1}{s} \\ \frac{1}{s^2} \end{pmatrix} \tag{1.1.7}$$

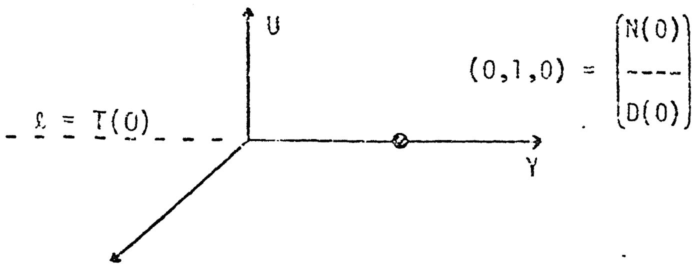
and the coprime factorization

$$\begin{pmatrix} N(s) \\ \text{---} \\ D(s) \end{pmatrix} = \begin{pmatrix} s \\ \text{-}1\text{-} \\ s^2 \end{pmatrix} . \tag{1.1.8}$$

Now, for an arbitrary $s \in \mathbb{C}$, (1.1.8) is a point in $\mathbb{C}^3 - \{0\}$ although $T(s)$ does not determine this point canonically. Rather, $T(s)$ determines the line through

$$\begin{pmatrix} N(s) \\ D(s) \end{pmatrix} .$$

as depicted below:



(1.1.9)

Since $T(\infty) = U$, T extends to a map of the extended complex plane

$$T : \mathbb{C} \cup \{\infty\} = \mathbb{P}^1 \to \mathbb{P}^2 \quad . \tag{1.1.10}$$

And one easily checks, using the homogeneous coordinates in (1.1.8), that $T(\mathbb{C} \cup \{\infty\})$ is the curve defined in our example, viz., the locus of $f(x,y,z) = yz - x^2$. Moreover, if $\mathbb{P}^1 \subset \mathbb{P}^2$ is the space of lines in Y, (1.1.9), then $\mathbb{P}^1 \cap T(\mathbb{C} \cup \{\infty\}$ is easily computed, under T^{-1} it is the set $\text{sing}(T)$ of poles of T:

$$T^{-1}(T(\mathbb{C} \cup \{\infty\}) \cap \mathbb{P}^1) = \text{sing}(T) \tag{1.1.11}$$

and thus consists of one point of multiplicity 2, (see Professor Martin's lectures for the geometry of a general transfer function).

1.1.12. Theorem (Bézout [9]). *If* $X_1, X_2 \subset \mathbb{P}^2$ *are irreducible curves of degree* d_1, d_2, *then, counting multiplicities,*

$$\#(X_1 \cap X_2) = d_1 \cdot d_2 \quad .$$

We shall prove this in the case where X_2 is a line $\mathbb{P}^1$. By a change of coordinates in $\mathbb{C}^3$, X_2 corresponds to the set of lines in the plane $z = 0$. And, by a change of notation, if X_1 is the locus of $f(x,y,z)$, homogeneous of degree d, then Euler's relation is

$$f(x_1,x_2,x_3) = d \cdot \sum_{i=1}^{3} \frac{\partial f}{\partial x_i} \cdot x_i \quad . \tag{1.1.13}$$

Intersecting $f(x_1,x_2,x_3) = 0$ with $x_3 = 0$ gives the equation, of degree d,

$$d \cdot \sum_{i=1}^{2} \frac{\partial f}{\partial x_i} \cdot x_i = 0$$

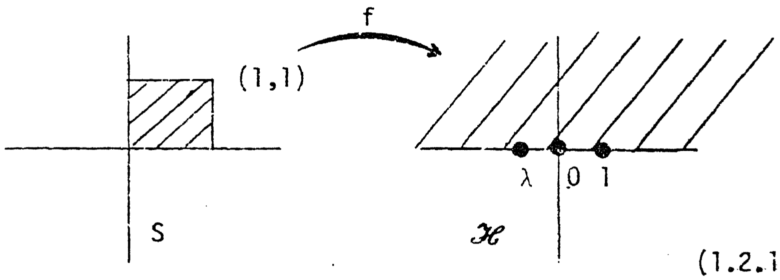
defining, counting multiple roots, d lines in the (x_1,x_2) plane.

1.1.14. Remark. $\mathbb{P}^2$ provides a model for a (non-Euclidean) geometry, where the points are just the points of $\mathbb{P}^2$ and the lines are just the loci of linear functionals on $\mathbb{C}^3$, i.e., planes in $\mathbb{C}^3$. Thus, for example, 2 distinct points P_1, P_2 regarded as lines in $\mathbb{C}^3$ determine a line $\mathbb{P}^1$ in $\mathbb{P}^2$, viz. the plane in $\mathbb{C}^3$ spanned by P_1 and P_2. Moreover, any 2 lines in $\mathbb{P}^2$ intersect in a point.

1.2 Riemann Surfaces and Fields of Meromorphic Functions

Thus, by homogenizing curves in $\mathbb{C}^2$, we take a lot of the mystery out of the points at ∞. Indeed, one can give a beautiful expression for the intersection number of 2 curves. Plane projective curves also arise in potential theory and in the calculus.

In two papers published in 1869 [1], H. A. Schwartz considered the problem of finding, for purposes of solving Dirichlet problems, conformal maps of bounded regions to the unit disk or, equivalently, to the upper half plane. For example, Schwartz considers the problem of finding a conformal map of the unit square onto the upper half plane, $\mathscr{H}$, where f maps 3 corners



(1.2.1)

to points $0, 1, \lambda$ and the 4th corner to ∞. In particular f is meromorphic, as it should be, for f can be extended to a doubly periodic function on $\mathbb{C}$, by the Schwartz Reflection Principle. Actually, it is easier to construct a holomorphic map $g: \mathscr{H} \to S$. The Schwartz-Christoffel formula applies in this case to give the elliptic integral

$$g(P) = \int_{P_0}^{P} \frac{dz}{\sqrt{4z(1-z^2)}} \quad . \qquad (1.2.2)$$

Such integrals had already been the subject of deep research by Fagnano, Euler, Gauss, Abel, Jacobi and others, being first encountered in the computation of the arclength of an ellipse. In particular, Euler had shown that elliptic integrals, such as (1.2.2), satisfy addition formulae

$$g(P) + g(Q) = g(R) \tag{1.2.3}$$

where R is a rational function of P and Q, generalizing the familiar trigonometric addition formulae gotten from considering the lengths of arcs on a circle. Indeed, there are group theoretic ideas underlying (1.2.3) too.

That is, consider the meromorphic function f (cf. (1.2.1)) which inverts g. As we have noted f extends to a doubly periodic meromorphic function on $\mathbb{C}$ and hence to a meromorphic function on the torus, or more properly the elliptic curve,

$$\mathcal{E} = \mathbb{C}/\{n + im\}, \quad n, m \in \mathbb{Z},$$

gotten by identifying the (oriented) horizontal edges of the unit square and by identifying the (oriented) vertical edges. One therefore has a nontrivial meromorphic function,

$$f : \mathcal{E} \to \mathbb{C} \tag{1.2.4}$$

and a holomorphic 1-form on $\mathcal{E}$,

$$\frac{dz}{\sqrt{4z(1-z^2)}} \tag{1.2.5}$$

which turns out to be invariant under multiplication on the group $\mathcal{E}$. This can also be seen from the method of substitution applied to the integral (1.2.2). That is, substitute $y^2 = 4z(1-z^2)$ and consider integrating dz/y over paths defined on the algebraic curve, $y^2 = 4z(1-z^2)$. Homogenizing this curve we obtain

$$y^2x = 4z(x^2 - z^2), \tag{1.2.6}$$

and hence a cubic curve $X \subset \mathbb{P}^2$. One can see a beautiful geometric definition of the group law on X: choose 2 points P_1, P_2 on X and consider the line $\ell(\simeq\mathbb{P}^1)$ in $\mathbb{P}^2$ which they determine. By Bézout's Theorem, ℓ intersects X in a third point, $P_3 = (P_1 \cdot P_2)^{-1}$! Moreover, dz/y is an invariant holomorphic 1-form on X and from this one may obtain (1.2.3). However, more is true; X admits a non-constant meromorphic function derived from dz/y, viz. f. In fact, the field of meromorphic functions on X is easily seen to be $\mathbb{C}(y,z)$ where y and z are related as above. Again using the form dz/y and a formula relating the degree of X to the topology of X one may show that $X \simeq \mathcal{E}$ as complex manifolds!

Remark. As a sketch of the proof, one sees from the fact that X is a nonsingular cubic in $\mathbb{P}^2$ that X is not simply-connected and thus integrals $\int_\gamma dz/y$ where γ is a closed path on X are not necessarily 0, although the proper form of Cauchy's Theorem is still valid; viz., if $\gamma_1 \sim \gamma_2$ (are homologous) then the path integrals taken over γ_1, γ_2 are equal. And, although X is not simply-connected, one knows that there is a basis $\{\gamma_1, \gamma_2\}$ for the closed curves on X modulo homology. Thus there are two basic "periods" of dz/y,

$$\int_{\gamma_1} \frac{dz}{y} \quad \text{and} \quad \int_{\gamma_2} \frac{dz}{y} \quad . \tag{1.2.7}$$

Now, if $P_0 \in X$ is the identity (or any point) then one might consider the quantities

$$\int_{P_0}^{P} \frac{dz}{y} \ ,$$

for all $P \in X$. This quantity is not a well-defined complex number, as the integral depends on the choice of path. If γ and $\tilde{\gamma}$ are 2 paths joining P_0 and P, then

$$\int_\gamma \frac{dz}{y} - \int_{\tilde{\gamma}} \frac{dz}{y}$$

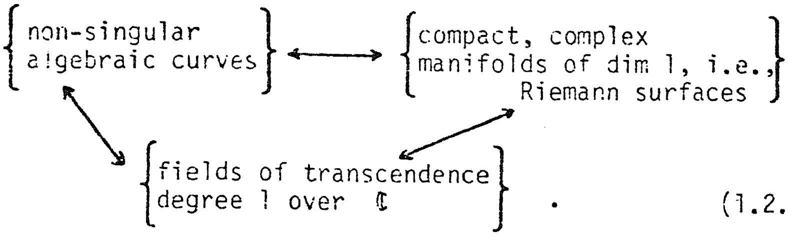
is an integral around a closed path, based at P_0, on X and is therefore (by Cauchy's Theorem) an integer combination of the periods (1.2.7). Thus, if Λ is the lattice in $\mathbb{C}$ generated by the periods (1.2.7) one has an isomorphism

$$X \to \mathbb{C}/\Lambda = \mathcal{E} \ , \qquad \text{defined via}$$

$$P \mapsto \left(\int_{P_0}^{P} \frac{dz}{y}\right) \quad . \tag{1.2.8}$$

To conclude the remark, if one instead considered integrals with a rational integrand (or, more generally, of the form $dz/\sqrt{z^2 + az + b}$) the curve X in $\mathbb{P}^2$ turns out to be $\mathbb{P}^1$, as a conic in $\mathbb{P}^2$, which is simply connected, while the form dz/y is meromorphic and the usual residue calculus applies. This explains the ease with which rational integrals may be calculated as well as the relative difficulty involved in calculating elliptic integrals.

Summarizing, one has the interconnection between elliptic curves, complex tori, and certain fields of meromorphic functions. This is a special case of what has been properly referred [7] to as "the amazing synthesis." That is, one may consider three formally distinct classes of objects: non-singular projective curves (of any degree), complex compact manifolds of dimension 1, and fields of meromorphic functions. Then, the amazing synthesis is that any one of these objects determines the other two. Schematically,



. (1.2.9)

The deeper part of this correspondence is that from an abstract Riemann surface S one may recover the embedding of S into projective space and the equations defining this curve, or equivalently, that one may construct the field of meromorphic functions on S. Above, the meromorphic function f on the curve $\mathcal{E}$ was constructed via potential theory, i.e., in order to solve the Dirichlet problem. As Riemann demonstrated with liberal use of the "Dirichlet principle," such transcendental techniques can be used to construct non-trivial meromorphic functions on an arbitrary Riemann surface. Briefly, the intuition runs as follows.

First of all (and we will consider analytic equivalence in 1.3), a compact Riemann surface is topologically a sphere with g handles, where if the surface is given as a curve of degree d in $\mathbb{P}^2$ the genus g is given by $(d-1)(d-2)/2$. Thus, the elliptic curve $\mathcal{E}$ has genus $(3-1)(3-2)/2 = 1$ and is a sphere with 1 handle, i.e., a torus.

Next, a meromorphic function f on S has as many poles as zeroes. Where $f = u + iv$ is analytic, u and v satisfy Laplace's equation in light of the Cauchy-Riemann equations. Therefore, f gives rise to a time-invariant flow with inessential singularities on S where u = const. defines the equipotential curves and v = const. defines the lines of force. Conversely, Riemann's idea was to construct such an f by regarding, intuitively, S as a surface made of a conductive material and by placing the poles of a battery at each pole-zero pair of f. This can be made somewhat more precise by a much more careful description of S and an appeal to the Dirichlet principle. Indeed, the application of modern harmonic theory to the (Riemann-Roch) question of existence of meromorphic functions on S is one of the most beautiful sides of the "amazing synthesis." For a more detailed account of the intuitive discussion hinted at above, be sure to browse in F. Klein's book [ ].

In closing this section, we would like to make contact with what is perhaps a more familiar description of a Riemann surface, viz. as a branched cover of the extended complex plane $\mathbb{P}^1$. For example, at least the finite part of the Riemann surface of the relation $y^2 = 4z(1-z^2)$ can be obtained by forming the branch cuts between -1 and 0 and 1 and $+\infty$ and sewing two copies of the plan less these cuts together in the appropriate fashion. One can get at the whole Riemann surface more easily by considering the graph of the relation $y^2 = 4z(1-z^2)$. Explicitly, introduce homogeneous coordinators $([y,\tilde{y}],[z,\tilde{z}])$, and homogenize the relation, obtaining the curve $\mathscr{E}$

$$y^2 \tilde{z}^3 - 4\tilde{y}^2 z(\tilde{z}^2 - z^2) \qquad (1.2.10)$$

in $\mathbb{P}^1 \times \mathbb{P}^1$. However, we get more than just the curve $\mathscr{E} \subset \mathbb{P}^1 \times \mathbb{P}^1$, we also obtain 2 rational functions on $\mathscr{E}$

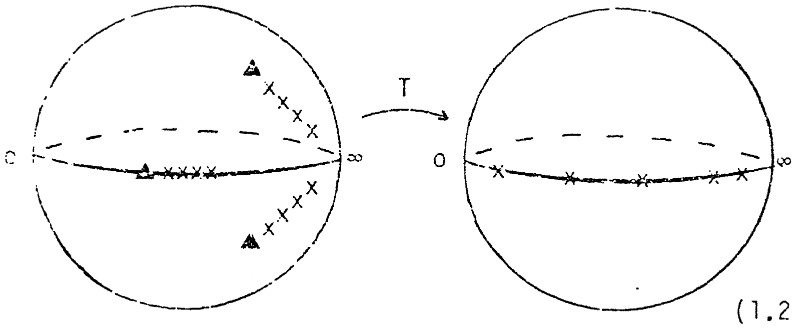
$$\mathrm{proj}_1 : \mathscr{E} \to \mathbb{P}^1 \ , \qquad \mathrm{proj}_2 : \mathscr{E} \to \mathbb{P}^1$$

which are, of course, the algebraic functions y and z, on the Riemann surface $\mathscr{E}$ of $y^2 = 4z(1-z^2)$. Notice that $y: \mathscr{E} \to \mathbb{P}^1$ exhibits $\mathscr{E}$ as a 2-fold cover of $\mathbb{P}^1$, branched at the 4 points $z = 0, \pm 1$, and ∞.

This is, at the very least, reminiscent of root loci. That is, for a scalar transfer function $T(s)$ one may regard, as in (1.1.7) etc., $T(s)$ as a branched cover of the Riemann spheres

$$T : \mathbb{P}^1 \to \mathbb{P}^1 \ ,$$

as depicted below



(1.2.11)

Here, the ▲'s denote $T^{-1}(\infty)$ and the x's denote the motion of the closed-loop root loci, as a function of $-1/k$ toward the open loop zeroes--one finite real zero and a branched zero at ∞. Indeed, for a scalar gain $K = kI$ and square multivariable tranfer function $T(s)$, an extension of these ideas has been given by A. MacFarlane and I. Postlethwaite.

Example [9]. Consider the transfer function

$$T(s) = \frac{1}{(1.25)(s+1)(s+2)} \begin{pmatrix} s-1 & s \\ -6 & s-2 \end{pmatrix}$$

and the scalar output gain

$$K = \begin{pmatrix} k & 0 \\ 0 & k \end{pmatrix}$$

In order to study the locus of roots of the closed-loop characteristic polynomial (see Professor Byrnes's lectures), it is enough to study the locus of roots of $\det(I + kT(s))$ or, setting $k = -1/g$, the Riemann surface X defined by

$$0 = \det(gI - G(s)) = g^2 - \mathrm{tr}G(s)g + \det G(s) .$$

Clearing denominators, one obtains

$$0 = f(s,g) = (1.25)(s+1)(s+2)g^2 - (2s-3)g + \frac{4}{5} = 0$$

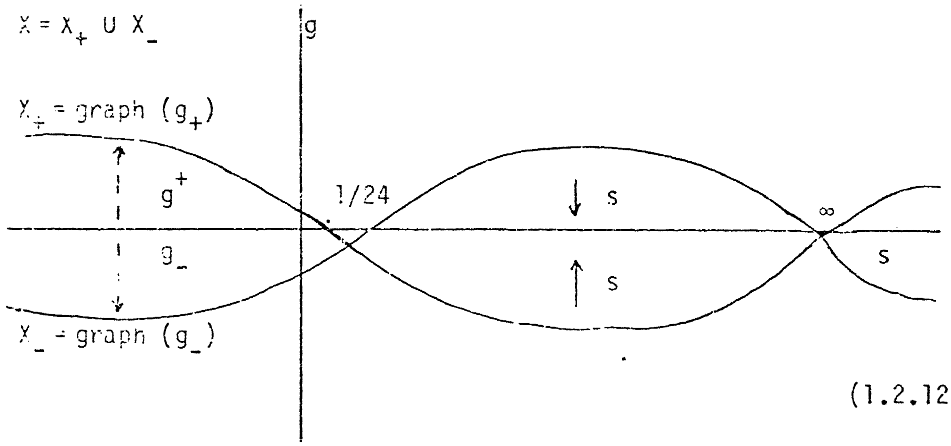
leading to the algebraic functions

$$g \pm (s) = \frac{(2s-3) \pm \sqrt{1 - 24s}}{(2.5)(s+1)(s+2)} .$$

In this way, one has

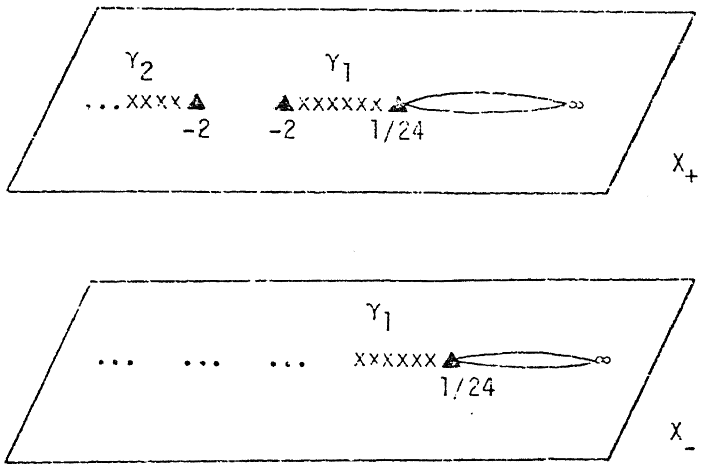
$$\begin{array}{ccc} \mathbb{P}^1 & \times & \mathbb{P}^1 \\ & \cup & \\ & X & \\ {\scriptstyle s}\swarrow & & \searrow{\scriptstyle g} \\ \mathbb{P}^1 & & \mathbb{P}^1 \end{array}$$

where $s : X \to \mathbb{P}^1$ is a 2-sheeted cover of $\mathbb{P}^1$, branched at $\lambda = 1/24$ and at $s = \infty$, as depicted below.



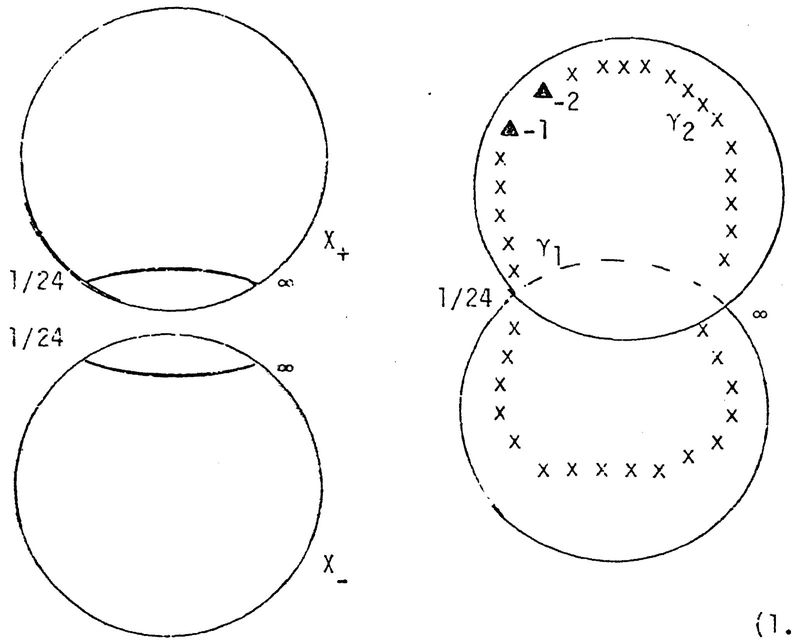
(1.2.12)

Now the study of root loci, is the study of the loci of s on X, for each fixed real positive gain k -- i.e., for each fixed real negative value of $g = -1/k$. Thus, the root locus is simply the arc on X given by g^{-1} (negative real axis) and to see this concretely it's perhaps easiest to study the pair of arcs γ_1, γ_2 given by $s(g^{-1}$ (negative real axis)). On the 2 copies X_+, X_- of the s-plane, branched at $s = 1/24$, $s = \infty$, one sees that these loci start at the open loop poles, $s = -1$, $s = -2$ and move to ∞, the only open loop zero, as follows (note $g_\pm$ is real iff s is real and $s < 1/24$.



(1.2.13)

Thus γ_2 moves, as $0 < g < \infty$, from the pole -2 on X_+ to the branch point ∞, while γ_1 moves from the pole -1 on X_+ to the branch point 1/24, where γ_1 changes sheets, moving to ∞ on X_-. We can describe $X = X_+ \cup X_-$ topologically as a sphere, for



(1.2.14)

One now easily finds the regions of stability:

(a) for $0 \le k < 1.25$, the closed loop system is asymptotically stable.

(b) for $1.25 \le k \le 2.5$, the system is unstable with one pole (on γ_2) in the left-half plane.

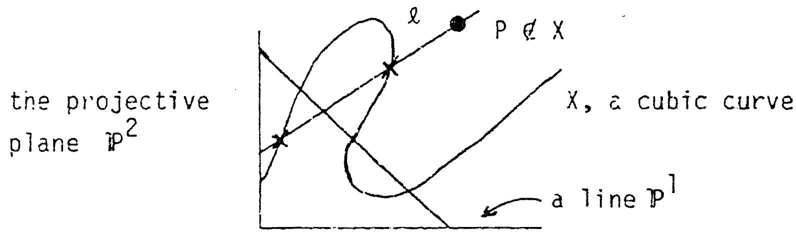
(c) for $2.5 < k < \infty$, the system is stable.

Remark. Branched covers of $\mathbb{P}^n$ by complex manifolds of dimension n play a role in the study of root locus, when one allows arbitrary gains K; see Professor Byrnes's lectures.

Now, there is an alternate route to representing a plane curve as a branched cover of $\mathbb{P}^1$, recall that one may homogenize and projectivize, obtaining the algebraic curve X in $\mathbb{P}^2$ defined by $y^2x - 4z(x^2 - z^2) = 0$, as in (1.2.9). Then choosing any line $\mathbb{P}^1$ and a point P not on $\mathbb{P}^1$ or X, the branched cover of $\mathbb{P}^1$ is gotten by a "central projection" based at P. That is, by Bézout's Theorem any line ℓ through P intersects X in 3 points (counting multiplicity) and $\mathbb{P}^1$ in a single point and therefore defines a function,

$$f : X \to \mathbb{P}^1 \tag{1.2.15}$$

which sends these three points to the corresponding point on $\mathbb{P}^1$. One may calculate that there are 6 branch points on $\mathbb{P}^1$ for which multiplicities occur in $\ell \cap X$, where ℓ joins the branch point to P. [This is as it should be, for $\mathbb{P}^1 \simeq S^2$ is simply connected and therefore does not admit a non-trivial connected covering space.] Note that f has the form $f(x) = [q(x),p(x)]$ in homogeneous coordinates and thus corresponds to the coprime factorization of meromorphic function $f = q/p$ on X.



(1.2.16)

1.3 Invariants

In the final part of this section, we want to return to Cayley's Problem, especially the question of finite generation of the ring of invariants $R^{SL(n,\mathbb{C})}$. Set the notation: $V_{n,r}$ = {r-th degree forms in n-variables}, R is the ring of polynomials on $V_{n,r}$, and $SL(n,\mathbb{C})$ acts on $\mathbb{C}^n$ and therefore on $V_{n,r}$ by composition. $f \in R$ is said to be invariant under $SL(n,\mathbb{C})$ if, and only if, (1.1.2) holds and $S(n,r)$ denotes $R^{SL(n,\mathbb{C})}$. Now, for $n = 2$ the explicit structure of $S(2,r)$ is known for $r = 2,\ldots,8$, the case $r = 2$ being Gauss's Theorem, while the case $r = 8$ was only recently (1964) obtained by Shioda. Gord[illegible] and later Clebsch and Gordan, was able to prove that the ring of $SL(2,\mathbb{C})$ invariants is finitely generated for all r.

Remark. Part of this problem is rather straightforward; i.e., if $R = \Sigma_{m\geq 0} R_m$ is grading of R into homogeneous polynomials of degree m, then since $SL(2,\mathbb{C})$ acts on $V_{2,r}$ by linear transformations $SL(2,\mathbb{C})$ acts on each R_m. In fact, this action is the symmetric tensor representation of $SL(2,\mathbb{C})$ on the space $\mathscr{S}^m(\mathscr{S}^2(\mathbb{C}^2))$ of symmetric tensors. The invariants in R_m correspond to the subspace of $\mathscr{S}^m(\mathscr{S}^2(\mathbb{C}^2))$ on which $SL(2,\mathbb{C})$ acts as the identity and this representation can be decomposed as in the Clebsch-Gordan formula. Moreover, the action of $SL(2,\mathbb{C})$ on $V_{2,r}$ is just the standard irreducible representation of dimension $r+1$. This explains, for example, the absence of any invariants of degree 1 in the ring $\mathbb{C}[\Delta]$. It is now, however, a proof that Δ generates $S(2,2)$. It should be remarked that for $n > 2$, the action of $SL(n,\mathbb{C})$ on R_m is the object of study in the "first main theorem of invariant theory" [2].

Now, in 1892 David Hilbert proved that $S(n,r)$ is finitely generated and, even better, gave a proof that revolutionized commutative algebra. Before sketching a proof, we would like to point out the connection with the construction of moduli (or parameter) spaces--in this case, the moduli space of homogeneous forms. That is, one is interested (as in the case of constructing the space of systems) in regarding $V_{n,r}/SL(n,\mathbb{C})$, the set of equivalence classes of forms modulo a special linear change of coordinates, as a variety or as a manifold in a natural way, viz., so that the map

$$\pi: V_{n,r} \to V_{n,r}/SL(n,\mathbb{C}) \tag{1.3.1}$$

is algebraic or smooth. First of all, the orbits must be closed in $V_{n,r}$ as they are the inverse images of the closed points of $V_{n,r}/SL(n,\mathbb{C})$. Second, if $V_{n,r}/SL(n,\mathbb{C})$ is an affine variety, there must be enough functions

$$f:V_{n,r}/SL(n,\mathbb{C}) \to \mathbb{C} \tag{1.3.2}$$

to separate points and, moreover, this algebra of such f's must be finitely-generated. Such f's are, however, invariant polynomials on $V_{n,r}$ since π in (1.3.1) is assumed to be algebraic. Thus, two necessary conditions for an affine quotient to exist are:

(a) all orbits are closed

(b) $R^{SL(n,\mathbb{C})} = S(n,r)$ is finitely-generated.

Notice that if one had, instead, a compact group G acting on a vector space V, then (a) would be trivial, whereas by "averaging over G" one can always construct enough G-invariant functions to separate orbits. In fact, the existence of a process for averaging over $SL(n,\mathbb{C})$ underlies the validity of Hilbert's Theorem. This fact was brought out quite clearly by Nagata, who gave satisfactory answers to Hilbert's 14th Problem, which is a natural generalization of Cayley's Problem.

1.3.3. Theorem (Hilbert). $S(n,r)$ *is finitely-generated for all* n *and all* r.

Sketch of Proof (from [8]). One first of all has the Hilbert basis theorem: each ideal of R, the ring of polynomials on $V_{n,r}$, is finitely generated--for a proof of this fact, one may refer to Chapter 2, Theorem 2.9. Next, one introduces the Reynolds operators (i.e., averaging over $SL(n,\mathbb{C})$): if V is an $SL(n,\mathbb{C})$-module, then the submodule $V^{SL(n,\mathbb{C})}$ of invariants has a unique $SL(n,\mathbb{C})$-invariant complement $V_{SL(n,\mathbb{C})}$. Alternatively, one has a projection

$$R:V \to V^{SL(n,\mathbb{C})} \tag{1.3.4}$$

commuting with the action of $SL(n,\mathbb{C})$. R is called the Reynolds operator, and could be represented symbolically in a seductive (but formal) way,

$$RV = \int_{SL(n,\mathbb{C})} gv\, dg.$$

By uniqueness, one may deduce that, for I an ideal of $R^{SL(n,\mathbb{C})} = S(n,r)$,

$$(R/IR)^{SL(n,\mathbb{C})} \simeq R^{SL(n,\mathbb{C})}/I \quad . \tag{1.3.5}$$

That is, $I \to IR$ is an injection of the lattice of ideals of $R^{SL(n,\mathbb{C})}$ into the lattice of ideals of R. Hence, $R^{SL(n,\mathbb{C})}$ is Noetherian, by the Hilbert Basis Theorem. In particular, the ideal $\Sigma_{m>0} R_m^{SL(n,\mathbb{C})}$ of $R^{SL(n,\mathbb{C})}$ is finitely generated, say by $x_1,\ldots,x_\ell$. One next proves by induction that monomials in the x_i's generate each homogeneous piece $R_m^{SL(n,\mathbb{C})}$ and therefore $R^{SL(n,\mathbb{C})}$ is finitely generated over $\mathbb{C}$.

It should be emphasized that Hilbert's proof preceded and to a large extent motivated the introduction of chain conditions into ring theory and it should be remarked that the detailed structure of $S(2,6)$ was the subject of E. Noether's thesis.

Finally, one rather interesting and tractable case is $n = 2$, $r = 4$. Here, it is known [8] that $S(2,4) = \mathbb{C}[P,Q]$, where $\deg P = 2$, $\deg Q = 3$. In fact, if $f(x,y) = a_0x^4 + a_1x^3y + a_2x^2y^2 + a_3xy^3 + a_4y^4 \in V_{2,4}$, then Q is defined via

$$Q(f) = \det \begin{pmatrix} a_0 & a_{1/4} & a_{2/6} \\ a_{1/4} & a_{2/6} & a_{3/4} \\ a_{2/6} & a_{3/4} & a_4 \end{pmatrix} \quad . \tag{1.3.6}$$

viz. as the determinant of a Hankel matrix! Moreover, the $SL(2,\mathbb{C})$ action on the space of 3×3 non-singular Hankels can be obtained in terms of control-theoretic scaling actions on the space of Hankel, as in Professor Brockett's lecture. Now, the structure of $S(2,4)$ (indeed of $S(2,2g+2)$) is also of interest in Riemann surface theory. That is, any elliptic curve $\mathcal{E}$ is a 2-sheeted branched cover, with 4 branch points of $\mathbb{P}^1$. In this way, the moduli space of elliptic curves can be represented as the moduli space of 4 unordered points on $\mathbb{P}^1$, up to equivalence under projective automorphisms, i.e., the group $GL(2,C)/\{\alpha I\}$ acting on lines in $\mathbb{C}^2$. Notice, however, that 4 lines in $\mathbb{C}^2$ determine, up to a multiplicative constant, a homogeneous quartic polynomial $f(x,y)$, i.e., a point $f \in V_{2,4}$, while projective equivalence corresponds to equivalence modulo $GL(2,\mathbb{C}) \supset SL(2,\mathbb{C})$.

In this way, an analysis of the $\mathbb{C}^* = \{\lambda Id\}$ action on $V_{2,4}/SL(2,\mathbb{C})$ leads to the construction of the (moduli) space of all elliptic curves. By counting dimensions one sees that such a space must have dimension 1, for $\dim \mathbb{C}^* = 1$ and $\mathrm{tr.deg.}\ \mathbb{C}[P,Q] = 1$. This existence of this one parameter family of elliptic curves (these turn out to be points in $\mathbb{C}$) illustrates the fact that there are too many conformally distinct yet topologically equivalent Riemann surfaces. In fact, Riemann asserted that there are 3g-3 parameters which describe all Riemann surfaces of genus $g > 1$. Another nice extension by Mumford of the work of Hilbert and Nagata enables one, for example, to construct such moduli spaces and therefore to speak about their dimension.

We remark that such problems arise frequently in control theory; for example, in the construction of moduli and canonical forms for linear dynamical systems. Here, one might ask, for fixed n, m, and p and for an arbitrary minimal triple (F,G,H) of these dimensions: do there exist canonical forms (F_C,G_C,H_C) for the action of $G\ell(n)$, via change of basis in the state space, such that the entries of (F_C,G_C,H_C) are algebraic in (F,G,H)? Since the entries of (F_C,G_C,H_C), as it were, are invariant functions (for this $G\ell(n)$ action) one might ask in particular for an explicit description of the ring of invariants. A description of the functions $f(F,G,H)$, invariant under the $G\ell(n)$ action on mixed tensors (F), vectors (G), and co-vectors (H), is well-known classically [5]; viz. the ring of such f's is generated by the entries of the matrices, HF^iG! However, it turns out that, because of the geometry of the moduli space $\{(F,G,H)\}/G\ell(n)$--or, equivalently the geometry of the corresponding space of Hankel matrices--neither algebraic, nor even continuous canonical forms exist (see Professor Hazewinkel's lectures).

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2. MODULES OVER NOETHERIAN RINGS AND PRINCIPAL IDEAL DOMAINS

One of the fundamental steps in the study of automata and linear system theory is the introduction of reduced, or minimal, realizations. These are doubly interesting, since they are unique up to isomorphism and lead to an implementation of the system using a minimum number of certain components.

We all know that in order to carry out such a reduction one must take the subset of the state space consisting of the reachable states. As we shall see when we study the realization theory of linear systems, the size of the realization is directly related to the number of generators of the state module. If we, therefore, believe in the interest and applicability of linear models with coefficients belonging to a ring--and there is good reason to do so--it is vital to know over which rings this reduction process will lead to a physically realizable system (i.e., one with a finitely generated state module) or, even better, to a smaller system (i.e., one with a state module having fewer generators).

2.1 Noetherian Rings and Modules: Fundamental Results

Let R be a commutative ring.

2.1.1 Definition. *A module* M *is Noetherian if every submodule of* M *is finitely generated.*

It follows, of course, that M itself is finitely generated. Since a ring can be viewed as module over itself, its submodules being the ideals, (2.1) subsumes the following

2.2.2 Definition. *A ring* R *is Noetherian if every ideal is finitely generated.*

We shall first prove some elementary properties of Noetherian modules, then show how they relate to Noetherian rings; we shall afterwards prove that a lot of the state modules we shall find in system theory fall into this category.

First of all, there is a characterization available for Noetherian modules.

2.1.3 Theorem. *A module* M *is Noetherian if, and only if, every strictly increasing sequence of submodules*

$$N_1 \subset N_2 \subset \ldots \subset N_i \subset M,$$

is finite.

Proof. Assume first of all that M is Noetherian; then the submodule $N = \bigcup_i N_i$ of M is finitely generated, and these generators lie in one of the N_i's, which is therefore equal to all of N.

Conversely, let N be a submodule of M, S the set of its finitely generated submodules; S is not empty, since it contains $\{0\}$. Let us show that it has a maximal element: indeed, since S is non-empty, we can choose a submodule N_0 in S; if it is not maximal, it is contained in a strictly larger submodule N_1, which is itself either maximal or contained in a strictly larger submodule N_2, etc., ...; the chain thus constructed being finite by assumption, S contains a maximal element. But this maximal element must be N itself, for otherwise we would add another generator of N to it, thereby constructing a larger finitely generated submodule of N. N is, therefore, in S, hence finitely generated.

This property, very useful in practice, is called the *Ascending Chain Condition* (or *A.C.C.*).

2.1.4 Lemma. *The submodules and quotient modules of a Noetherian module are themselves Noetherian.*

Proof. The relation $N \subset M$ for submodules being transitive, the case for submodules follows directly from the definition of Noetherian modules (2.1.1).

Let $L = M/N$, and $L_0 \subset L_1 \subset \ldots$ be a strictly increasing sequence of submodules of L; let $M_0 \subset M_1 \ldots$ be a sequence of representative elements of the equivalence classes in M (i.e., $L_i = M_i/N$); it is strictly increasing. M being Noetherian by assumption, (2.1.3) implies that the sequence is finite. So the original sequence $\{L_i\}$ is finite too, and L is Noetherian.

This lemma has a converse:

2.1.5 Lemma. *Suppose we have three modules and module homomorphisms* $L \xrightarrow{g} M \xrightarrow{h} N$ *such that* $\operatorname{im} g = \ker h$ *(we thus have what is called an exact sequence). Then if both* L *and* N *are Noetherian, so is* M.

Proof. Let $L' = \operatorname{im} g$, $N' = \operatorname{im} h$. (1.4) implies that L', isomorphic to a quotient module of L, and N', being a submodule of N, are both Noetherian. We can write an exact sequence

$$0 \to L' \to M \to N' \to 0 \quad .$$

Let $M' \subset M$ be a submodule of M; we must show that M' is finitely generated. We have an exact sequence

$$0 \to L' \cap M' \to M' \to M'/M' \cap L' \to 0$$

and $L' \cap M'$ and $M'/M' \cap L'$, submodules respectively of L' and N', are Noetherian; they are, therefore, finitely generated, by say $\{\ell_i\}$ and $\{n_j\}$ respectively.

Let x be an element of M'; its image in $M'/M' \cap L'$ is $\sum_j x_j n_j$, $x_j \in R$. If $\{\bar{n}_j\}$ designates a set of pre-images of $\{n_j\}$ in M', then the element $x - \Sigma x_j \bar{n}_j$ of M' is in the kernel of the projection $M' \to M'/M' \cap L'$.

Since the sequence is exact, it is in the image of the injection $L' \cap M' \to M'$, so $x - \Sigma x_j \bar{n}_j = \Sigma y_i \ell_i$, and

$$x = \Sigma y_i \ell_i + \Sigma x_j \bar{n}_j \quad .$$

M' is, therefore, generated by $\{\ell_i, \bar{n}_j\}$, which is a finite set.

2.1.7 Corollary. *A finite direct sum of Noetherian modules is Noetherian.*

Proof. It follows directly from (2.1.5) by induction on the number of direct summands.

We are now in a position to prove the important

2.1.8 Theorem. *Let* R *be a Noetherian ring. Then a module* M *is Noetherian if, and only if, it is finitely generated.*

Proof. It follows directly from definition 2.1.1 that a Noetherian module is finitely generated. Conversely, let M be a finitely generated module; we have an exact sequence

$$0 \to N \to F \to M \to 0$$

where F is the free R-module built on a set of generators of M; F is therefore a finitely generated free module over a Noetherian ring, hence, by (2.1.7), a Noetherian module. M, being a quotient of a Noetherian module, is itself Noetherian by (2.1.4) (note that this also implies that N is Noetherian, hence finitely generated).

2.2 Examples of Noetherian Rings

In particular, the state module of a finite dimensional linear system, being finitely generated, will be Noetherian whenever the ring is Noetherian. It becomes now urgent to exhibit some Noetherian rings, and to show that a large number of the rings we encounter in system-theoretic applications fall indeed in that category.

2.2.1 Definition. *A Principal Ideal Domain (P.I.D.) is an integral domain in which each ideal is principal (i.e. is generated by a single element).*

Since each ideal in a P.I.D. has a single generator, a P.I.D. is an example of a Noetherian ring; so $\mathbb{Z}$, for example, is Noetherian; and so is a field, of course. We can greatly enlarge the class by using the following:

2.2.2 Hilbert Basis Theorem. *A polynomial ring in finitely many unknowns over a Noetherian ring is also Noetherian.*

Proof. Since $R[x_1,\ldots,x_n] = R[x_1,\ldots,x_{n-1}]\,[x_n]$, it is clear by induction that we need only consider the case of polynomial ring in a single indeterminate $R[x]$.

Let I be an ideal in $R[x]$, and A_i the set of leading coefficients of polynomials of degree i in I. Since I is an ideal, A_i is an ideal too; furthermore, $f(x) \in I \Rightarrow xf(x) \in I$, so

$$A_0 \subset A_1 \subset \ldots$$

R being Noetherian, this sequence of ideals becomes eventually constant; let A_n be the maximal element for this chain. By a second application of the Noetherian assumption, we get that each A_i is generated by a finite set of generators $\{a_{ij}\}$, leading coefficients of a set of polynomials $\{f_{ij}\}$ of degree i in I.

Let us show that these polynomials $\{f_{ij}\}$ generate I, and consider an arbitrary polynomial $g(x)$ in I:

$$g(x) = g_m x^m + \ldots + g_1 x + g_0$$

We shall see that there exists a linear combination $h(x)$ of the $f_{ij}(x)$ such that $g(x) - h(x)$ be of strictly lower degree than $g(x)$, thereby establishing the desired result by induction: Since A_n is the maximal ideal in the chain, $g_m \in A_m \subset A_n$, so

$$g_m = \sum_j r_j d_{nj} \quad , \qquad r_j \in R$$

$$h(x) = \begin{cases} x^{m-n}(\sum r_j f_{nj}(x)) & \text{if } m > n \\ \sum f_j f_{mj}(x) & \text{otherwise} \end{cases}$$

is of degree m and has leading coefficient g_m. Thus $h(x)$ is a linear combination over $R[x]$ of the $f_{ij}(x)$'s, and $g(x) - h(x)$ has strictly lower degree.

Polynomial rings over fields or over the integers are therefore Noetherian.

2.3 On Duality and the Structure of Modules over Noetherian Rings

As an exercise in making use of the Noetherian assumption, let us establish two interesting results about Noetherian modules. The first one will be a structure theorem analogous in spirit to the Jordan-Hölder theorem for groups (after all, groups are but modules over the Noetherian ring $\mathbb{Z}$!) The second will give us an introduction to duality theory useful for future lectures in system theory.

2.3.1 Theorem. *Let R be a Noetherian ring and M a finitely generated R-module. Then there is a sequence of submodules*

$$0 = M_0 \subset M_1 \subset \ldots \subset M$$

such that for each i the module M_i/M_{i-1} is isomorphic to R/p_i, where p_i is a prime ideal of R.

Proof. Let S be the set of submodules of M for which the theorem holds. We can select out of S a sequence of strictly increasing elements

$$0 = M_0 \subset M_1 \subset \ldots \subset M_r \subset \ldots$$

By the Noetherian assumption, such a sequence is necessarily finite and ends with a maximal element, say M_r. If $M_r = M$, then we are done. Otherwise, let us show that $N = M/M_r$ contains a submodule isomorphic to R/p for some prime ideal p.

This will be achieved by a second use of the Noetherian assumption. But first note that for a module to contain a submodule isomorphic to R/p is equivalent to saying that p is the annihilator of some element x of the module (i.e., $p = \{r \in R | rx = 0\}$): the map

$$r \mapsto rx$$

which sends R onto the cyclic submodule generated by x has p for kernel, hence that cyclic submodule is isomorphic to R/p; conversely, if the module contains a copy of R/p, then p is the annihilator of its generator.

Let, therefore, F be the family of ideals other than R which annihilate elements of N, and, once again, let I be a maximal element of F. Let us prove that I is prime. Say x is the element in N annihilated by I; then, if $ab \in I$ but $b \notin I$, $bx \neq 0$; any element in I annihilates x, hence bx too, so I contains the annihilator of bx and is equal to it, being a maximal element in F. But $ab \in I \Rightarrow abx = 0$: a annihilates bx, hence is in I. I is therefore prime.

We can now return to the main line of the argument: $N = M/M_r$ contains a submodule isomorphic to R/p, which corresponds to a submodule N' of M containing M_r and such that N'/M_r be isomorphic to R/p; the sequence

$$0 = M_0 \subset M_1 \subset \ldots \subset M_r \subset N'$$

is therefore a strictly increasing sequence in S, contradicting the maximality of M_r.

2.3.2 Definition. *The dual* M *of a module* M *over a ring is the set of module-homomorphisms from* M *into* R.

As long as we limit ourselves to free modules over an integral domain, the theory remains the same as that of vector-space duality: the dual of a finitely generated free module is a finitely generated free module of same rank, and the proofs are the same. When the module is not free anymore, the issue, of course, becomes different; we however still have:

2.3.3 Theorem. *Let* R *be a Noetherian integral domain,* M *a finitely generated* R*-module. Then* M^* *is finitely generated as an* R*-module too.*

Proof. M, being finitely generated, is a quotient of a finitely generated free module L. But $\mathrm{Hom}(\cdot,R)$ is a contravariant left exact functor. Hence

$$L \to M \to 0 \Rightarrow 0 \to M^* \to L^*$$

and M^* is a sub-module of a finitely generated free module. It follows from the Noetherian assumption and (2.1.8) that M^* is a Noetherian module, hence finitely generated.

2.3.4 Definition. *The Krull dimension of a ring* R *is the length* n *of the longest chain*

$$p_0 \subset p_1 \subset \ldots \subset p_n \neq R$$

of prime ideals in R *(infinite if there is no maximal chain).*

2.3.5 Definition. *Let* R *be a finitely generated algebra over a field* k *(i.e., a quotient of a polynomial ring over* k*), which is itself an integral domain. Let* K *be its quotient field. Then the transcendence degree of* R *is the dimension of* K *as a vector-space over* k.

2.3.6 Definition. *Let* M *be an* R*-module. We shall say that* $hd(M) \leq n$ *if there exists a projective resolution of* M

$$0 \to P_n \to \ldots \to P_1 \to P_0 \to M \to 0$$

Then the global dimension of R *is* $\sup \{hd(M) | M : \text{module over } R\}$.

By a projective module is meant a module P which is complemented in a free module, i.e., one for which there exists a splitting

$$M \simeq P \oplus Q$$

where M is free. Note that a module is projective if, and only if, it may be realized as the image of a projection operator

$$M \to P$$

defined on a free module. One can check that if P is itself finitely generated, then M may be taken to be finitely generated and free.

2.3.7 Theorem. *In the case of polynomial rings over a field, all three notions of dimension are equivalent.*

Remark. In particular, a polynomial ring has finite Krull dimension. However, Negata has given an example of a Noetherian ring with infinite Krull dimension.

les Over a Principal Ideal Domain

he ring not only is Noetherian but is a P.I.D., then ay even more about the generators of a submodule:

eorem. *Every submodule of a finitely generated free er a P.I.D. R is a free R-module. (The finitely gen- sumption is not necessary but makes the proof shorter).*

f. Let L be a free module, $\{e_i\}$ a basis for L, corresponding coordinate functions. Let M be a sub- L. The image of M in R by projection p_i is an R, which is principal by assumption, say Ra_i. Let element of M such that $p_i(m_i) = a_i$ (if $a_i = 0$ take

us show that $\{m_i\}$ generates M: if $m \in M$ and $_ia_i$ let $m' = \Sigma r_i m_i$; then $m - m'$ projects to 0 on rdinate, hence is 0.

hermore, the $\{m_i\}$'s are free:

$$\sum r_i m_i = 0 \Rightarrow p_i(\sum r_i m_i) = 0 \Rightarrow r_i a_i = 0, \forall i$$

is different from 0 for $m_i \neq 0$, it follows that

efinition. *An element $m \neq 0$ in M is said to have f there exists $r \neq 0$ in R such that $rm = 0$. If no in M has torsion, then M is called torsion-free.*

emma. *A finitely generated torsion-free module M over al domain R can be embedded in a finitely generated le.*

of. Let K be the quotient field of R, and let $a_n\}$ be generators of M. Let $\{b_1,\ldots,b_\ell\}$ be a basis vector space $M \otimes_R K$ over K. Then

$$a_i = \sum (r_{ij}/s_{ij}) b_j \quad , \qquad r_{ij}, s_{ij} \in R$$

be a common multiple of the s_{ij}'s. Then

$$\{b_1/s,\ldots,b_\ell/s\}$$

being linearly independent over K generates a free R-module which contains M.

2.4.4 Corollary. *Every finitely generated torsion-free module over a P.I.D. is free.*

Proof. This is a direct consequence of (2.4.3) and (2.4.1).

Remark. The finitely generated assumption is crucial. Indeed, the quotient field K of R is a torsion free R-module, but is not free.

The structure theorem we established for Noetherian rings also takes a more powerful form:

2.4.6 Theorem. *Let* R *be a P.I.D. Then any finitely generated module* M *over* R *is isomorphic to a direct sum of sub-modules* R/p^n, p *prime.*

The structure theorem for finitely-generated modules over a P.I.D. is very powerful in studying the algebra of linear maps and, of course, linear systems defined on such modules. Suppose, for example, that we wish to study the linear system

$$x(t+1) = Ax(t) + Bu(t)$$

$$y(t) = Cx(t) \qquad (2.4.7)$$

where $u \in \mathbb{Z}^{(m)}$, $y \in \mathbb{Z}^{(p)}$, $x \in M$ a finitely-generated module over $\mathbb{Z}$. Of course, (A,B,C) we assumed to be $\mathbb{Z}$-linear maps. If (2.4.7) is observable, then M is necessarily free. For, observability implies that M may be imbedded, by successive observations, in the direct sum

$$\bigoplus_{i=1}^{\infty} \mathbb{Z}^{(p)}$$

and, therefore, has no non-zero torsion elements.

As a second illustration, consider an R-linear map

$$P : R^{(n)} \to R^{(n)}$$

which is a projection, i.e. $P^2 = P$. If R is a field, then it is a significant fact in linear algebra that one may choose a basis of $R^{(n)}$ so that the matrix of P, with respect to this basis, has only 1's and 0's on the diagonal and 0's elsewhere. This is not true for all R, but we can give a proof if R is a P.I.D. For, consider $M = P(R^{(n)}) \subset R^{(n)}$ it is finitely generated, as the image of $R^{(n)}$, and has no torsion, so M must be free and one can choose a basis:

$$\text{span } \{x_1, \dots, x_r\} = M, \qquad \text{over } \mathbb{R}$$

Fortunately, one can actually extend this basis, since the same statements are valid for $N = \text{image } (I-P)$ and $N \cap M = (0)$, $N + M = R^{(n)}$.

This basic result is also true for polynomial rings over a field but is a much deeper result than one might first suspect--it used to be known as the Serre conjecture, and has been proven by Quillen and Suslin.

As a final observation, we suppose given a R-linear map

$$T : R^{(n)} \to R^{(\ell)}$$

and ask whether T is injective. Passing to the fraction field K of R, one has an extended K-linear map

$$T_K : R^{(n)} \otimes_R K \to R^{(\ell)} \otimes_R K \quad , \quad \text{or simply}$$

$$T_K : K^{(n)} \to K^{(\ell)}$$

a K-linear map of K-vector spaces, where the question is answered quite easily. Since a non-zero element of M is zero in $M \otimes_R K$ only if it is a torsion element, for a map of free modules over a P.I.D., T is injective if, and only if, T_K is injective.

2.4.8 <u>Example</u>. Consider $T : \mathbb{Z} \to \mathbb{Z}$, $T(z) = 2z$. Then $T : \mathbb{Q} \to \mathbb{Q}$ is both injective and surjective, while T itself is only injective.

3. DIFFERENTIABLE MANIFOLDS, VECTOR BUNDLES, AND GRASSMANNIANS

This section discusses first some of the elements of the theory of differentiable manifolds, then discusses that powerful tool "partitions of unity" and then proceeds to say a few things about vector bundles. One particular family of manifolds, the Grassmann manifolds, have proved to be very important in linear system theory and one particular vector bundle over the manifolds enjoys a similar status. The last two almost telegraphic subsections are intended to indicate that this phenomenon is not peculiar to system theory: these manifolds and bundles play an equally distinguished role in the general theory of vector bundles itself, a feat which may help to understand the role they play in system theory.

There are many books and and lecture notes in which the theory of manifolds and vector bundles is clearly explained. Some of the present writer's favorites are:

M. F. Atiyah, *K-Theory*, Harvard Lecture Notes, Fall 1964. (Published by Benjamin)

S. Helgason, *Differential geometry, Lie groups and symmetric spaces*, Acad. Pr., 1978. (on press)

F. Hirzebruch, *Introduction to the theory of vector bundles and K-theory*, Lectures at the University of Amsterdam and Bonn, University of Amsterdam, 1965.

D. Husemoller, *Fibre bundles*, McGraw-Hill, 1966.

J. W. Milnor, J. D. Stasheff, *Characteristic classes*, Princeton University Press, 1974.

The last one named is especially recommended. Finally at a more introductory level recommended

L. Auslander, R. E. MacKenzie, *Introduction to differentiable manifolds*, Dover (reprint) 1977.

3.1 Differentiable Manifolds

3.1.1 Definition (Differentiable maps). Let $U \subset \mathbb{R}^m$ and $V \subset \mathbb{R}^n$ be open subsets. A mapping $\phi : U \to V$ is differentiable if the coordinates $y_i(\phi(x))$ of $\phi(x)$ are differentiable functions of $x = (x_1, \ldots, x_m) \in U \subset \mathbb{R}^m$, $i = 1, \ldots, n$. Here a function is said to be differentiable if all partial derivatives of all orders exist and are continuous. The differentiable mapping ϕ if it is 1-1, onto and if ϕ^{-1} is also differentiable.

3.1.2 Definition (Charts). Let M be a Hausdorff topological space. An *open chart* on M is a pair (U,ϕ) consisting of an open subset U of M and a homeomorphism ϕ of U onto some open subset of an $\mathbb{R}^m$; the number m is called the *dimension* of the chart.

3.1.3 Definition (Differentiable manifolds). Let M be a Hausdorff space. A *differentiable structure* on M consists of a collection of open charts (U_i,ϕ_i), $i \in I$ such that the following conditions are satisfied

$$\bigcup_{i \in I} U_i = M \tag{3.1.4}$$

for all $i,j \in I$ the mapping $\phi_j \circ \phi_i^{-1}:\phi_i(U_i \cap U_j) \rightarrow$

$\phi_j(U_i \cap U_j)$ is a diffeomorphism. (3.1.5)

The collection (U_i,ϕ_i), $i \in I$ is maximal with respect to properties (3.1.4) and (3.1.5) (3.1.6)

A *differentiable manifold* is a Hausdorff topological space together with a differentiable structure.

Locally it is just like $\mathbb{R}^n$, but globally not. The charts permit us to do (locally) calculus and analysis as usual. It is possible that one and the same topological space admits several inequivalent differentiable structures where inequivalent means "non diffeomorphic"--a notion which is defined below in 3.2.

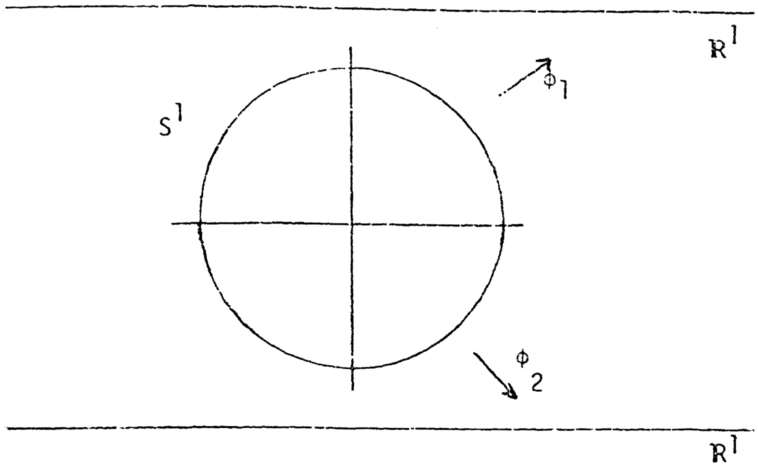
If M is connected as a topological space, all the charts (U_i,ϕ_i) necessarily have the same dimension which is then also (by definition) the dimension of the differentiable manifold M.

Often a differentiable structure is defined by giving a collection of charts (U_i,ϕ_i) such that only (3.1.4) and (3.1.5) are satisfied. Then there is a unique larger collection of charts such that also (3.1.6) holds. (Easy exercise.)

3.1.7 Example: The circle. Consider the subset of $\mathbb{R}^2$ defined by

$$S^1 = \{(x_1,x_2)\ x_3^2 + x_2^2 = 1\} \qquad (3.1.8)$$

Let $U_1 = \{x \in S^1 | x \neq (0,-1)\}$, $U_2 = \{x \in S^1 | x \neq (0,1)\}$.



Now define $\phi_1 : U_1 \to \mathbb{R}^1$ by $\phi_1(x_1,x_2) = \frac{x_1}{1+x_2}$ and $\phi_2 : U_2 \to \mathbb{R}^1$ by $\phi_2(x_1,x_2) = \frac{x_1}{1-x_2}$. These are both homeomorphisms. The inverse of ϕ_1 is given by the formula $x \mapsto (x_1,x_2)$ with

$x_1 = \frac{2x}{x^2+1}$, $x_2 = \frac{1-x^2}{1+x^2}$ and the inverse of ϕ_2 by the very similar formula: $x \mapsto (x_1,x_2)$, $x_1 = \frac{2x}{1+x^2}$, $x_2 = \frac{x^2-1}{1+x^2}$

The map $\phi_2 \cdot \phi_1^{-1} : \phi_2(U_1 \cap U_2) = \mathbb{R}^1 \backslash \{0\} \to \phi_2(U_1 \cap U_2) = \mathbb{R}^1 \backslash \{0\}$ is given by $x \to x^{-1}$ and hence is a diffeomorphism, so that (U_1,ϕ_1) and (U_2,ϕ_2) do indeed define a differentiable structure on S^1.

3.1.9 Trivial Example: $\mathbb{R}^n$ Itself. Let $M = \mathbb{R}^n$ and define a chart (U,ϕ) by $U = M = \mathbb{R}^n$, ϕ = identity map : $U \to \mathbb{R}^n$. This one element collection of charts satisfies, of course, (3.1.4) and (3.1.5), and hence defines a differentiable structure on $\mathbb{R}^n$.

3.1.10 Constructing differentiable manifolds 1. Embedded manifolds. The example above illustrates one way in which differentiable manifolds often arise. Namely the topological space M is given as a "smooth" subset of some $\mathbb{R}^n$ and the differentiable structure is induced from the natural differentiable structure of $\mathbb{R}^n$. Indeed, apart from a factor 2 the maps ϕ_1 and ϕ_2 of example 3.1.7 above arise by projecting the circle from $(0,-1)$ onto the line $x_2 = 1$ in $\mathbb{R}^2$ and by projecting the circle from $(0,1)$ onto the line $x_2 = 1$ in $\mathbb{R}^2$.

Abstractly a smoothly embedded differentiable manifold of dimension m is a subset $M \subset \mathbb{R}^n$ (for some n) such that for each $x \in M$ there is a differentiable map $\psi: V \to \mathbb{R}^n$ defined on some open subset $V \subset \mathbb{R}^m$ such that

ψ maps V homeomorphically onto some open neighborhood U of x in M (3.1.11)

for each $y \in V$ the matrix $\left(\frac{\partial \psi_i}{\partial y_j}(y)\right)$, $i = 1,\ldots,n$, $j = 1,\ldots,m$ has rank m. (3.1.12)

It is not difficult to prove (using the implicit function theorem) that the pairs (U, ψ^{-1}) for varying x now define a differentiable structure on M; i.e., that these pairs constitute a collection of charts which satisfy (3.1.4) (3.1.5). Inversely it is a theorem (Whitney) that every differentiable manifold with a countable basis arises in this way (up to diffeomorphism).

3.1.13 Constructing differentiable manifolds 2. Gluing. A second very frequently used method of obtaining a differentiable manifold is by a gluing procedure. Suppose that we have for each i in some index set I (often a finite set) some open set $U_i \subset \mathbb{R}^m$. Suppose moreover that for each $i,j \in I$, $i \neq j$, there are defined open subsets $U_{ij} \subset U_i$ and $U_{ji} \subset U_j$ and a diffeomorphism $\phi_{ij}: U_{ij} \to U_{ji}$. Suppose more over that the following compatibility conditions hold

$U_{ii} = U_i$, ϕ_{ii} = identity for all $i \in I$

and for all $i,j,k \in I$

$$U_{ij} \cap \phi_{ij}^{-1}(U_{jk}) \subset U_{ik} \text{ and } \phi_{jk} \cdot \phi_{ij} = \phi_{ik} \text{ on } U_{ij} \cap \phi_{ij}^{-1}(U_{jk}). \quad (3.1.14)$$

(Note that this implies that $\phi_{ij} = \phi_{ji}^{-1}$). Then we define a topological space M by taking the disjoint union $\amalg U_i$ and then identifying $x \in U_i$ with $y \in U_j$ iff $y = \phi_{ij}(x)$, $x \in U_{ij}$, $y \in U_{ji}$. This is an equivalence relation because of (3.1.14). Let M be the topological space $\amalg U_i/\sim$ with the quotient topology, where $\sim$ denotes the equivalence relation just defined.

Let $\phi_i : U_i \to \amalg U_i \to \amalg U_i/\sim$ be the obvious map. Suppose that M is Hausdorff (this is not automatically the case), then the (U_i, ϕ_i) are a collection of charts satisfying (3.1.4) and (3.1.5) so that they define a differentiable structure on M.

3.1.15 <u>Example: real n-dimensional projective space</u>. Let $I = \{0,1,\dots,n\}$ and for each $i \in I$ let $U_i = \mathbb{R}^n$, and for each $i \in I$ let $\alpha_i : U_i \to \mathbb{R}^{n+1}$ be the embedding $\alpha_i(x_1,\dots,x_n) = (x_1,\dots,x_i,1,x_{i+1},\dots,x_n)$. Label the coordinates of $\mathbb{R}^{n+1}$ by $0,1,\dots,n$. Thus $\alpha_i(U_i) = \{y = (y_0,\dots,y_n) \in \mathbb{R}^{n+1} | y_i = 1\}$. Let $i,j \in I$, $i \neq j$ and define U_{ij} as $\alpha_i^{-1}V_{ij}$ where $V_{ij} = \{y \in \mathbb{R}^{n+1} | y_i = 1, y_j \neq 0\}$, and define $\phi_{ij} : U_{ij} \to U_{ji}$ as the composite $\alpha_j^{-1} \cdot \psi_{ij} \cdot \phi_i$, where $\psi_{ij} : V_{ij} \to V_{ji}$ is defined by $\psi_{ij}(y_0,\dots,y_n) = (y_j^{-1}y_0,\dots,y_j^{-1}y_n)$. (Note that indeed $\psi_{ij}(V_{ij}) = V_{ji}$, so that $\phi_{ij}(U_{ij}) = U_{ji}$.

The compatibility conditions (3.1.14) hold and the topological space M is Hausdorff. Thus then gluing data define a differentiable manifold which is denoted $\mathbb{P}^n(\mathbb{R})$ and called real n-dimensional projective space.

Consider the differentiable manifold $X = \mathbb{R}^{n+1}\setminus\{0\}$. For each $y \in X$, $y = (y_0,\dots,y_n)$ choose an i such that $y_i \neq 0$. Now define $\pi : X \to \mathbb{P}^n(\mathbb{R})$ by assigning to y the equivalence class of $\alpha_i^{-1}(y_0y_i^{-1},\dots,y_{i-1}y_i^{-1},1,y_{i+1}y_i^{-1},\dots,y_ny_i^{-1})$. Note that

$\pi(y)$ does not depend on the choice of i. It is now an easy exercise to check that $\pi(y) = \pi(y')$ if and only if there is an $\lambda \neq 0$ such that $y_i' = \lambda y_i$, $i = 0,\ldots,n$. Thus, the construction above defines as a differentiable manifold structure on the set of all lines through the origin of $\mathbb{R}^{n+1}$.

3.1.16 Grassmann manifolds. Let $1 \leq k < n$, $k,n \in \mathbb{N}$. Then $\mathcal{G}_{k,n}$ is by definition the set of all k-dimensional subspaces of $\mathbb{R}^n$. This set can be given a differentiable manifold structure in a manner rather similar to the one used above in (3.1.15). For explicit details see section 4 of this chapter.

3.1.17 Morphisms of manifolds: differentiable mappings. Let M and N be two differentiable manifolds. Let (U_i,ϕ_i) and (V_j,ψ_j) be collections of charts for M and N respectively such that (3.1.4) and (3.1.5) hold. A map $\phi : M \to N$ is a *morphism of differentiable manifolds* or a *differentiable mapping* if for all $i \in I$, $j \in J$ the map

$$\psi_j \circ \phi \circ \phi_i^{-1} : \phi_i(U_i \cap \phi^{-1}(V_j) \to \psi_j(V_j)$$

is a differentiable map in the sense of 3.1.1 above. A differentiable mapping ϕ which is 1-1 and onto and such that ϕ^{-1} is also a differentiable mapping is called a *diffeomorphism*.

3.1.18 Example. Give $X = \mathbb{R}^{n+1}\setminus\{0\}$ the differentiable structure defined by the one element collection of charts $U = X$, ϕ = identity. Then $\pi : X \to \mathbb{P}^n(\mathbb{R})$ as defined in example 3.1.15 above is a differentiable mapping.

3.1.19. Differentiable map and gluing data. Suppose the two differentiable manifolds M and N have been obtained by means of the procedure discussed above in 3.1.13 from the local pieces U_i and patching data ϕ_{ij} (resp. local pieces V_k and patching data $\psi_{k\ell}$). Then a frequently used method of specifying a differentiable map $\alpha: M \to N$ is as follows. For each i and k let there be given an open subset $U_{ik} \subset U_i$ and a differentiable map (in the sense of 3.1.1)

$$\alpha_{ik} : U_{ik} \to V_k, \qquad \bigcup_k U_{ik} = U_i$$

Suppose that the following compatibility condition holds where appropriate

$$\psi_{k\ell} \circ \alpha_{ik} = \alpha_{j\ell} \circ \phi_{ij} \tag{3.1.20}$$

i.e. if $n \in U_{ik}$ and $y \in U_{j\ell}$ and $\phi_{ij}(n) = y$, then $\alpha_{ik}(n) \in V_k$, $\alpha_{j\ell}(y) \in V_\ell$ and $\psi_{k\ell}(\alpha_{ik}(n)) = \alpha_{j\ell}(k)$. Then the α_{ik} combine to define a differentiable map $\alpha : M \to N$ as is easily checked.

3.1.21 Example. Consider $\mathbb{P}^1(\mathbb{R})$ as defined in 3.1.15 above. Now define $\alpha : \mathbb{P}^1(\mathbb{R}) \to \mathbb{R}^2$ as follows

$$\alpha_1 : U_1 = \mathbb{R} \to \mathbb{R}^2, \quad x_0 \to \left(\frac{2x_0}{x_0^2+1}, \frac{1-x_0^2}{x_0^2+1}\right)$$

$$\alpha_0 : U_0 = \mathbb{R} \to \mathbb{R}^2, \quad x_i \to \left(\frac{2x_2}{x_2^2+1}, \frac{x_1^2-1}{x_1^2+1}\right)$$

Recall that $U_{10} = \{x_0 \in \mathbb{R} | x_0 \neq 0\}$, $U_{01} = \{x_1 \in \mathbb{R} | x_1 \neq 0\}$ and that the gluing map ϕ_{10} is given by $\phi_{10}(x_0) = x_0^{-1}$. And we check that on U_{10}

$$\alpha_0\phi_{10}(x_0) = \alpha_0(x_0^{-1}) = \left(\frac{2x_0^{-1}}{x_0^{-2}+1}, \frac{x_0^{-2}-1}{x_0^{-2}+1}\right)$$

$$= \left(\frac{2x_0}{1+x_0^2}, \frac{1-x_0^2}{1+x_0^2}\right) = \alpha_1(x_0)$$

so that the compatibility condition (3.1.20) is fulfilled, and the α_0, α_1 do indeed combine to define a differentiable map $\alpha : \mathbb{P}^1(\mathbb{R}) \to \mathbb{R}^2$. Note that $\alpha(\mathbb{P}^1(\mathbb{R})) = S^1 = \{(x_1, x_2) \in \mathbb{R}^2 | x_1^2 + x_2^2 = 1\}$. The map α is also 1-1 and surjective onto S^1 and the inverse map $\alpha^{-1} : S^1 \to \mathbb{P}^1(\mathbb{R})$ is also differentiable. Thus α induces a diffeomorphism of $\mathbb{P}^1(\mathbb{R})$ with the circle S^1.

3.1.22. Products. Let M and N be differentiable manifolds of dimension m and n respectively. Then the cartesian product $M \times N$ has a natural differentiable structure defined as follows. Let (U_i, ϕ_i), $i \in I$ be a collection of open charts for M such that (3.1.4) and (3.1.5) hold; and let (V_j, ψ_j), $j \in J$ be a similar collection for N. Then the open sets $U_i \times V_j$, $i \in I$, $j \in J$ cover the topological space $M \times N$ and the maps ϕ_i and

ψ_j combine to define a homeomorphism $\phi_i \times \psi_j : U_i \times V_j \to \phi_i(U_i) \times \psi_j(V_j) \subset \mathbb{R}^m \times \mathbb{R}^n$. This defines a collection of charts $(U_i \times V_j, \phi_i \times \psi_j)$, $i \in I$, $j \in J$ which satisfies (3.1.4) and (3.1.5) and hence defines a differentiable structure on $M \times N$.

If both M and N are embedded manifolds, cf. 3.1.10 above, say, $M \subset \mathbb{R}^r$, $N \subset \mathbb{R}^s$, then $M \times N \subset \mathbb{R}^r \times \mathbb{R}^s = \mathbb{R}^{r+s}$ is naturally again an embedded manifold.

If both M and N are obtained by a local pieces and gluing data construction $M \times N$ can be described in a similar way. Indeed, if (U_i, U_{ij}, ϕ_{ij}) describe M and $(V_k, V_{k\ell}, \psi_{k\ell})$ the manifold N then M N is described by the local pieces and gluing data $(U_i \times V_k,\ U_{ij} \times V_{k\ell},\ \phi_{ij} \times \psi_{k\ell})$.

3.1.23. Example. $\mathbb{P}^1(\mathbb{R}) \times \mathbb{P}^1(\mathbb{R})$. According to the recipe above $\mathbb{P}^1(\mathbb{R}) \times \mathbb{P}^1(\mathbb{R})$ is obtained by gluing together four local pieces

$$U_1 \times V_1 = \mathbb{R} \times \mathbb{R},\ U_1 \times V_0 = \mathbb{R} \times \mathbb{R},\ U_0 \times V_1 = \mathbb{R} \times \mathbb{R},$$
$$U_0 \times V_0 = \mathbb{R} \times \mathbb{R}$$

by means of the following six diffeomorphsims (and their inverses)

$$\mathrm{id} \times \psi_{10} : U_1 \times V_{10} \to U_1 \times V_{01},\quad (x_0, y_0) \to (x_0, y_0^{-1})$$

$$\phi_{10} \times \mathrm{id} : U_{10} \times V_1 \to U_{01} \times V_1,\quad (x_0, y_0) \to (x_0^{-1}, y_0)$$

$$\phi_{10} \times \psi_{10} : U_{10} \times V_{10} \to U_{01} \times V_{01},\ (x_0, y_0) \to (x_0^{-1}, y_0^{-1})$$

$$\phi_{10} \times \psi_{01} : U_{10} \times V_{01} \to U_{01} \times V_{10},\ (x_0, y_1) \to (x_0^{-1}, y_1^{-1})$$

$$\phi_{10} \times \mathrm{id} : U_{10} \times V_0 \to U_{01} \times V_0,\quad (x_0, y_1) \to (x_0^{-1}, y_1)$$

$$\mathrm{id} \times \psi_{10} : U_0 \times V_{10} \to U_0 \times V_{01},\quad (x_1, y_0) \to (x_1, y_0^{-1})$$

Let us use this description to define a morphism $\alpha : \mathbb{P}^1(\mathbb{R}) \times \mathbb{P}^1(\mathbb{R}) \to \mathbb{P}^3(\mathbb{R})$ as follows. Recall that $\mathbb{P}^3(\mathbb{R})$ is built out of four pieces $W_i = \mathbb{R}^3$, $i = 0,1,2,3$; cf. 3.1.15. We define α by means of the maps

$$\alpha_1 : U_1 \times V_1 \to W_3,\ (x_0,y_0) \to (x_0y_0, x_0, y_0)$$

$$\alpha_2 : U_1 \times V_0 \to W_2,\ (x_0,y_1) \to (x_0, x_0y_1, y_1)$$

$$\alpha_3 : U_0 \times V_1 \to W_1,\ (x_1,y_0) \to (y_0, x_1y_0, x_1)$$

$$\alpha_4 : U_0 \times V_0 \to W_0,\ (x_1,y_1) \to (y_1, x_1, x_1y_1)$$

It is now easy to check that the compatibility conditions 3.1.20 are satisfied. For example that $\alpha_2 \cdot (\mathrm{id} \times \psi_{10}) = \chi_{32} \cdot \alpha_1$ is illustrated by the diagram below (there χ_{32} is the gluing diffeomorphism $W_{32} \to W_{23}$ of 3.1.15 above and we use (for convenience) the embedding $W_i \to \mathbb{R}^4$ which we also used in 3.1.15).

$$\begin{array}{ccccc}
(x_0,y_0) & \overset{\alpha_1}{\mapsto} & (x_0y_0, x_0, y_0) & \leftrightarrow & (x_0y_0, x_0, y_0, 1) \\
\downarrow \mathrm{id} \times \psi_{10} & & \downarrow \chi_{32} & & \downarrow \\
(x_0, y_0^{-1}) & \overset{\alpha_2}{\mapsto} & (x_0, x_0y_0^{-1}, y_0^{-1}) & \leftrightarrow & (x_0, x_0y_0^{-1}, 1, y_0^{-1})
\end{array}$$

The morphism constructed above in such painful detail is a very well known one. If we view $\mathbb{P}^n(\mathbb{R})$ as the set of all lines through the origin in $\mathbb{R}^{n+1}$, i.e. as equivalence classes of points in $\mathbb{R}^{n+1}$ under the equivalence relation $(x_0,\ldots,x_n) \sim (x_0^1,\ldots,x_n^1)$ iff $\exists\lambda \neq 0$ such that $x_i^1 = \lambda x_i$, $i = 0,\ldots,n$, then $\alpha : \mathbb{P}^1(\mathbb{R}) \times \mathbb{R}^1(\mathbb{R}) \to \mathbb{P}^3(\mathbb{R})$ is induced by $((x_0,x_1),(y_0,y_1)) \to (x_0y_0, x_0y_1, x_1y_0, x_1y_1)$ and from this the explicit local pieces description above is easily deduced.

3.1.24. <u>Submanifolds</u>. Let M be a differentiable manifold of dimension n. A subset $N \subset M$ is a submanifold of dimension $p \leq n$ if there exists for every $x \in N$ an open chart $\phi : U \to \mathbb{R}^n$ such that $\phi(x) = 0$ and the $V = \{x \in U \mid \phi_{p+1}(x) = \ldots = \phi_n(x) = 0\}$ together with the restriction of ϕ to V (as a map to $\mathbb{R}^p$) form a system of open charts for N. The differentiable manifold N is said to be a *regular submanifold* of M if for every $x \in M$ there is a U as above such that moreover $V = N \cap U$. (V as above).

An example is $S^1 = \{(x_1 x_2) | x_1^2 + x_2^2 = 1\} \subset \mathbb{R}^2$. This is a regular submanifold (Exercise: prove this.) This winding line (with irrational winding angle) on a torus is an example of a nonregular submanifold. (The torus T^2 is the differentiable manifold $S^1 \times S^1$ and $\mathbb{R}$ can be seen as a subset of $S^1 \times S^1$ by mapping t to $(e^{2\pi i t}, e^{2\pi i \alpha t})$, α irrational: note that the induced topology on $\mathbb{R}$ from this injection into T^2 is not the original topology of $\mathbb{R}$. $N \subset M$ is a regular submanifold the induced topology on N is indeed original topology (belonging to the differentiable structure) of N.

3.1.25. Analytic manifolds. Similarly to differentiable manifolds one can define *analytic manifolds* by replacing everywhere differentiable map by analytic map. Thus an analytic manifold is locally like $\mathbb{R}^m$ and the local coordinate transition mapping $\psi_j \circ \phi_i^{-1}$, cf. 3.1.3, are analytic, i.e. they admit (locally) convergent power series expansions.

To define *complex manifolds* replace $\mathbb{R}$ by $\mathbb{C}$ everywhere and require that the coordinate transition mappings $\phi_j \circ \phi_i^{-1}$ are holomorphic.

3.2. Partitions of unity

A powerful and often used tool in differential topology are partitions of unity.

3.2.1 Some definitions and facts from general topology. Recall that a covering $\{U_i, i \in I\}$ of a topological space X is said to be locally finite if for every $x \in X$ there is an open neighbourhood V containing x such that $U_i \cap V \neq \phi$ for only finitely many i. Recall also that a topological space is *paracompact* if every covering admits a locally finite refinement. A space is *normal* if for all closed $A, B \subset X$ there are open $U, V \subset X$ such that $A \subset U$, $B \subset V$ and $U \cap V = \phi$. A locally compact Hausdorff space with countable base is paracompact and every paracompact space is normal.

3.2.2. Convention. We shall assume from now on that every differentiable manifold is paracompact. This is not automatically the case, though it is not easy to construct counterexamples. If M is built up out of countably many $U_i \subset \mathbb{R}^m$ by a local pieces and gluing data procedure as in 3.1.13 above it is automatically paracompact (by the remarks made above). Thus manifolds like spheres, projective space, Grassmannians are all paracompact.

3.2.3 Theorem. *Let* M *be a paracompact differentiable manifold and let* $\{U_i \mid i \in I\}$ *be a locally finite open covering of* M. *Assume that all* $\bar{U}_i$ *are compact. Then there exists a collection* $\{\phi_i \mid i \in I\}$ *of differentiable functions on* M *such that*

$$\mathrm{Supp}(\phi_i) \subset U_i \tag{3.2.4}$$

$$\phi_i(x) \geq 0 \quad \text{for all } x \in M \tag{3.2.5}$$

$$\sum_i \phi_i(x) = 1 \quad \text{for all } x \in M \tag{3.2.6}$$

Here $\mathrm{Supp}(\phi)$ is the closure of the set of all $x \in M$ such that $\phi(x) \neq 0$. Note that in the sum (3.2.6) for all x there are only finitely many i such that $\phi_i(x) \neq 0$ (because the covering is locally finite and because of (3.2.6)) so that this sum makes sense.

3.3 Vectorbundles

3.3.1 Definition (real vector bundles). An n-dimensional real vector bundle over a topological space X is a topological space E together with a continuous map $\pi : E \to X$ (called the projection on X) such that

For each $x \in X$, $\pi^{-1}(x)$ is (equipped with a structure of) a real n-dimensional vector space (3.3.2)

For every $x \in X$ there is an open neighborhood U of x such that $\pi^{-1}(U)$ is isomorphic to $U \times \mathbb{R}^n$, (3.3.3)

where with this last phrase we mean that there is a homeomorphism $\phi : \pi^{-1}(U) \to U \times \mathbb{R}^n$ such that the following diagram commutes

$$\begin{array}{ccc} \pi^{-1}(U) & \xrightarrow{\quad\phi\quad} & U \times \mathbb{R}^n \\ \big\downarrow \pi & & \big\downarrow p_1 \\ U & \xrightarrow{\quad\mathrm{id}\quad} & U \end{array}$$

where p_1 is the projection onto the first factor, and such that moreover $\phi : \pi^{-1}(x) \to \{x\} \times \mathbb{R}^n$ is an isomorphism of vector $x \in U$, (where, of course, $\{x\} \times \mathbb{R}^n$ is given the vectorspace structure arising from identifying $\{x\} \times \mathbb{R}^n$ with $\mathbb{R}^n$ in the obvious way).

The vectorspace $\pi^{-1}(x) \subset E$ is called the *fibre* of the vector bundle over x and is often denoted E_x.

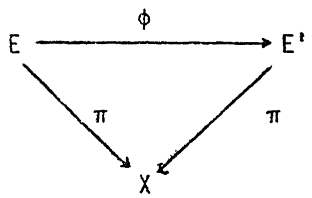
3.3.4 <u>Example (trivial bundle).</u> $E = X \times \mathbb{R}^n \xrightarrow{\pi} X$, where π is projection on the first vector.

3.3.5 <u>Example (Tangent bundle of S^2).</u> Consider $S^2 = \{(x_1,x_2,x_3) \mid x_1^2+x_2^2+x_3^2 = 1\}$ and consider in $S^2 \times \mathbb{R}^3$ the subspace E defined by

$$E = \{(x,v) \in S^2 \times \mathbb{R}^3 \mid x_1v_1 + x_2v_2 + x_3v_3 = 0\} \qquad (3.3.6)$$

and define $\pi : E \to S^2$ by $(x,v) \mapsto x$. For each fixed $x \in E$ the set $\pi^{-1}(x) = E_x$ consists of all v satisfying the equation $x_1v_1 + x_2v_2 + x_3v_3 = 0$. Now give E_x the vectorspace structure of this subspace of $\mathbb{R}^3$. We check that property (3.3.3) holds. Let $x \in S^2$, then at least one of the x_i is $\neq 0$, say, x_1. Let $U = \{x \in S^2 \mid x_1 \neq 0\}$. Now define $\phi : U \times \mathbb{R}^2 \to \pi^{-1}(X)$ by $(x,(w_1,w_2)) \mapsto (x,(-x_1(x_2,w_1 + x_3w_2),w_1w_2))$. This ϕ is an isomorphism as required in (3.3.3).

3.3.7 <u>Homomorphisms of vector bundles.</u> Let $\pi : E \to X$, $\pi' : E' \to X$ be two vector bundles over X. A *homomorphism of vector bundles* is a continuous map $\phi : E \to E'$ such that the following diagram commutes



and such that the induced map $\phi_x : E_x \to E'_x$ are homomorphisms of vector spaces. The homomorphism ϕ is called an *isomorphism* if the maps $E_x \to E'_x$ are all isomorphisms.

Thus, for example, the map ϕ in (3.3.3) above is an isomorphism of the vector bundle $\pi : \pi^{-1}(U) \to U$ with the bundle $p_1 : U \times \mathbb{R}^n \to U$. A vector bundle which is isomorphic to one as in example 3.3.4 is called *trivial*.

3.3.8 Constructing vector bundles 1: local pieces and gluing data. Property (3.3.3) shows that every vector bundle can be obtained (up to isomorphism) by gluing trivial bundles together. In detail this goes as follows. Let X be a topological space and $\{U_i\}$, $i \in I$ an open covering of X. Suppose we have for each $i,j \in I$ a continuous map

$$\phi_{ij} : U_i \cap U_j \to G\ell_n(\mathbb{R}) \tag{3.3.9}$$

where $G\ell_n(\mathbb{R})$ is the (Lie) group of all invertible real $n \times n$ matrices.

We now require the ϕ_{ij} to be compatible in the following sense

$$\phi_{ii}(x) = I_n, \text{ the } n \times n \text{ unit matrix for all } n \in U_i$$
(3.3.10)
$$\phi_{jk}(x)\phi_{ij}(x) = \phi_{ik}(x) \quad \text{for all } x \in U_i \cap U_j \cap U_k$$

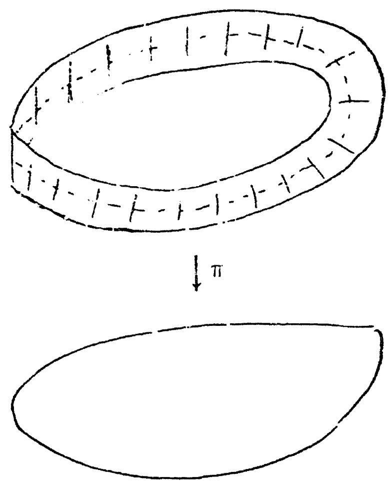
From these data we can construct a vector bundle E over X as follows. Take the disjoint union $\cup! U_i \times \mathbb{R}^n$. Now define an equivalence relation $\sim$ as follows. The element $(x,v) \in U_i \times \mathbb{R}^n$ is equivalent to $(y,w) \in U_j \times \mathbb{R}^n$ if $x = y$ in X and $\phi_{ij}(x)v = w$. Let $E = \cup! U_i \times \mathbb{R}^n/\sim$ and let π be induced by $(x,v) \to x$. The local trivialization maps required in (3.3.3) are given by $U_i \times \mathbb{R}^n \subset \cup! U_i \times \mathbb{R}^n \to E$, and these also define the vectorspace structures on the fibres.

3.3.12 Example. Consider $\mathbb{P}^1(\mathbb{R})$ as the set of all lines through zero in $\mathbb{R}^2$, i.e. as the set of all ratios $(x_0:x_1)$, $x_0,x_1 \in \mathbb{R}$ $(x_0,x_1) \neq (0,0)$. Let $U_0 = \{x \in \mathbb{P}^1(\mathbb{R}) | x_0 \neq 0\}$, $U_1 = \{x \in \mathbb{P}^1(\mathbb{R}) | x_1 \neq 0\}$. Define $\phi_{01} : U_0 \cap U_1 \to G\ell_1(\mathbb{R})$ by $\phi(x_0:x_1) = x_1^{-1}x_0^1$. Set $\phi_{10} = \phi_{01}^{-1}$ and the compatibility conditions (3.3.11) hold. Let E be the resulting vector bundle. We claim that E is nontrivial. Indeed suppose E were trivial, then there would be an isomorphism $\phi: E \to \mathbb{P}^1(\mathbb{R}) \times \mathbb{R}$ compatible with the projections and hence there would be a map $s : \mathbb{P}^1(\mathbb{R}) \to E$ defined by $s(x) = \phi^{-1}(x,1)$ which satisfies

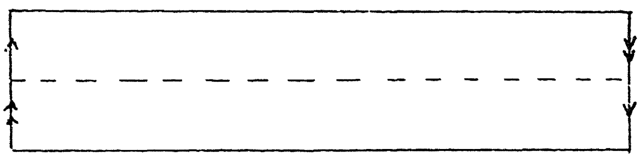
$$\pi \cdot s = id \tag{3.3.13}$$

and which is moreover such that $s(x) \neq 0 \in E_x$ for all x. Now $U_0 = \{(1:x_1) | x_1 \in \mathbb{R}\}$, $U_1 = \{(x_0:1) | x_0 \in \mathbb{R}\}$. From the construction of E we know that a map s satisfying (3.3.13) is given by two functions $f_1 : x_1 \to f_2(x_1)$, $f_0 : x_0 \to f_0(x_0)$ such that moreover $x_1^{-1} f_1(x_1) = f_0(x_1^{-1})$ for $x_1 \neq 0$. The requirement $s(x) \neq 0$ $\forall x$ means that $f_i(x_i) \neq 0$ $\forall x_i$. Hence by continuity $f_1(x_1)$ has the same sign for all x_1 and $f_0(x_0)$ has the same sign for all x_0. This, however, is incompatible with $x_1^{-1} f_1(x_1) = f_0(x_1^{-1})$.

A picture of this bundle is the so-called Möbius band



where π is the projection on the central circle. (The Möbius band is obtained by taking a rectangular strip of paper twisting it around once and gluing the ends together (as indicated below).



3.3.14 Linear constructions. Linear algebra or more precisely the category of finite dimensional vector spaces has many constructions which assign a new vector space to a set of one or more old vector spaces. Such a functor T is called continuous if the associated map $T : \mathrm{Hom}(V,W) \to \mathrm{Hom}(T(V),T(W))$ is continuous, where for simplicity we have taken a covariant functor in one variable. These constructions extend to constructions for vector bundles by simply performing the construction pointwise for every fibre. Thus given two vector bundles E,F over X one has e.g. the new vector bundles

$E \oplus F$, the direct sum of E and F

$E \otimes F$, the tensor product of E and F

$\mathrm{Hom}(E,F)$, the bundle over X where fibre over x is $\mathrm{Hom}(E_x,F_x)$

E^*, the dual bundle over X whose fibre over x is $\mathrm{Hom}(E_x,\mathbb{R})$

$\lambda^i(E)$, the i-th exterior power of X

A similar remark holds with respect to the natural isomorphisms of linear algebra. So one has e.g. $\mathrm{Hom}(E,F) \simeq E^* \otimes F$.

3.3.15 Sections. Let E be vector bundle over X. A continuous *section* of E is a continuous map $s : X \to E$ such that $\pi \cdot s = \mathrm{id}_X$. The set of sections forms a vector space (pointwise addition and scalar multiplication) which is denoted ΓE or $\Gamma(E;X)$.

In example 3.3.12 we showed that for every section s of the Möbius band bundle there is an $x \in \mathbb{P}^1(\mathbb{R})$ such that $s(x) = 0$ thus proving that this bundle is nontrivial. (A trivial bundle clearly has sections which are everywhere nonzero. Exercise: Let E be an n-dimensional vector bundle over X. Suppose that there are n continuous sections $s_1,\dots,s_n$ such that $s_1(x),\dots,s_n(x)$ are linearly independent vectors in E_x for all $x \in X$. Prove that E is trivial.)

It is worth noting that $\Gamma\mathrm{Hom}(E,F)$ is the vector space of vector bundle homomorphisms $E \to F$ (cf. 3.3.14 and 3.3.7; Exercise: Prove this.)

3.3.16 Example. Tangent bundle of a manifold. Let M be an m-dimensional differentiable manifold. Let (U_i,ϕ_i), $i \in I$ be a collection of charts such that (3.1.4),(3.1.5) hold. We now construct a bundle over M by the local pieces and patching data descriptions of 3.3 above. To this end define

$$\phi_{ij} : U_i \cap U_j \to G\ell_n(\mathbb{R})$$

by the formula

$$\phi_{ij}(x) = J(\phi_j \cdot \phi_i^{-1})(\phi_i(x))$$

where the symbol on the right is the Jacobian matrix of the diffeomorphism $\phi_j \cdot \phi_i^{-1}$ evaluated at $\phi_i(x)$. Note that the compatibility condition (3.3.11) follows from the chain rule.

The fibre of this bundle over $x \in M$ is called the tangent space of M and x and is denoted T_xM.

The bundle itself is denoted $TM \to M$, or simply TM. We can view the whole bundle $TM \to M$ as obtained by a local pieces and gluing data procedure as follows.

Consider the open pieces $\phi(U_i)$, $i \in I$ (where the U_i are as above). Now consider the pieces

$$\phi_i(U_i) \times \mathbb{R}^n, \quad i \in I$$

and we write an element of this set as a 2n-tuple

$$(x_1,\ldots,x_n,a_1,\ldots,a_n)^T$$

The total space TM of the tangent bundle of M is now obtained by gluing together the $\phi(U_i) \times \mathbb{R}^n$ by means of the isomorphisms

$$\phi_{ij} : \phi_i(U_i \cap U_j) \times \mathbb{R}^n \to \phi_j(U_i \cap U_j) \times \mathbb{R}^n$$

$$(x,a) \to ((\phi_i \cdot \phi_j^{-1})(x), \quad J(\phi_j \cdot \phi_i^{-1})(x)(a))$$

These identifications are compatible with the projections ,

$$\phi_i(U_i) \times \mathbb{R}^n \to \phi_i(U_i), \quad (x,a) \to x .$$

and thus the whole bundle $TM \to M$ is described.

Note that these considerations make it clear that TM is itself a differentiable manifold and that $\pi : TM \to M$ is a differentiable map. We can thus speak of differentiable sections.

3.3.17 Vector fields. Let $TM \to M$ be the tangent bundle of a differentiable manifold M. A differentiable section (cf. 3.3.15 above) of this bundle is called a vector field. In terms of local pieces and gluing data such a section thus is given by differentiable functions

$$a(i) : \phi_i(U_i) \to \mathbb{R}^n$$

(the local pieces of the section are then given by $\phi_i(U_i) \to \phi_i(U_i) \times \mathbb{R}^n$, $x \to (x, a(i)(x))$. Then functions must then satisfy the compatibility condition

$$\mathcal{J}(\phi_j \cdot \phi_i^{-1})(x)(a(i)(x)) = a(j)(\phi_j \cdot \phi_i^{-1}(x)) \; . \qquad (3.3.18)$$

3.3.20 Derivations. Let A be an algebra over a field K. (Take $K = \mathbb{R}$ or $\mathbb{C}$ if desired.) A *derivation* of A is a K-linear map $D : A \to A$, such that $D(fg) = f(Dg) + (Df)g$.

3.3.21 Vector fields as derivations. Let M be a differentiable manifold and let $S(M)$ be the ring of differentiable functions on M. Let s be a differentiable section of the tangent bundle $TM \to M$. We claim that s defines a derivation of $S(M)$. Indeed let s be given by the function $s(i) : \phi_i(U_i) \to \mathbb{R}^n$. A differentiable function on M can be viewed as a collection of functions $f(i) : \phi(U_i) \to \mathbb{R}$, $f(i) = f \cdot \phi_i^{-1}$, satisfying the compatibility condition

$$f(j)(\phi_j \cdot \phi_i^{-1}(x)) = f_i(x), \quad x \in \phi_i(U_i \cap U_j) \qquad (3.3.22)$$

Now define the collection of functions $g(i) : \phi(U_i) \to \mathbb{R}$ by the formula

$$g(i)(x) = \sum_{k=1}^{n} s(i)(x)_k \frac{\partial f(i)}{\partial x_k}(x) = \frac{\partial f(i)}{\partial x} s(i)(x)$$

where $s(i)(x)_k$ is the k-th component of column vector $s(i)(x)$, and $\frac{\partial f(i)}{\partial x}$ is the row vector $\left(\frac{\partial f(i)}{\partial x_i}(x), \ldots, \frac{\partial f(i)}{\partial x_n}(x)\right)$. We now claim that the $g(i)$ satisfy the compatibility condition (3.3.22). Indeed from (3.3.22) we find by the chain rule that (writing y for $(\phi_j \cdot \phi_i^{-1})(x)$)

$$\frac{\partial f(j)}{\partial y}(y) = \frac{\partial f(i)}{\partial x}(x) \quad \mathcal{J}(\phi_i \cdot \phi_j^{-1})(y)$$

Therefore

$$g(j)(y) = \frac{\partial f(j)}{\partial y} s(j)(y)$$
$$= \frac{\partial f(i)}{\partial x}(x)\, \mathcal{J}(\phi_i \cdot \phi_j^{-1})(y)\, \mathcal{J}(\phi_j \cdot \phi_i^{-1})(x)\, s(i)(x)$$
$$= g(i)(x)$$

because $\mathcal{J}(\phi_i \cdot \phi_j^{-1})(y)\, \mathcal{J}(\phi_j \cdot \phi_i^{-1})(x) = I_n$ and the compatibility relation (3.3.18). Inversely every derivation defines a vector field.

3.3.24 <u>The Lie bracket</u>. Let D_1, D_2 be two derivations of an algebra over $\mathbb{R}$ (or any other field). Then (as easily checked)

$$[D_1, D_2] = D_1 D_2 - D_2 D_1$$

is again a derivation. Now let s_1, s_2 be vector fields on a differentiable manifold M, with corresponding derivations D_1, D_2. Then the vector field corresponding to the derivation $[D_1, D_2]$ is denoted by $[s_1, s_2]$ is called the *Lie bracket* of the vector fields s_1 and s_2. The vector field $[s_1, s_2]$ can be calculated in terms of local pieces as follows: Let s_1 and s_2 be given locally by the functions $s_2(i), s_1(i) : \phi_i(U_i) \to \mathbb{R}^n$. Then $[s_1, s_2]$ is given by the functions

$$a(i) : {}_i(U_i) \quad \mathbb{R}^n,$$
$$a(i)(x) = (\mathcal{J}s_1(x)(x)s_2(i)(x)) - (\mathcal{J}s_2(i)(x)s_1(i)(x))$$

which in slightly less precise notation can be written

$$\frac{\partial s_1(i)}{\partial x} s_2(i) - \frac{\partial s_2(i)}{\partial x} s_1(i)$$

3.3.26 <u>Exercise</u>. Check that the $a(i)$ of (3.3.25) satisfy the compatibility relation (3.3.18) and that the derivation operator defined by these $a(i)$ according to (3.3.23) is indeed the derivation $D_1 D_2 - D_2 D_1$.

3.3.27 <u>Constructing homomorphisms by local pieces and patching data</u>. Let E and F be two vector bundles over a topological space X, both given in terms of local pieces and gluing data. Then often, a homomorphism $E \to F$ is easiest described in terms of local pieces too. Suppose for simplicity that the local pieces describing E and F are with respect to the same covering U_i.

(This can always be assured by taking a common refinement of the coverings defining E and F.) Let $\mathbb{R}^{m\times n}$ be the space of $m \times n$ matrices, let E and F be described by

$$\phi_{ij} : U_i \cap U_j \to G\ell_n(\mathbb{R}), \; \psi_{ij} : U_i \cap U_j \to G\ell_m(\mathbb{R})$$

then a homomorphism $\alpha : E \to F$ is unique described by a family of maps

$$\alpha_i : U_i \to \mathbb{R}^{m\times n}$$

such that for all $i,j \in I$ and $x \in U_i \cap U_j$

$$\psi_{ij}(x)\alpha_i(x) = \alpha_j(x)\phi_{ij}(x) \tag{3.3.28}$$

3.3.29 Metrics. If V is a vector space, let $Q(v)$ be the vector space of all quadratic forms on V. This is an example of a continuous functor in the sense of 3.3.14 above. Thus given a vector bundle E over X there is an associated vector bundle $Q(E)$ whose fibre over x is the space $Q(E_x)$ of all quadratic forms on E_x. A *metric* on E is now a section s of $Q(E)$ such that $s(x)$ is positive definite for all $x \in X$.

In more down to earth terms this means the following. Let E be built out of trivial pieces with respect to the covering U_i. Let $\phi_{ij} : U_i \cap U_j \to G\ell_n(\mathbb{R})$ be the gluing maps. Then a metric on E consists of continuous maps

$$s_i : U_i \to P_n$$

where P_n is the space of all positive definite quadratic forms on $\mathbb{R}^n$ such that

$$\phi_{ij}^T(x)s_j(x)\phi_{ij}(x) = s_i(x) \tag{3.3.30}$$

for all $x \in U_i \cap U_j$, where the upper T denotes "transpose."

It remains to show that every vector bundle over suitable, say, paracompact or compact, spaces admits a metric. This goes as follows. Let the covering $\{U_i\}$ be locally finite and let $\{\psi_i\}$ be a partition of unity with respect to $\{U_i\}$. For each $i \in I$ choose some positive definite form Q_i and define

$$s_j : U_j \to P_n, \quad s_j(x) = \sum_k (\phi_{kj}(x)^T)^{-1}\psi_k(x)Q_k\phi_{kj}(x)^{-1}$$

(Note that the expression on the right hand side as a converse linear combination of positive definite quadratic forms is positive definite). These mappings satisfy the compatibility condition (3.3.30) and hence define a metric.

3.3.31 Subbundles and quotient bundles are direct summands. Let $E \xrightarrow{\pi} X$ be a vector bundle. A *subbundle* is a subset $F \subset E$ such that the restriction of π to F makes F a vector bundle and such that $F \hookrightarrow E$ is a homomorphism of vector bundles. If $F \hookrightarrow E$ is a subbundle we can consider the union $\bigcup_x E_x/F_x$ with the induced topology. There is a natural projection onto X defined by $E_x/F_x \ni v \mapsto x$ and using the obvious quotient vector space structures on E_x/F_x the result is a vector bundle over X which is called a *quotient bundle* and is denoted E/F.

Now let $F \subset E$ be a subbundle. Let s be a metric on E. For each $x \in X$ let $G_x = \{v \in E_x | \langle v, F_x \rangle_x = 0\}$ where $\langle\, ,\rangle_x$ denotes the inner product on E_x defined by $s(x)$. Then $\cup G_n$ is a subvector bundle of E and $E = F \oplus G$ so that every subbundle is a direct summand. Analogously if $\alpha : E \to F$ is a homomorphism of vector bundles such that $E_x \to F_x$ is surjective for all x, then there exists a vector bundle homomorphism $\beta : F \to E$ such that $\alpha \cdot \beta = id_F$.

3.3.32 Finite generation of vector bundles. Let $\pi : E \to X$ be an m-dimensional vector bundle over a compact space X. Let $\{U_i | i = (1,\ldots,n)\}$ be a finite open covering of X such that E is trivial over all U_i. For each i let $s_{ij} : U_i \to E|_{U_i}$ be m sections of $\pi^{-1}(U_i) \to U_i$ such that for all $x \in U_i$ the vectors $s_{ij}(x)$, $u = 1,\ldots,m$ form a basis for $\pi^{-1}(x) = E_x$. Now let $\{\phi_i\}$ be a partition of unity with respect to U_i. Then we claim that the maps

$$\phi_i s_{ij}, \quad i \in I, \quad j = 1,\ldots,m$$

(defined by $\phi_i s_{ij}(x) = \phi_i(x) s_{ij}(x)$ if $x \in U_i$, $\phi_i s_{ij}(x) = 0$ if $x \notin U_i$) are continuous sections and are such that for each $x \in X$ the $(\phi_i s_{ij})(x)$ generate E_x. Indeed for each $x \in X$ there is an i_0 such that $\phi_{i_0}(x) \neq 0$ and then the

$\phi_{i_0}(x)\phi_{i_0 j}(x)$, $u = 1,\ldots,m$ generate E_x.

3.3.33 <u>Corollary.</u> *Every vector bundle over a compact space is quotient of a finite dimensional trivial bundle.*

Indeed we have seen above that if $E \to X$ is a vector bundle, there exists a finite number of sections $s_1,\dots,s_r$ such that the $s_1(x),\dots,s_r(x)$ generate E_x for all x. Now define $\alpha : X \times \mathbb{R}^r \to E$ by $\alpha(x_1(a_1,\dots,a_r)) = \Sigma a_i s_i(x)$. Then $\alpha : X \times \mathbb{R}^r \to E$ is a homomorphism of vector bundles (exercise) and surjective making E a quotient of $X \times \mathbb{R}^r$ (and hence by 3.3.21) also a direct summand.

3.3.34 <u>Differentiable bundles.</u> A differentiable vector bundle is a vector bundle $\pi : E \to X$ such that E, X are differentiable manifolds and π is a differentiable mapping. Analytic bundles are defined similarly. An example of a differentiable bundle is the tangent bundle $TM \to M$ of a differentiable manifold.

3.3.35 <u>Vector bundles and projective modules.</u> Let M be a differentiable manifold. Then $S(M)$ denotes the ring of differentiable functions on M (pointwise multiplication and addition). Now let $E \to M$ be a differentiable vector bundle over M. Let $s : M \to E$ be a differentiable section of E and $f \in S(M)$. Then for all $m \in M$, $f(m)s(m) \in E_m$ is well defined and this makes the vector space of all differentiable sections a module over the ring $S(M)$. By 3.3.33 and 3.3.31 (or rather their differentiable analogues (which also hold) then modules are direct summand of free modules (the module of sections of $M \times \mathbb{R}^r \to M$ is, of course, $S(M)^r$) and hence then modules are projective modules. Thus giving us a correspondence between differentiable vector bundles over M and finitely generated projective modules over $S(M)$.

Similarly vector bundles over a suitable topological space X correspond to finitely generated projective modules over the ring of continuous functions on X and in algebraic geometry algebriac vector bundles over an affine variety $\mathrm{Spec}(R)$ correspond to finitely generated projective modules over R.

3.3.36 <u>The pullback construction. (Constructing vector bundles 2).</u> Let $\pi : E \to X$ be a vector bundle and let $f : Y \to X$ be a continuous map. Consider

$$E' = \{(e,y) \in Y \times E \mid \pi(e) = f(y)\}$$

There is a natural projection $\pi' : E' \to Y$ defined by $\pi(e,y) = y$. for a fixed $y \in Y$ we have

$$(\pi')^{-1}(y) = \{(e,y) \mid \pi(e) = f(y)\} = E_{f(y)} \times \{y\}$$

which we give the vector space structure of $E_{f(y)}$. Then $\pi' : E' \to Y$ is a vector bundle over Y which is called the pull-back of E along f and which is denoted $f^!E$.

In words $f^!E$ is the vector bundle over Y whose fibre over $y \in Y$ is the fibre of E over $f(y)$.

If E is obtained by patching together local trivial pieces over U_i, $i \in I$ by means of gluing data

$$\phi_{ij} : U_i \cap U_j \to G\ell_m(\mathbb{R})$$

then $f^!E$ is obtained by patching together trivial pieces over the open subsets $f^{-1}(U_i)$, $i \in I$ by means of the gluing data

$$f^{-1}(U_i) \cap f^{-1}(U_j) \xrightarrow{f} U_i \cap U_j \xrightarrow{\phi_{ij}} G\ell_m(\mathbb{R}) .$$

From both descriptions it is obvious that if $g : Z \to Y$ is another continuous map then

$$(f\cdot g)^!E = g^!(f^!E) \qquad (3.3.37)$$

3.3.38 Bundle morphisms covering a continuous map. Let $E \to M$ and $F \to N$ be two vector bundles and let $f : M \to N$ be a continuous map. A bundle morphism covering f is a continuous map $\tilde{f} : E \to F$ such that the following diagram is commutative

$$\begin{array}{ccc} E & \xrightarrow{\tilde{f}} & F \\ \downarrow & & \downarrow \\ M & \xrightarrow{f} & N \end{array}$$

and such that the induced maps $\tilde{f}_x : E_x \to F_{f(x)}$ are homomorphisms of vector spaces. There is an obvious 1-1 correspondence between bundle morphisms $E \to F$ covering f and homomorphisms of vector bundles over M from E to $f^!F$. (Exercise)

By now it should be obvious how to describe a bundle morphism covering a continuous map in terms of local pieces and gluing data (Exercise)

3.3.39 Example (Jacobians). Let M and N be differentiable manifolds of dimension m and n, $f : M \to N$ a differentiable map. Let (U_i,ϕ_i) and (V_i,ϕ_i) be coordinate charts for M and N and suppose that $f(U_i) \subset V_i$. Let $f(i) = \psi_i \cdot f \cdot \phi_i^{-1}$:

$\phi_i(U_i) \to \psi_i(V_i)$. Recall (cf. 3.3.16) that the tangent bundles TM and TN can be obtained by gluing together the $\phi_i(U_i) \times \mathbb{R}^m$ and $\psi_i(V_i) \times \mathbb{R}^n$. Define

$$df(i) : \phi_i(U_i) \times \mathbb{R}^m \to \psi_i(V_i) \times \mathbb{R}^n$$

by the formula

$$(x,v) \to (f(i)(x), Jf(i)(x)(v))$$

Note that this is compatible with the gluing data for TM and TN so that the df(i) combine to define a differentiable map

$$df : TM \to TN$$

which is (obviously) a morphism of bundles covering f. The induced maps

$$df_x : T_xM \to T_{f(x)}N$$

is called the differential of f at $x \in M$. If M and N are themselves open subsets of $\mathbb{R}^m$ and $\mathbb{R}^n$ then $df_x : \mathbb{R}^m \to \mathbb{R}^n$ is given by the Jacobian matrix of f at x.

3.3.40 Submanifolds (2). Let M,N be differentiable manifolds. A differentiable mapping $f : M \to N$ is *regular* at x if df_x has rank max(m,n). The manifold M is a submanifold of N if $M \subset N$ set theoretically, $\dim M \le \dim N$ and the inclusion $M \to N$ is a regular differentiable map.

3.4 On Homotopy

3.4.1 Definitions. Two continuous maps $f,g : X \to Y$ are called *homotopic* if there exists a continuous map $F : X \times [0,1] \to Y$ such that $F(x,0) = f(x)$, $F(x,1) = g(x)$ for all $x \in X$.

For $t \in [0,1]$, let $F_t(x) = F(x,t)$. Then the intuitive picture is that f can be continuously deformed into g via the F_t, $0 \le t \le 1$, ($F_0 = f$, $F_1 = g$).

3.4.2 Theorem. *Let* $\pi : E \to X$ *be a vector bundle and let* $f,g : Y \to X$ *be two homotopic continuous maps. Then the pullback bundles* $f^!E$ *and* $g^!E$ *are isomorphic over* Y.

3.5 Grassmannians and Classifying Vector Bundles

3.5.1 Grassmann manifolds. Consider the set $G_n(\mathbb{R}^{n+k})$ of n-dimensional subvector spaces of $\mathbb{R}^{n+k}$. Let $\mathbb{R}^{n\times(n+k)}_{reg}$ be the set of all $n\times(n+k)$ matrices of rank n. There is a natural map $\mathbb{R}^{n\times(n+k)}_{reg} \to G_{n}(\mathbb{R}^{n+k})$ which assigns to an $n\times(n+k)$ matrix A of rank n the n-dimensional subspace of $\mathbb{R}^{n+k}$ spanned by the rows of A. We give $G_n(\mathbb{R}^{n+k})$ the quotient topology. There is a natural differentiable manifold structure on $G_{n,n+k}$ which is described in detail in section 4 of this Introduction (in terms of local pieces and gluing data).

There is a natural embedding $\varepsilon_k : G_n(\mathbb{R}^{n+k}) \hookrightarrow G_n(\mathbb{R}^{n+k+1})$ induced by the map $\mathbb{R}^{n\times(n+k)}_{reg} \to \mathbb{R}^{n\times(n+k+1)}_{reg}$ which adds a column of zeros to an $n\times(n+k)$ matrix A of rank n. We let G_n denote the inductive limit space $\varinjlim_k G_n(\mathbb{R}^{n+k})$. The space G_n can perfectly well be seen as the space of all n-dimensional vector subspaces of $\mathbb{R}^\infty = \{(x_1,x_2,x_3,\ldots) | x_i \in \mathbb{R}$, all but finitely many n_i are zero$\}$.

3.5.2 The "universal" bundle ξ_n. Define

$$\xi_n = \{(x,v) \in G_n(\mathbb{R}^{n+k}) \times \mathbb{R}^{n+k} | v \in x\} \tag{3.5.3}$$

There is a natural projection $\xi_n \to G_n(\mathbb{R}^{n+k})$ defined by $(x,v) \to x$ and it is easily seen that this makes ξ_n into a vector bundle whose fibre over $x \in G_n(\mathbb{R}^{n+k})$ "is" the vector space x. A description of this vector bundle in terms of local pieces and gluing data can be found in section 3.4.5 of Professor Hazewinkel's lectures in this volume.

3.5.4 Exercise (easy). Let $\varepsilon_k : G_n(\mathbb{R}^{n+k}) \to G_n(\mathbb{R}^{n+k+1})$ be the embedding described above in 3.5.1 and let ξ_n and ξ'_n be the universal bundles as described above in 3.5.2 over $G_n(\mathbb{R}^{n+k})$ and $G_n(\mathbb{R}^{n+k+1})$ respectively. Then $\varepsilon_k^! \xi'_n = \xi_n$. (This also justifies the notation used).

3.5.5 Classifying vector bundles. Let $\alpha : X \to G_n(\mathbb{R}^{n+k})$ be a continuous map. Then this gives a vector bundle $\alpha^! \xi^n$ over X and homotopic maps give rise to isomorphic vector bundles.

Moreover if k is big enough (and X compact) all vector bundles over X are (up to isomorphism) obtained in this way. The construction which assigns a map into some Grassmann manifold to a bundle over X goes as follows. Let $E \to X$ be a vector bundle. Then there is an $r \in \mathbb{N}$ and a surjective homomorphism of vector bundles $\phi : X \times R^r \to E$ (cf. 3.3.33 above). Now define $f(x)$ to be the n-dimensional subspace of $\mathbb{R}^r$ consisting of all vectors which are orthogonal to the kernel of $\phi_x : \mathbb{R}^r \to E_x$.

These remarks form the bare bones of the classifying theorem for vector bundles which states that over suitable spaces X

$$[X,G_n] = B_n(X) \quad (3.5.6)$$

where $[X,G_n]$ is the set of homotopy classes of continuous maps $X \to G_n = G_n(\mathbb{R}^\infty)$ and where $B_n(X)$ is the set of isomorphism classes of n-dimensional vector bundles over X. Roughly one can say that if one knows the n-dimensional universal vector bundle ξ_n over $G_n(R^{n+k})$, k large, that one knows all n-dimensional vector bundles.

4. VARIETIES, VECTOR BUNDLES, GRASSMANNIANS AND INTERSECTION THEORY

In this chapter we will define some of the basic ideas and objects needed for the application of algebraic geometry in systems theory. The material parallels the development of differential topology developed in section 3. We will describe the concepts of affine space, affine varieties, projective spaces and projective varieties. The Grassmannian manifolds will be developed with some care and the various representations that have proven so useful in linear systems theory will be given.

4.1 Affine Spaces and Affine Algebraic Varieties

Let k be an algebraically closed field (for our purposes we can almost always assume that k is the field of complex numbers $\mathbb{C}$). Let k^n denote the point set of n-tuples. We say that a subset X of k^n is *closed* if there are finitely many polynomials $g_1(x_1,\ldots,x_n),\ldots g_m(x_1,\ldots,x_m)$ such that $X = \{x \in k^n : g_1(x) = \ldots = g_m(x) = 0\}$. The set of all closed sets defines a topology on k^n, called the Zariski topology. (The fact that this is a topology is nontrivial--a consequence of the Hilbert Basis Theorem). A closed subset $X \subset k^n$ is given the induced topology and is called an affine algebraic set.

Let $g(x_1,\ldots,x_n)$ be a polynomial and define a set $X_g = \{x \in X : g(x) \neq 0\}$. Clearly the sets X_g are open and form a basis for the topology of X. A regular function on X_g is a function f with domain X_g and range k such that f can be written as $h(x)/g^m(x)$ for some polynomial h and all x. So f is represented by a rational function having no poles on X_g. Let U be an arbitrary open set in X and f a map from U to k. Since U is open, it's the union of X_g's and we say that f is regular if the restriction of f to each X_g is regular. This sequence appears over and over in geometry. We define something simple, then build an object from the simple things and extend the definition.

A closed algebraic set $X \subset k^n$ along with its regular functions on open sets is an affine algebraic variety. An open subset of X together with the ring of regular functions is called a quasi-affine algebraic variety. In the special case that $X = k^n$ the affine variety is denoted by $\mathbb{A}^n$--the affine space of dimension n.

Let $U \subseteq X \subseteq \mathbb{A}^n$ and $V \subseteq Y \subseteq \mathbb{A}^m$ be open subsets of affine varieties X and Y. A map g from U to V is a morphism from U to V if there exist m regular functions $g_1,\ldots,g_m$ defined on U such that $g(x) = (g(x_1),\ldots,g(x_n))$.

In particular, the "coordinate ring" R_X of functions regular on all of X may be thought of in the following seemingly coordinate-dependent way. If $X \subset \mathbb{A}^N$ is an affine algebraic set, then R_X consists of the ring of functions which are restrictions to X of polynomials on $\mathbb{A}^n$. The point is that the ring R_X is intrinsic, i.e., independent of the particular presentation $X \subset \mathbb{A}^N$. Thus, R_X contains not only X as an abstract object (Hilbert Nullstellensatz) but also all possible embeddings of X in affine space. For $X = \mathbb{A}^n$, $R_X = k[x_1,\ldots,x_n]$ which is Noetherian, since k is, by the Hilbert Basis Theorem (2.2.2). More generally, $X \subset \mathbb{A}^n$ gives rise to an algebra homomorphism, restriction,

$$\rho_X : k[x_1,\ldots,x_n] \to R_X \tag{4.1.1}$$

which exhibits R_X as a quotient of $k[x_1,\ldots,x_n]$. Therefore, by lemma 2.1.5, R_X is Noetherian. In this light, it is interesting to examine the geometric content of the ascending chain condition. For affine space $\mathbb{A}^n$, any subvariety $X \subset \mathbb{A}^n$, gives rise to an ideal I_X, viz. the kernel of ρ_X

$$\ker \rho_X = \{f \in k[x_1,\ldots,x_n] \mid f|_X = 0\} \tag{4.1.2}$$

By the Hilbert Basis Theorem, $I_X = (f_1,\ldots,f_m)$ and one sees that X is in fact defined by the equations

$$f_1(x) = \ldots = f_m(x) = 0 \tag{4.1.3}$$

Moreover, this correspondence reverses inclusion; that is to say, if $X \subset Y$ then $I_Y \subset I_X$. Therefore, the ascending chain condition on ideals implies the descending chain condition on subvarieties of $\mathbb{A}^n$. This is true, by similar reasoning, for any affine variety Z.

4.1.4 <u>Theorem</u>. *If Z is an affine algebraic variety, then every descending chain $Z_1 \supset Z_2 \supset \ldots \supset Z_m \supset \ldots$ of subvarieties of Z terminates.*

In one considers the special case in which Z_i is obtained from Z_{i-1} by imposing an additional algebraic constraint

$$f_i(z) = 0 ,$$

then (4.1.4) asserts that no Z can satisfy infinitely many independent constraints. The key to formalizing this notion of independence lies in the concept of dimension.

4.2 Projective space, projective varieties, and quasi-projective varieties

Again let k be an algebraically closed field. Define an equivalence relation on $k^{n+1}\setminus\{(0,\ldots,0)\}$ defining $x \sim y$ iff there is a $\lambda \in k$ such that $\lambda x = y$. Denote the point set of equivalence classes by $\mathbb{P}^n(k)$. Recall that a polynomial g is homogeneous if there is an integer m such that $g(\lambda x) = \lambda^m g(x)$ for all x. We say that a subset X of $\mathbb{P}^n(k)$ is closed if there is a finite set of homogeneous polynomials $g_1,\ldots,g_m$ such that $X = \{[x] \in \mathbb{P}^n(k): g_1(x) = \ldots = g_m(x) = 0\}$. Note that because of homogeneity $g(x) = 0$ implies $g(\lambda x) = 0$ and hence the definition is well founded. The set of closed sets defines a topology on X and this topology is also referred to as the Zariski Topology.

The projective spaces can be developed more prosaically, if k is $\mathbb{C}$, as a compact differentiable manifold. Let V be $\mathbb{C}^{n+1}$ considered as a vector space over $\mathbb{C}$. Let $\mathbb{P}^n(\mathbb{C})$ denote the set of one dimensional subspaces of V. We define open sets in $\mathbb{P}^n(\mathbb{C})$ as follows. Let W be a subspace of V of dimension n and let $U = \{\ell \in \mathbb{P}^n(\mathbb{C}): \ell \cap W = \{0\}\}$. We say that U is open in $\mathbb{P}^n(\mathbb{C})$ and we let $\mathbb{P}^n(\mathbb{C})$ have the topology generated by the U's. This definition coincides with the previous definition for if W is of dimension n then W is the kernel of a non-zero linear functional and hence is the zero set of a homogeneous polynomial. The other direction is more difficult.

We will see later when we discuss the Grassmannian manifolds that the U's can be identified with the affine spaces $\mathbb{C}^n$ exhibiting $\mathbb{P}^n(\mathbb{C})$ as a complex manifold. $\mathbb{P}^1(\mathbb{C})$ can be identified with Riemann sphere or with the real sphere S^2. In a later section we will develop the Grassmannians with more detail. We will also show that $\mathbb{P}^n(\mathbb{C})$ is compact as a manifold in the manifold topology. Note that the Zariski topology is a subtopology of the manifold topology.

Closed subsets of $\mathbb{P}^n(k)$ are called projective varieties and if V is an open subset of a projective variety X then we call V a quasi-projective variety.

We need to extend the definition of regular functions to projective varieties. Note that we have $n+1$ "canonical" embeddings of $\mathbb{A}^n$ into $\mathbb{P}^n(k)$. Define $j_i(x_1,\dots,x_n) = [(x_1,\dots,x_{i-1},1,x_i,\dots,x_n0)]$. The map j_i is continuous with respect to the Zariski topology and the image of j_i is an open subset of $\mathbb{P}^n(k)$ (it coincides with the open sets defined in the Grassmannian setup).

Let X be a projective variety contained in $\mathbb{P}^n(k)$ and let U be an open subset of X. The set $j_i^{-1}(X) \subseteq \mathbb{A}^n$ is closed for each i and $j_i^{-1}(U)$ is an open subset of $j_i^{-1}(X)$. A regular function f from U to k is defined to be a map such that the composite map from $j_i^{-1}(U) \xrightarrow{j_i} U \to k$ is a regular function on $j_i^{-1}(U)$ for all i.

Morphisms between quasi-projective varieties are defined similarly. First let U be a quasi-projective variety such that $U \subseteq X \subseteq \mathbb{P}^n(k)$ and let V be a quasi-affine variety defined by $V \subseteq Y \subseteq \mathbb{A}^m$. A morphism f from U to V is a map such that there are regular functions $f_1,\dots,f_m$ from U to k such that $f(x) = (f_1(x),\dots,f_m(x))$ for all $x \in U$. Now let W be quasi-projective and defined by $W \subseteq Z \subseteq \mathbb{P}^m(k)$. A morphism f from U to W is a map from U to W with the following properties. Define W_i by $W_i = W \cap j_i(\mathbb{A}^m)$. Let $U_i = f^{-1}(W_i)$. The map f is a morphism iff for each i the induced map from $U_i \to W_i \to j_i^{-1}(W_i) \subseteq \mathbb{A}^m$ is a morphism into the quasi-affine variety $j_i^{-1}(W_i)$. One can easily show that the identity maps are morphisms and that the composition of morphisms is a morphism. Thus we have defined a category whose objects are quasi-projective varieties and whose morphisms are regular maps. Denote the category by qpSch(k).

Let $U \subseteq X \subseteq \mathbb{A}^n$ be a quasi-affine variety and let X be specified by the polynomials $g_i(x_1,\dots,x_n)$ $i = 1,\dots,m$. We must embed X as a closed subset of some $\mathbb{P}^k$. To do this we introduce homogeneous coordinates and let $\hat{g}_i(\lambda, x_1, \ldots, x_n)$ be the corresponding homogeneous polynomial. Let $X^\wedge$ be the zero set of the $\hat{g}_i$ $i = 1, \ldots, m$ in $\mathbb{P}^n$. Then j_0 embeds X as an open subset of $\hat{X}$ and hence j_0 embeds U as an open subset of $\hat{X}$. Thus each quasi-affine variety can be identified with a quasi-projective variety.

Let $U \subseteq X \subseteq \mathbb{A}^n$ and $V \subseteq Y \subseteq \mathbb{A}^m$ be quasi-affine varieties. The set $X \times Y$ is easily seen to be an algebraic subset of $\mathbb{A}^{n+m}$ by identifying it with the set $\{(x_1, \ldots, x_n, y_1, \ldots, y_m): g_i(y) = 0 \quad i = 1, \ldots, m, \quad f_i(x) = 0, \quad i = 1, \ldots, n\}$. The reader should convince himself that the topology of $X \times Y$ is not the product topology. (Examine, for example, the Zariski closed sets in A^2 as compared to closed sets in the product topology of $A^1 \times A^1$.) The product set $U \times Y$ is open in $X \times Y$ and hence $U \times V$ is also quasi-affine.

In order to show that the product of quasi-projective varieties admits a quasi-projective structure we must work a bit harder. Let $\mathbb{P}^n(k)$ and $\mathbb{P}^m(k)$ be given and consider the point set $\mathbb{P}^n \times \mathbb{P}^m$. Let $N = (n+1)(m+1) - 1$. Let the coordinates in $\mathbb{P}^N$ be given by w_{ij}, $i = 1, \ldots, n+1$ $j = 1, \ldots, m+1$. Define $\phi: \mathbb{P}^n \times \mathbb{P}^m \to \mathbb{P}^N$ by $\phi(x,y) = (\ldots, w_{ij}, \ldots)$ where $w_{ij} = x_i y_j$. The image of ϕ is a closed subset of $\mathbb{P}^N$ given by $w_{ij} w_{k\ell} = w_{i\ell} w_{kj}$ $\quad i,k = 1, \ldots, n+1 \quad j, \ell = 1, \ldots, m+1$ and ϕ is one to one. We give $\mathbb{P}^n \times \mathbb{P}^m$ the projective variety structure of its image in $\mathbb{P}^N$. Now let $U \subseteq X$ and $V \subseteq Y$ be quasi-projective then $U \times V$ is given the structure of $\overline{\phi(U \times V)}$. A useful result about products and morphisms is the Closed Graph Theorem for algebraic varieties.

4.2.1 <u>Theorem</u>. *A function* f *from an algebraic variety* X *to an algebraic variety* Y *is a morphism iff the graph of* f *is closed in* $X \times Y$.

A topological space X is called *reducible* if X can be written as a union of closed subsets $X_1 \cup X_2$ with $X_1 \neq X$ and $X_2 \neq X$. The space X is called *irreducible* if X is not reducible. If $U \subseteq X$ is an open subset of the topological space X and $\bar{U} = X$ (where the bar denotes topological closure) then X

is irreducible if and only if U is irreducible (Elementary). A quasi-projective variety X is said to be irreducible if the underlying space is irreducible. Let X be an affine variety and $A[X]$ the k-algebra of regular functions (pointwise addition and multiplication) on X, then one easily checks that X is irreducible iff $A(X)$ has no zero divisors. (If $f,g \in A[X]$ are not identically zero on X and $f(x)g(x) = 0$ for all $x \in X$, $X_f = \{x \in X | f(x) = 0\}$, $X_g = \{x \in X | g(x) = 0\}$ are closed subsets of X satisfying $X = X_f \cup X_g$, $X_f \neq X$, $X_g \neq X$.) Using this we see that the affine spaces $\mathbb{A}^n$ are irreducible. Then by the remarks made above we see that open subsets of $\mathbb{A}^n$ are irreducible and that $P^n(k)$ and its open subsets are irreducible.

If X is an irreducible variety and $U \subset X$ is open, then $\bar{U} = X$. (If $\bar{U}$ were not equal to X then $\bar{U} \cup (X-U) = X$ would show X to be reducible.) For irreducible varieties we have Weyl's irrelevancy principle. Let U be an open subset of an irreducible (quasi) affine variety $X \subset \mathbb{A}^n$, and suppose that $f(x_1,\ldots,x_n)$ is a polynomial over k such that $f(x) = 0$ for all $x \in U$, then $f(x) = 0$ for all $x \in X$. Indeed $f(x) = 0$ defines a closed subset Y of $\mathbb{A}^n$ and we have by hypothesis $U \subset Y$, hence $X \subset \bar{U} \subset \bar{Y} = Y$. Similarly if U is an open subset of an irreducible (quasi) projective variety $X \subset \mathbb{P}^n(k)$ and $g(x)$ is a homogeneous polynomial in $x_0,\ldots,x_n$ such that $g(x) = 0$ for all $x \in U$, then $g(x) = 0$ for all $x \in X$.

Let X be a variety and suppose X is the union of sets $\cup S_i$. We say the union is irredundant iff $S_i \subset S_j$ implies $S_i = S_j$. We have the following theorem

4.2.2 Theorem. *Every algebraic variety is the finite irredundant union of irreducible closed varieites. The decomposition is unique up to permutation.*

The proof of Theorem 4.2.2 follows the following line. Suppose $V = W_1 \cup W_2$ where W_1 and W_2 are closed varieties. If the assertion is false for V then it is false for W_1 or W_2. Applying the theorem again we produce a sequence $W_1 \supset W_3 \supset W_4 \ldots$. The sequence is infinite decreasing and hence corresponds to an infinite increasing sequence of ideals in the coordinate ring. Since the coordinate ring is Noetherian we have a contradiction and hence V can be written as a finite union of irreducible closed subvarieties. The uniqueness of the decomposition can be shown as follows: Suppose $V = \cup W_i$ and $V = \cup V_i$ then $V_j = \cup(W_i \cap V_j)$ and since V_j is irreducible $V_j = W_i \cap V_j$ for some j. On the other hand, $W_i = W_i \cap V_k$ for some k and hence $V_j = V_k = W_i$ thus there is a one to one correspondence between the W_i's and V_i's.

Let V be irreducible. Then the coordinate ring is an integral domain and we can define its field of fractions K_X. Now K_X is a vector space over k and hence has a dimension n. The number $n = \dim K_X$ is the transcendence degree of K_X. We define the *degree of* F to be the number $\dim K_X$. In section 2.3 this is discussed further. We comment here that the dimension of the tangent space at a *nonsingular point* x is the same as the degree of V. This can be discovered by considering the ring of derivations of the coordinate ring and considering the derivations as vector fields as in section 3.3.21.

4.3 Algebraic Vector Bundles

In 4.3.1 and 4.3.3 we review some of the material developed in 3.3 in the algebraic geometric setting. In the remaining sections we study the relationship between subvarieties of a variety X and vector bundles on X.

4.3.1 Definition (algebraic vector bundle). An algebraic vector bundle of dimension n over a (quasi-projective) variety X consists of a surjective morphism of varieties $\pi : E \to X$ and an n-dimensional k-vector space structure on each $\pi^{-1}(x) \subset E$, $x \in X$ such that for every $x \in X$ there exists an open neighborhood $x \in U \subset X$ and an isomorphism (of varieties) $\phi: \pi^{-1}(U) \cong U \times \mathbb{A}^n$ which satisfies

(i) $p_U\phi = \pi|U$, where $\pi|\pi^{-1}(U)$ is the restriction of $\pi : E \to X$ to $\pi^{-1}(U)$.

(ii) for every $y \in U$, $\phi : \pi^{-1}(y) \to y \times \mathbb{A}^n$ is a linear isomorphism of k-vector spaces where $y \times \mathbb{A}^n$ is given the obvious k-vector space structure.

We shall often write E_x for $\pi^{-1}(x)$; E_x is called the fibre of E at x.

Let $E_1 \overset{\pi_1}{\to} X$, $E_2 \overset{\pi_2}{\to} X$ be two algebraic vector bundles over the variety X. A homomorphism $\phi : E_1 \to E_2$ of vector bundles over X is a morphism $\phi : E_1 \to E_2$ such that $\pi_2\phi = \pi_1$ and such that the induced maps $\phi_x : E_{1x} \to E_{2x}$ are k-linear homomorphisms of the k-vector spaces E_{1x} into the k-vector spaces E_{2x}. A homomorphism of vector bundles $\phi : E_1 \to E_2$ is an isomorphism of vector bundles if there is a homomorphism $\psi : E_2 \to E_1$ such that $\psi\phi = 1_{E_1}$, $\phi\psi = 1_{E_2}$.

4.3.2 Definition (algebraic sections). A section of the algebraic vector bundle $E \overset{\pi}{\to} X$ is a morphism $s : X \to E$ such that $\pi s = 1_X$. Giving a section of $E \to X$ is equivalent to giving a homomorphism of the trivial one dimensional vector bundle $X \times \mathbb{A}^1 \to X$ into $E \overset{\pi}{\to} X$. The correspondence is as follows: Let $s_1 : X \to X \times \mathbb{A}^1$ be the section $x \mapsto (x,1)$ then $\phi \to \phi s_1$ establishes a one-one onto correspondence between homomorphisms $\phi : X \times \mathbb{A}^1 \to E$ and sections $X \to E$.

4.3.3 Patching data description of bundles and bundle homomorphisms

The definition of vector bundle in 4.4.1 says that every algebraic vector bundle over a variety X can be described (e.g. obtained) by the following data

(i) a (finite) covering $\{U_\alpha\}$ of X by open sets $U_\alpha \subset X$

(ii) for every α a trivial bundle $U_\alpha \times \mathbb{A}^n$ over U_α

(iii) for every α and β an isomorphism of trivial vector bundles

$$\phi_{\alpha\beta} : (U_\alpha \cap U_\beta) \times \mathbb{A}^n \to (U_\beta \cap U_\alpha) \times \mathbb{A}^n$$

where the isomorphisms $\phi_{\alpha\beta}$ are required to satisfy the conditions

(iv) $\phi_{\alpha\beta}\phi_{\beta\alpha} = 1$

(v) $\phi_{\beta\gamma}(x)\phi_{\alpha\beta}(x) = \phi_{\alpha\gamma}(x)$ for every $x \in U_\alpha \cap U_\beta \cap U_\gamma$ where $\phi_{\alpha\beta}(x)$ is the isomorphism $x \times \mathbb{A}^n \to x \times \mathbb{A}^n$ induced by $\phi_{\alpha\beta}$.

We note that giving an isomorphism $\phi_{\alpha\beta} : (U_\alpha \cap U_\beta) \times \mathbb{A}^n$ $(U_\beta \cap U_\alpha) \times \mathbb{A}^n$ is equivalent to giving a morphism $U_\alpha \cap U_\beta \to$ $G\ell_n$ where $G\ell_n$ is the quasiaffine algebraic variety over k with nonzero determinant.

Let E_1 and E_2 be two algebraic vector bundles over the variety X obtained by gluing together trivial bundles $U_\alpha \times \mathbb{A}^n$, resp. $U_\alpha \times \mathbb{A}^m$, where $\{U_\alpha\}$ is an open covering of X (We can take the same covering for E_1 and E_2 by taking if necessary to common refinement of two open coverings).

Let $\phi^1_{\alpha\beta}$ and $\phi^2_{\alpha\beta}$ be the gluing isomorphisms for E_1 and E_2 respectively. A homomorphism $\psi : E_1 \to E_2$ can now be described as follows: ψ consists of homomorphisms $\psi_\alpha : U_\alpha \times \mathbb{A}^n \to U_\alpha \times \mathbb{A}^m$ of trivial bundles such that for every α and β we have

$$\phi^2_{\alpha\beta}(x)\psi_\alpha(x) = \psi_\beta(x)\phi^1_{\alpha\beta}(x) \qquad \text{for all } x \in U_\alpha \cap U_\beta$$

Note that giving a homomorphism $\psi_\alpha : U_\alpha \times \mathbb{A}^n \to U_\alpha \times \mathbb{A}^m$ is equivalent to giving a morphism $U_\alpha \to M(m,n)$, where $M(m,n)$ is the affine algebraic variety of all $m \times n$ matrices with coefficients in k.

Let $E \xrightarrow{\pi} X$ be an algebraic vector bundle over the variety X and let $f : Y \to X$ be a morphism of varieties. We are going to construct a vector bundle $f^!E$ over ϕ. The so-called pull-back (along f) of E. Suppose E is given by patching data $\phi_{\alpha\beta} : U_\alpha \cap U_\beta \to G\ell_n$, then $f^!E$ over ϕ is given by the patching data $f^{-1}(U_\alpha) \cap f^{-1}(U_\beta) \xrightarrow{f} U_\alpha \cap U_\beta \to G\ell_n$.

Similarly if $\psi : E_1 \to E_2$ is a homomorphism of vector bundles given by the local homomorphisms determined by morphisms $\psi_\alpha : U_\alpha \to M(m,n)$, then we define $f^!\psi : f^!E_1 \to f^!E_2$ by means of morphisms $(f^!\psi)_\alpha : f^{-1}(U_\alpha) \xrightarrow{f} U_\alpha \to M(m,n)$.

4.3.4 Subvarieties of X and algebraic vector bundles on X.

Suppose $Z \subset X$ is an irreducible subvariety of an irreducible quasi-projective variety X and for simplicity assume

$$\mathrm{codim}(Z) = \dim X - \dim Z = 1 .$$

If X is smooth, e.g. if X is an open subspace of an algebraic submanifold of $\mathbb{P}^n(\mathbb{C})$, then Z may be locally defined as the zeroes of a single analytic function. More formally, we may cover the manifold X by charts such that on each U_α

$$Z \cap U_\alpha = f_\alpha^{-1}(0) \qquad (4.3.5)$$

for $f_\alpha : U_\alpha \to \mathbb{C}$ an analytic function.

A central question in the classification of subvarieties of a given variety X is whether each codimension 1 subvariety Z may be defined as the locus of a single algebraic or analytic function f. Now, the description (4.3.5) of Z leads to the data

$$\{U_\alpha\} \text{ a cover of } X, \quad g_{\alpha\beta} = f_\alpha/f_\beta : U_\alpha \cap U_\beta \to \mathbb{C} - \{0\} \qquad (4.3.6)$$

But since $g_{\alpha\beta}\, g_{\beta\gamma} = g_{\alpha\gamma}$, (4.3.6) itself constitutes the *local pieces and gluing data* (see 3.3.8) for an analytic rank 1 vector bundle, or preferably an analytic line bundle

$$\pi : L \to X \quad . \qquad (4.3.7)$$

Moreover, the description (4.3.5) also yields an analytic section of the line bundle L, viz. s is given on each U_α by

$$s_\alpha : U_\alpha \to U_\alpha \times \mathbb{C}$$
$$s_\alpha(p) = (p, f_\alpha(p)) \ . \qquad (4.3.8)$$

By (4.3.5), Z arises as the zeroes of the section s. In particular, Z arises as the zeroes of a globally defined analytic function f if, and only if, L is trivial. [We remark that, with more work (see [4], Chap. III) one may show that in (4.3.5) the U_α may be taken to be Zariski open and the f_α to be regular algebraic functions.] As an example, it is fairly easy to show that an algebraic line bundle $L \to \mathbb{A}^n$ an affine space is (algebraically) trivial.

Now, more generally, consider a subvariety Z of X with

$$\operatorname{codim}(Z) = \dim X - \dim Z = r \geq 1.$$

Again, one may cover X by $\{U_\alpha\}$ for which there exist suitable functions $f_\alpha^1,\ldots,f_\alpha^r$ such that

$$Z \cap U_\alpha = \{z \mid f_\alpha^1(z) = \ldots = f_\alpha^r(z) = 0\} . \qquad (4.3.5)'$$

And, on each intersection $U_\alpha \cap U_\beta$ one has

$$f_\beta^i = \sum_j g_{ij} f_\alpha^j$$

(provided we choose the f_α^j generation for the ideal of analytic functions on U_α vanishing on $Z \cap U_\alpha$), leading to the data

$$\{U_\alpha\} \text{ a cover of } X, \quad g_{\alpha\beta} = (g_{ij}) : U_\alpha \cap U_\beta \to G\ell(r,\mathbb{C}) \qquad (4.3.6)'$$

Now, (4.3.6)' gives the local pieces and gluing data for an analytic rank r vector bundle

$$V \to X,$$

which is trivial if and only if V is definable as the common zeroes of r globally defined analytic functions on X. In a less restrictive setting, if Z is the (complete) intersection of r hypersurfaces Z_i in X,

$$Z = \bigcap_{i=1}^{r} Z_i$$

then

$$V \simeq \bigoplus_{i=1}^{r} L_i \quad .$$

In particular, one is naturally led to the study of algebraic and geometric invariants of vector bundles on X from quite simple considerations involving subvarieties and their intersections or from studying the solution set to a system of simultaneous algebraic equations.

In the next section we will consider the Grassmann variety of p-planes in n-space, developing the algebraic analogues of sections (3.5). In (4.5) some of the basic tools for intersection theory on manifolds will be briefly reviewed.

4.4 Grassmann Manifolds

Let V be a finite dimensional vector space of dimension n over the complex numbers and let $G_p(V)$ be the set of all p-dimensional subspaces of V. The set $G_p(V)$ admits a manifold structure with the following charts. Write $V = U \oplus W$ with $U \in G_p(V)$ and $W \in G_{n-p}(V)$. For $A \in L(U,W)$ define $U_A = \{u + Au : u \in U\}$. The map $A \mapsto U_A$ is a one-to-one map from $L(U,W)$ into $G_p(V)$. It is not onto for we can describe the set of U_A's as exactly those elements of $G_p(V)$ that have zero intersection with W. Let $S_W = \{U_A : A \in L(U,W)\}$. If a basis for U and W is chosen so that A has a matrix representation then S_W along with the map that takes U_A onto the matrix A is a suitable chart. As W ranges over all complements of U the sets S_W form a cover for $G_p(V)$. If ϕ_W is the map from S_W to $L(U,W)$ an easy calculation shows that

$$\phi W_1 \cdot \phi \bar{W}_2^1(A) = A_2(I + A_1)^{-1}$$

where A_1 and A_2 are the unique matrices such that $Au = A_1u + A_2u$ with $A_1u \in U$ and $A_2u \in W_1$. The mapping is defined whenever $A \in \phi_{W_2}(S_{W_1} \cap S_{W_2})$ and being rational it is differentiable. The sets S_W with the maps ϕ_W form an atlas for the manifold.

An important fact about these charts is that S_W is an open dense subset of $G_p(V)$. In fact even more is true because of the fact that the complement of S_W is the subspaces that intersect W. This implies that the complement is algebraic and hence that S_W is Zariski open. The mapping that send A to U_A is thus an embedding of $L(U,W)$ into $G_p(V)$ as an open dense subset.

Let $G\ell(V)$ be the group of all linear automorphisms of V. Any group of automorphisms of V acts naturally on $G_p(V)$ by linear transformation of subspaces. Let α be any element of $G\ell(V)$ and partition α as

$$\begin{bmatrix} \alpha_{11} & \alpha_{12} \\ \alpha_{21} & \alpha_{22} \end{bmatrix}.$$

where $\alpha_{11} \in L(U,U)$, $\alpha_{12} \in L(W,U)$, $\alpha_{21} \in L(W,W)$. Then α maps the subspace U_A to the subspace

$$\alpha(U_A) = \{(\alpha_{11} + \alpha_{12}A)u + (\alpha_{21} + \alpha_{22}A)u : u \in U\} .$$

The space $\alpha(U_A)$ is in S_W iff $(\alpha_{11} + \alpha_{12}A)^{-1}$ exists and in that case

$$\alpha(U_A) = U_{(\alpha_{21} + \alpha_{22}A)(\alpha_{11} + \alpha_{12}A)^{-1}} .$$

The $G\ell(V)$ action thus acts locally as a generalized linear fractional transformation. The local behavior of the action is very familiar.

On the other hand, given any two p dimensional subspaces U and W there is a linear automorphism that maps U onto W. Thus the action is transitive and we have that $G_p(V)$ is the homogeneous space $G\ell(V)/H$ for some H. Let U be a fixed element of $G_p(V)$ and W an arbitrary complement. The isotropy subgroup of U is just those transformations with $\alpha_{21} = 0$. Thus we can count dimensions either by the homogeneous space or by the chart.

If we select on V a positive definite bilinear form we choose in each subspace U an orthonormal basis and extend it to basis of V by the Gram-Schmidt process. This shows that the group of orthonormal matrices acts transitively on $G_p(V)$ and thus $G_p(V)$ is compact since $O(n)$ is compact. This also implies that $G_p(V)$ is projective variety.

Let $U \in G_p(V)$ then each basis of U determines an $n \times p$ matrix B of rank n. Furthermore if B_1 and B_2 are such matrices there is an $p \times p$ matrix P invertible matrix such that $B_1 = B_2P$. Conversely if B_1 and B_2 are $n \times p$ matrices of rank p and there exists a P such $B_1 = B_2P$ the column space of B_1 is the column space of B_2. We have established that their one-to-one correspondence between the orbits of $G\ell(p)$ acting on $n \times p$ matrices of rank p and the $G_p(V)$. The Plucker coordinates of a matrix B is the $\binom{n}{p}$-tuple of determinants of $p \times p$ submatrices of B. It is easy to see that if $B_1 = B_2P$ then Plucker coordinates of B_1 is scalar multiple of the Plucker coordinates of B_2. Thus we can associate with each point in $G_p(V)$ a line in $\mathbb{C}^{\binom{n}{p}}$. It can be shown, of course, that distinct points map onto distinct lines and that the embedding satisfies a homogeneous algebraic equation and hence $G_p(V)$ is an algebraic subset of

$$P^{\binom{n}{p}-1} .$$

Thus, $G_p(V)$ is a projective algebraic variety.

The Grassmannian manifolds carry a natural algebraic vector bundle that can be described as follows. Let

$$\eta = \{(x,v) : (x,u) \in G_p(V) \times V \quad \text{and} \quad v \in x\}.$$

η is a subvariety of $G_p(V) \times V$ and can be shown by the methods of 4.3.1 to be an algebraic vector bundle where the projection $\pi : \eta \to G_p(V)$ is onto the first coordinate. It can be shown that this bundle possesses no sections, but there is no particularly enlightening proof available.

However, if we construct the dual bundle η^* whose fibres are the spaces dual to the fibres of η. Then η^* has a full complement of sections. For let V have a basis $e_1,\ldots,e_n$ and an algebraic innerproduct. Define a section s_i of η^* by $s_i(x)(y) = \langle y,e_i\rangle$ where $x \in G_p(V)$ and y X. The s_i's are linearly independent as sections for consider

$$(\sum \alpha_i s_i)(x)(y) = \langle y, \sum \alpha_i e_i\rangle = 0$$

implies that $\sum \alpha_i e_i = 0$ and hence that the s_i's are independent. Every holomorphic section can be written as a linear combination of the s_i's.

The question whether η or η^* is the natural bundle on $G_p(V)$ depends somewhat on one's background. Traditionally differential geometers consider η to be natural and algebraic geometers prefer η^*.

4.5 Intersections of Subvarieties and Submanifolds

Consider 2 subvarieties X_1, X_2 of $\mathbb{P}^2(\mathbb{C})$ defined by homogeneous functions

$$f_1(x,y,z) = 0 , \qquad f_2(x,y,z) = 0$$

of degress d_1 and d_2, respectively. Bézout's Theorem (1.1.12) asserts that, unless f_1, f_2 have a common factor, the number of points in $X_1 \cap X_2$ counted with multiplicity is given by

$$\#(X_1 \cap X_2) = \deg X_1 \cdot \deg X_2 = d_1 d_2 . \qquad (4.5.1)$$

(1.1.12) was proved in the special case $d_1 = 1$; that is, where X_1 is a line in $\mathbb{P}^2$. We offer a second proof in this case which relies on the "principle of conservation of number."

Now, if f_2 is the product

$$f_2(x,y,z) = \prod_{i=1}^{d_2} \phi_i(x,y,z) \qquad (4.5.2)$$

of pairwise independent linear functionals of$_2$ (x,y,z), then X_1 is the union of d_1 distinct lines in $\mathbb{P}^2$; i.e. X_2 is reducible as

$$X_2 = \bigcup_{i=1}^{d_2} X_2^i \qquad (4.5.2)'$$

However, if X_1 and X_2 contain no common irreducible factors, then

$$\#(X_1 \cap X_2) = \sum_{i=1}^{d_2} \#(X_1 \cap X_2^i)$$

But,

$$\#(X_1 \cap X_2^i) = 1$$

since each pair of distinct lines in $\mathbb{P}^2$ intersect in a unique point.

Consider the case where f_2 is not a product as in (4.5.2). The space $V_{(d_2)}$ of homogeneous polynomials of degree d_2 in (x,y,z) is a finite dimensional vector space. In particular,

f_2 may be joined to a polynomial $\tilde{f}_2$ satisfying (4.5.2) by a path not passing through the 0 polynomial. Indeed, consider the path

$$tf_2 + (1-t)\tilde{f}_2 \qquad \subset V_{d_2} . \qquad (4.5.3)$$

This deformation from f_2 to $\tilde{f}_2$ also gives rise to a deformation of X_2 to a union of d_2 lines:

$$X_2(t) : X_2(0) = X_2 , \quad X_2(1) = \bigcup_{i=1}^{d_2} X_2^i .$$

The principle of conservation of number asserts that

$$\#(X_1 \cap X_2(t))$$

is independent of t (provided it remains finite), and therefore

$$\#(X_1 \cap X_2) = \sum_{j=1}^{d_2} \#(X_1 \cap X_2^j) = d_2 \qquad (4.5.4)$$

proving Bézout's Theorem for X_1 a line. If $\deg X_1 = d_1 > 1$, one may reiterate the above argument deforming X_1 to a union of d_1 lines, say $\tilde{X}_1 = \bigcup_{j=1}^{d_1} X_1^j$, and appealing to the basic principle, i.e.

$$\#(X_1 \cap X_2) = \#(\tilde{X}_1 \cap X_2) = \sum_{j=1}^{d} \#(X_1^j \cap X_2)$$

$$= \sum_{j=1}^{d_1} \sum_{i=1}^{d_2} \#(X_1^j \cap X_2^i) = d_1 d_2 .$$

Now, the successful application of the principle of conservation of number reposes on the introduction of an equivalence relation on submanifolds or subvarieties (of a fixed dimension) so that appropriate deformations of a submanifold do not change the equivalence class of the submanifold and so that intersection numbers, etc. depend only on the equivalence class. In the proof of Bézout's Theorem offered above, such deformations were affected by a continuous change in the coefficients of defining equations and the basic principle amounts to the continuous dependence of the roots on the coefficients on a defining equation. Such a program may be carried out in principle for general varieties, but is far beyond the scope of these notes. The more elementary topological approach employs the equivalence relations defined by homology and homotopy and we list some of the basic results below. For M a smooth manifold of dimension n, and for each r, $0 \le r \le n$, one introduces the r-th homology group of M, with integer coefficients in $\mathbb{Z}$ (or $\mathbb{Z}_2$), denoted by $H_r(M;\mathbb{Z})$ or $H_r(M;\mathbb{Z}_2)$. Each submanifold $N \subset M$ determines a $[N] \in H_r(M;\mathbb{Z})$; for example, for $M = \mathbb{P}^2(\mathbb{C})$ it is known that the only nonzero homology groups are

$$H_0(M;\mathbb{Z}) \simeq \mathbb{Z} = ([P])$$

$$H_2(M;\mathbb{Z}) \simeq \mathbb{Z} = ([X_1]) , \quad \deg X = 1$$

$$H_4(M;\mathbb{Z}) \simeq \mathbb{Z} = ([\mathbb{P}^2]) .$$

In this context, intersection of 2 submanifolds $X_{d_1}, X_{d_2} \subset \mathbb{P}^2(\mathbb{C})$ of dimension 2 (over $\mathbb{R}$) is determined by $[X_{d_1}], [X_{d_2}]$ in a bilinear manner. Thus the intersection theory in Bézout's Theorem amount to the evaluation of the bilinear form

$$i([X_{d_1}],[X_{d_2}]) = d_1 d_2 .$$

In general, let M be a orientable connected compact manifold. For each integer n, let

$$H^n(M,\mathbb{R}), \quad H_n(M,\mathbb{R})$$

denote the cohomology and homology vector spaces (with the real numbers $\mathbb{R}$ as coefficients.

For each pair (j,k) of integers, there is a bilinear mapping

$$H^j(M,\mathbb{R}) \times H^k(M,\mathbb{R}) \to H^{j+k}(M,\mathbb{R}) \tag{4.5.5}$$

called the *cup product*. If $\omega_1 \in H^j(M,\mathbb{R})$, $\omega_2 \in H^k(M,\mathbb{R})$, the $\omega_1 \cup \omega_2$. In particular for $k = m-j$ it maps

$$H^j(M,\mathbb{R}) \times H^{m-j}(M,\mathbb{R}) \to H^m(M,\mathbb{R}) = \mathbb{R} . \tag{4.5.6}$$

4.5.7 Poincaré Duality Theorem. *The bilinear mapping* (4.5.6) *is nondegenerate. In particular, it identifies* $H^{m-j}(M,\mathbb{R})$ *with the dual vector space of* $H^j(M,\mathbb{R})$, *and identifies* $H^{m-j}(M,\mathbb{R})$ *with* $H_j(M,\mathbb{R})$.

The cup-product (4.5.5) on cohomology then transforms (under this Poincaré duality isomorphism between homology and cohomology) into an algebraic operation on homology--the *intersection* pairing. If

$$j + k = m .$$

and $H_0(M)$ is identified with $\mathbb{R}$, the *intersection* operation defines a bilinear map

$$H_j(M,\mathbb{R}) \times H_k(M,\mathbb{R}) \to \mathbb{R} .$$

For $\alpha \in H_j(M,\mathbb{R})$, $\beta \in H_k(M,\mathbb{R})$, the real number

$$i(\alpha,\beta)$$

assigned to (α,β) be the operation is called the *intersection number* of the two homology classes α,β.

The above definition of "intersection number" is conceptually very simple, once one understands basic homology theory. To be useful, it must be supplemented by a method of computing it in more familiar geometric terms, for a suitably "generic" situation. Differentiable manifold theory offers such a possibility.

Let N,N' be compact orientable manifolds, with fixed orientation such that

$$\dim M = \dim N + \dim N' .$$

The spaces $H_n(N,\mathbb{R})$, $H_{n'}(N',\mathbb{R})$ have canonical generators ($n = \dim N$, $n' = \dim N'$), which are called the *fundamental homology classes* of the manifolds, denoted by $h_N, h_{N'}$. Let

$$\phi : N \to M, \qquad \phi' : N' \to M$$

be two continuous maps, and let

$$\phi_*(h_N) \in H_n(M,\mathbb{R}) , \qquad \phi_*(h_{N'}) \in H_{n'}(M,\mathbb{R}) ,$$

be the image of these fundamental cycles in the homology of M. The intersection

$$i[\phi_*(h_N),\phi_*(h_{N'})]$$

is called the *intersection number* of the maps ϕ,ϕ', denoted by

$$i(\phi,\phi').$$

Now suppose that ϕ, ϕ' are C^{∞} maps. Let $p \in N'$, $p' \in N'$ be two points such that

$$\phi(p) = \phi'(p') .$$

The maps are said to *intersect in general position* at this point if

$$M_{\phi(p)} = d\phi(N_p) \oplus d\phi'(N'_p) . \qquad (4.5.8)$$

(M_q denotes the tangent vector space to M at q; $d\phi$ denotes the induced linear maps on tangent vectors.)

Now, fixing an orientation for N means that it makes sense when a basis for each tangent space is "positively" or "negatively" oriented. Let us say that $\phi(N)$ and $\phi'(N')$ meet at $\phi(p)$ in a positive way if 4.5.8 is satisfied, and if putting together a positively oriented basis for N_p and N'_p, provides a positively oriented basis for $M_{\phi(p)}$. Otherwise (and if they meet in general position) they are said to meet at $\phi(p)$ in a *negative* way.

Suppose that $\phi(N)$ and $\phi'(N')$ meet in general position at each point of intersection. Then

4.5.9 Theorem

$$i(\phi,\phi') = \sum_{p \in \phi(N) \cap \phi'(N')} \pm 1 . \qquad (4.5.10)$$

Here, the sign + or - is chosen according to whether the submanifolds meet in a positive or negative way.

Determining the orientations of the intersections is often an obstacle to determining the intersection number using formula (4.5.10). Working in the categories of *complex analytic* instead of *real* manifold removes this obstacle. The manifold M has a *complex manifold structure* if a set of coordinate charts is given, setting up coordinates in $\mathscr{C}^m$, with the transition maps between the charts given by complex analytic functions. A map $\phi : N \to M$ between complex manifolds is complex if it is given, in terms of complex charts, by complex analytic functions. A submanifold $\phi : N \to M$ is said to be complex if the map is complex.

Such a complex structure on manifold M determines an orientation for the manifold M. In terms of this orientation, two complex submanifolds always *meet with positive orientation.* Thus, the sum on the right-hand side of (4.5.10) *only involves* *plus signs.* In particular, $i(\phi,\phi')$ *is equal to the number of intersections of the submanifolds* $\phi(N)$, $\phi'(M')$, provided they meet in general position.

Here is the situation of greatest importance in algebraic geometry.

$$M = \mathbb{P}_n(\mathbb{C})$$

the complex projective space, of *real* dimension $2n$. It is the quotient of $\mathbb{C}^{n+1}\setminus(0)$ under the dilation group. $\phi(N)$, $\phi(N')$ are subsets determined by *nonsingular*, irreducible algebraic subsets of M. $\mathbb{P}_n(\mathbb{C})$ is a complex manifold, and the algebraic subsets are complex submanifolds. For $n = 2$, this, of course, is just Bézout's Theorem which we proved by purely algebraic methods at the beginning of this section.

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5. LINEAR ALGEBRA OVER RINGS

The solution of linear equations, $AX = B$, and more generally the structure of R-linear transformations on R-modules requires us, in the end, to introduce and study quite a few auxiliary objects which one encounters in only a simplified form over fields. We begin with criterion for surjectivity and injectivity of an R-linear transformation

$$T : M \to N$$

of finitely-generated R-modules. These are always important properties to study, but particular use of these may be made in studying questions of reachability and observability.

5.1 Surjectivity of Linear Transformations, Nakayama's Lemma

We consider an R-linear map

$$T : M \to N \tag{5.1.1}$$

and would like to reduce our questions to a similar question over a field. However, as Example 2.4.8 shows, even when R is a PID, passing to the fraction field K gives us only some of the information we desire, viz. T_K is surjective if, and only if, the cokernel $N/T(M)$ is a torsion module.

Set $\max(R) = \{m | m \subset R$ is a maximal ideal of $R\}$, so that $m \in \max(R)$ if, and only if, R/m is a field. If T in (5.1.1) is surjective, then so is

$$\overline{T} : M/mM \to N/mN \tag{5.1.2}$$

5.1.3 Theorem. *T in (5.1.1) is surjective if, and only if, $\overline{T}$ in (5.1.2) is surjective for all* $m \in \max(R)$.

For example, $T : \mathbb{Z} \to \mathbb{Z}$ mapping z to $2z$ gives rise to the map,

$$0 = \overline{T} : \mathbb{Z}/2\mathbb{Z} \to \mathbb{Z}/2\mathbb{Z} ,$$

which fails to be surjective. Similarly

$$T : \mathbb{R}[x,y] \to \mathbb{R}[x,y]$$

$$Tf = (x^2 + y^2)f$$

fails to be surjective, since T "vanishes at the origin." That is, if $m_0 = \{f | f(0,0 = 0\}$, then T induces the 0 map

$$0 = \bar{T} : \mathbb{R}[x,y]/m_0 \to \mathbb{R}[x,y]/m_0 \quad .$$

Proof of 5.1.3. The examples above hint at an important special case, let $g \in \mathbb{R}$ and define $T_g : R \to R$ by $T_g(f) = gf$. Then T_g is surjective if, and only if, g is a unit in R. That is, g is a unit if, and only if, g is a unit in R/m for all m. For, g is a unit if, and only if, $g \notin m$ for any $m \in \max(R)$. Consider, on the other hand, those $g \in \bigcap_{m \in \max(R)} m = \mathrm{Jac}(R)$ --the Jacobson radical of R.

$$g \in \mathrm{Jac}(R) \text{ if, and only if, } 1 - gf \text{ is a unit for all } f \in R \; .$$

If $g \in \mathrm{Jac}(R)$, then $1\text{-}fg \equiv 1 \bmod(m)$, for all m, and is therefore a unit of R. Suppose that $1 - fg$ is always a unit, but that $g \notin m$, for some m; i.e., that $(g) + m = R$. Then, for some $f \in R$, $h \in m$, we have the equation

$$fg + h = 1 \;, \quad \text{or} \quad h = 1 - fg \;,$$

implying $m = R$.

Next consider $T : M \to M$ and suppose there exists an ideal I of R such that

$$TM \subset IM \;,$$

then there exists a relation

$$T^n + \sum_{i=1}^{n} r_i T^{n-i} = 0 \;, \quad \text{with} \quad r_i \in I.$$

For, if $\{x_1,\ldots,x_n\}$ generates M, consider

$$Tx_i = \sum_{j=1}^{n} a_{ij} x_j \;, \qquad a_{ij} \in I \;.$$

Equivalently,

$$\sum_{j=1}^{n} (\delta_{ij} T(x_j) - a_{ij} x_j) = 0 \;, \qquad \text{or}$$

$$\sum_{j=1}^{n} (\delta_{ij} T - a_{ij}) x_j = 0 \;.$$

By Cramer's Rule, $\det(\delta_{ij}T - a_{ij})$ annihilates all $x \in M$ and is therefore the 0 endomorphism.

In particular, if $T = I$ one has, setting $r = \sum a_i$.

If $IM = M$, then there exists $r \in R$ such that

(i) $r \equiv 1 \mod I$

(ii) $rM = 0$. (5.1.4)

If $mM = M$, for all $m \in \max(R)$, then $M = (0)$.

For, suppose $0 \neq x \in M$. Consider the ideal

$$\text{Ann}(x) = \{r \in R \mid rx = 0\} \subset R \quad .$$

Since $x \neq 0$, $\text{Ann}(x) \neq R$ and therefore, $\text{Ann}(x) \subset m$ for some m. By hypothesis, there exists $r \in R$ satisfying

$$r \equiv 1 \mod m, \quad \text{and} \quad rx = 0 \ .$$

But, the second equation asserts $r \in \text{Ann}(x) \subset m$, contrary to the first.

It is now an easy consequence that T is surjective if, and only if, $\overline{T} : M/mM \to N/mN$ is surjective, for all m, for all of the above applies to the module $N/\text{image } T$.

5.1.5 Corollary. [4] In particular, if one considers the linear system,

$$x(t+1) = Ax(t) + Bu(t) \qquad (5.1.6)$$

defined over R, then (5.1.6) is reachable, in the sense that the columns of $(B, AB, \ldots)$ span the state module, if and only if

$$x(t+1) = \overline{A}x(t) + \overline{B}u(t) \qquad (5.1.6)'$$

is reachable over R/m, for all $m \in \max(R)$.

Along the way, we have also developed enough algebra to prove the "Fundamental Theorem of Commutative Algebra,"

5.1.7 Nakayama's Lemma. *If* M *is finitely generated over* $R, I \subset \text{Jac}(R)$ *an ideal of* R *such that* $IM = M$, *then* $M = (0)$.

Proof. From (5.1.4) one has an $r \in R$ such that

$r \equiv 1 \mod I$ and $rM = (0)$.

The first equation asserts that $(1-4) \in Jac(R)$ and, by (3.3), r is a unit. The second equation now asserts that $M = (0)$.

This is especially useful when the ring R in question has only one maximal ideal, say m. (R,m) is said to be a local ring--for example, the ring of formal power series $\mathbb{R}[[x_1,\ldots,x_N]]$ is a local ring with $m = \{f|\text{the constant term of } f \text{ is } 0\}$, and the ring of germs at 0 of analytic functions in $\mathbb{R}^N$ is a local ring, contained in $\mathbb{R}[[x_1,\ldots,x_N]]$.

If R is local, then $Jac(R) = m$ and we have

5.1.8 Nakayama's Lemma. *If* M *is finitely generated over* R, *and* $mM = 0$, *then* $M = 0$. *In particular,* $\{x_1,\ldots,x_N\}$ *generates* M *if, and only if,* $\{\bar{x}_1,\ldots,\bar{x}_N\}$ *generates* M/mM.

Local rings will arise rather naturally when we study injectivity of R-linear maps in the next section.

5.2 Injectivity of Linear Transformations, Solvability of TX = Y, Localizations

In order to study injectivity as well as a particular equation $Tx = y$, we introduce a refinement of the idea of "evaluating T" at the point $m \in \max(R)$, viz. expanding T locally at m. For $m \in \max(R)$, denote the ring of fractions of R, with denominators in $R\setminus m$, by R_m (see [1], p. 36). Thus R_m consists of equivalence classes of pairs (f,g), $f \in R$, $g \in R\setminus m$, thought of as fractions f/g. Two pairs are equivalent if there exists $r \in R\setminus m$ such that

$$(f\tilde{g} - \tilde{f}g)r = 0 ,$$

that is, if the corresponding fractions are equal, and pairs are added and multiplies as fractions. As an exercise, one may check that $[(f,g)]$ is invertible in R_m if, and only if, $f \in R-m$. Therefore, each ideal I of R_m is contained in $\{[f,g]|f \in m\}$.

5.2.1 Lemma. R_m *is a local ring, with unqiue maximal ideal* $\{[f,g]|f \in m\}$.

If M is an R-module, then one can form the module of fractions, which is a module over the ring R_m. And, R-linear map

$T: M \to N$ induces an R-linear map $T_m : M_m \to N_m$. This is exactly the set-up we need.

5.2.2 Theorem. *The equation* $Tx = y$ *has a solution* $x \in R^{(n)}$, *for a given* $y \in R^{(\ell)}$, *if and only if, the equation*

$$T_m x = y \qquad (5.2.3)$$

has a solution over R_m, *for all* $m \in \max(R)$.

Proof. We need only prove sufficiency, set

$$I = \{r \in R | Tx = ry \text{ has a solution over } R\} .$$

If the ideal $I = R$, we're done, and if $I \neq R$ then $I \subset m$, for some maximal ideal m of R. Fix such an m and choose a solution $\tilde{x} \in R_m^{(n)}$ to equation (3.11). By clearing denominators, which lie in R-m, one has $r \in R_m$ such that $\tilde{x} = r^{-1}x$, x defined over R, and an $s \in R$ such that $t = rs \equiv 1 \bmod m$. Therefore,

$$T(tx) = sy$$

and $s \in I \subset m$, contrary to assumption.

Remarks 1. If R_m is Noetherian, then the solubility of (5.2.3) can be further reduced, first to the case of a complete local ring and finally [2] to the case of a local Artinian ring, viz. to the solution of (5.2.3) over R/m^k, for each $k \geq 1$.

2. If we consider the question of surjectivity, then Theorem 5.2.2, together with Nakayama's Lemma, implies Theorem 5.1.3 for free (or even projective) state modules. One need not, however, make such hypothesis on M. Indeed, one can show [1]:

Theorem 3.12. *Let* $T : M \to N$, *then*

(i) T *is surjective* $\Leftrightarrow$ $T_m : M_m \to N_m$ *is surjective, for all* m.

(ii) T *is injective* $\Leftrightarrow$ $T_m : M_m \to N_m$ *is injective, for all* m.

5.3 The Structure of Linear Transformations, The Suslin-Quillen Theorem.

We now turn to the structure of linear transformations

$$T : M \to M \ , \qquad M \simeq R^{(n)} \ .$$

If T is not invertible, is M isomorphic to a direct sum of kernel T with image T? In Example 4.8, image T can never be complemented in $\mathbb{Z}$, so we must refine our question. If image T is complemented in M, i.e., is the image of a projection, can we find a basis for image T and complete this, with a basis for $\ker T$, to find a basis for M? The first condition is satisfied, for example, when T itself is a projection and, again, we are led to the question:

(SQ 1) Is every projection $P : R^{(n)} \to R^{(w)}$ diagonalizable?

Suppose, on the other hand, that T is invertible. What does the first column of T look like? Clear $(2,4)^+$ cannot be the first column of an invertible $T \in M_2(\mathbb{Z})$. Indeed, by the classical expansion of a determinant into a linear combination of cofactors one sees that the existence of $r_i \in R$ such that

$$\sum_{i=1}^{n} a_i r_i = \text{unit of } R$$

is a necessary condition that $(a_1,\ldots,a_n)^t$ be the first column of an invertible matrix. By dividing if necessary, one may assume

$$\sum_{v=1}^{n} a_i r_i = 1 \ ,$$

that is, $(a_1,\ldots,a_n)$ is unimodular. If P is a rank 1 projection such that image P is free, then by choosing $(a_1,\ldots,a_n)$ to be a generator of image P one might attempt to follow the standard linear algebra route for constructing a T such TPT^{-1} is diagonal. That is, we construct T by setting $(a_1,\ldots,a_n)^+$ as the first row and complete T to an invertible matrix (by adding the basis vectors for $\ker T$). Thus we are led to ask

(SQ 2) Is every unimodular vector $(a_1,\ldots,a_n)$ the first column of an invertible matrix?

For $n=1,2$, (SQ 2) is trivially answered, in the affirmative, for any commutative ring $\mathbb{R}$.

5.3.1 Example. Consider $R = C(S^2)$ = ring of continuous, real-valued functions on the 2-sphere, and consider the free R module M, of rank 3, of $\mathbb{R}^3$-valued functions on S^2. Let $L \subset M$ be the R-submodule of those functions which point in $\pm$ the normal direction, so that L is spanned by the unimodular vector $v = (x,y,z)^t$, where $x^2 + y^2 + z^2 = 1$. Then v cannot be the first row of a unimodular matrix or, equivalently, if $P : M \to L$ is the projection on L, ker P does not admit a basis. In fact, to exhibit $w \in \ker P$ such that $w(x,y,z) \neq 0$ is to find a nowhere zero vector field on S^2, which is well-known to be contrary to fact.

Thus, the fact that one cannot "comb the hair on a tennis ball," has considerable impact on the linear algebra over $R = C(S^2)$. We note that (SQ 1) is equivalent to the more familiar form of these questions.

> (SQ 3) Is every finitely-generated projective module over R necessarily free?

The connection between (SQ 3) and "combing the hair on a tennis ball" can be made more precise, since the module ker P of tangent vector fields S^2 is the (finitely-generated, projective) module of continuous sections of a certain vector bundles on S^2, viz. the tangent bundle.

Set $R = \mathbb{C}[x_1,\ldots,x_N]$, then $R^{(1)}$ as a module over R is simply the module of algebraic, scalar valued functions on $\mathbb{A}^N$-- which may be regarded as the module of algebraic section of the trivial line bundle

$$\mathbb{A}^N \times \mathbb{C} \to \mathbb{A}^N$$

On the other hand, if $\pi : V \to \mathbb{A}^N$ is a vector bundle, then the additive group $\Gamma(\mathbb{A}^N;V)$ is an R-module, with multiplication $f \in \mathbb{R}$, $\gamma \in \Gamma(\mathbb{A}^N;V)$ defined pointwise

$$f\gamma(p) = f(p)\gamma(p)$$

in the fiber $\pi^{-1}(p)$. If V is trivial, of rank m, then $\Gamma(\mathbb{A}^N;V) \simeq R^{(m)}$. And, we have already noted the converse, for the case $m = 1$. Moreover, any homomorphism $V \to W$ induces, by composition an R-module map $\Gamma(\mathbb{A}^N;V) \to \Gamma(\mathbb{A}^N;W)$.

Thus, we have a correspondence:

$$\{\text{vector bundles on } \mathbb{A}^N\} \to \{\text{modules over } R\} \quad (5.3.2a)$$

such that

$$\{\text{homomorphisms of vector bundles}\} \to \{\text{homomorphism of modules}\}. \quad (5.3.2b)$$

Moreover, this correspondence gives an equivalence

$$\{\text{trivial vector bundles}\} \leftrightarrow \{\text{free modules over } R\} \quad (5.3.3a)$$

$$\{\text{homomorphisms of trivial vector bundles}\} \leftrightarrow \{\text{homomorphisms of free modules}\}$$

In particular, if a trivial vector bundle V of rank m splits

$$V \simeq W_1 \oplus W_2$$

into 2 subbundles, then the homomorphism

$$P_1 : V \to W_1 \subset V, \qquad \text{satisfying} \quad P_1^2 = P_1,$$

corresponds to a projection operator

$$\tilde{P}_1 : R^{(m)} \to R^{(m)} \quad (5.3.4)'$$

with image $\tilde{P}_1 \simeq \Gamma(\mathbb{A}^N;W_1)$, a finitely-generated projective R-module. And, conversely, each finitely generated, projective module gives rise to some subbundle W_1 of a trivial bundle, by definition. Now, it can be shown that every vector bundle W_1 is a direct summand in some trivial bundle V and thus the equivalence (5.3.3) extends to an equivalence

$$\{\text{vector bundles}\} \leftrightarrow \{\text{finitely generated, projective modules}\} \quad (5.3.5a)$$

$$\{\text{homomorphisms of vector bundles}\} \leftrightarrow \{\text{homomorphisms of finitely, generated projective modules}\} \quad (5.3.5b)$$

Thus, triviality of a vector bundle is equivalent to freeness of its module of sections, bringing us to ask, for $R = \mathbb{C}[x_1,\ldots x_N]$

(SQ4) Is every vector bundle on $\mathbb{A}^N$ trivial?

This question was raised by J-P. Serre and settled, in the affirmative, by A. Suslin and D. Quillen [5], [3].

5.3.6 Theorem (SQ) *For* $R = k[x_1,\ldots,x_N]$, *every finitely generated projective module is free; that is,* (SQ1),...,(SQ4) *hold for* R.

We will find all of these forms of Suslin-Quillen quite useful.

Thus, by extending these ideas we see that there exists projective, but not free, modules defined over $\mathbb{R} = C(S^2)$. By using the line bundle over S^1 derived from the Möbius band, this is also true for $C(S^1)$. These facts lie at the heart of the non-existence of continuous canonical forms for realizations, which is, of course, a question of linear algebra with parameters (see Professor Hazewinkel's lectures).

It is somewhat deeper that (SQ2) fails to hold for $R = H^\infty(\mathbb{D})$, this calculation comes from certain topological non-triviality of the space, $\max(H^\infty(\mathbb{D}))$, as in (SQ4).

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