Sum rule for products of Bessel functions: Comments on a paper by Newberger

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Recently, Newberger considered a series of Bessel functions with as a special case the form $\Sigma(n^j J_n^2(z))/(n+\mu)$. The interesting point is that he obtained new explicit expressions for the sum of the series. In this note we point out that some results of Newberger are not correct, especially the results obtained by the principle of analytic continuation. Our remarks include a correction for his important result for the series $\Sigma J_n(z) J_{n-m}(z)/(n+\mu)$.

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1. INTRODUCTION

Newberger[1] presented a sum rule for the infinite series of the form

$$S=\sum_{n=-\infty}^{\infty}\frac{(-1)^n n^j J_{\alpha-\gamma n}(z) J_{\beta+\gamma n}(z)}{n+\mu}, \tag{1.1}$$

where $j\in\mathbb{N}\cup\{0\}$, $\mu\in\mathbb{C}\setminus\mathbb{Z}$, $\alpha,\beta,z\in\mathbb{C}$, $\gamma\in(0,1]$. Initially, α and β are restricted to Re $(\alpha+\beta)>-1$. Under this last restriction, Newberger found interesting explicit expressions for the sum S. Afterwards he extended his results beyond this range of parameters α and β. As will be shown in this note, this last step yields incorrect results.

2. SYMMETRY RELATION FOR S; $\alpha,\beta\in\mathbb{Z}$, $\gamma=1$

An important observation is that S is not defined for all α and β, as stated after (1.1). This will be proved in Sec. 4. Here we consider $\gamma=1$ and integer values of α and β. Then the series is convergent and there is a symmetry rule. To show this we denote S of (1.1) by $S_j(\alpha,\beta,\gamma,\mu)$. Then we have

$$S_j(-\alpha,-\beta,1,-\mu)=(-1)^{j+\alpha+\beta+1}S_j(\alpha,\beta,1,\mu), \tag{2.1}$$

where we used

$$J_{-n}(z)=(-1)^n J_n(z),\quad n\in\mathbb{Z}.$$

The following important special case is considered by Newberger. We define $T_m(z,\mu)=(-1)^m S_0(m,0,1,\mu)$, or explicitly

$$T_m(z,\mu)=\sum_{n=-\infty}^{\infty}\frac{J_n(z)J_{n-m}(z)}{n+\mu},\quad m\in\mathbb{Z}. \tag{2.2}$$

This function arises in a lot of physical problems, for instance in plasma physics. Applying the symmetry rule (2.1) for this case we obtain

$$T_{-m}(z,\mu)=(-1)^{m+1}T_m(z,-\mu). \tag{2.3}$$

Newberger found [see his result (4.6)]

$$T_m(z,\mu)=\frac{(-1)^m\pi}{\sin\mu\pi}J_{m+\mu}(z)J_{-\mu}(z),\quad m\geqslant 0. \tag{2.4}$$

The addition $m\geqslant 0$ is not given by Newberger, but has to be made. To see this, verify the symmetry rule (2.3) for the above relation. It follows that (2.4) cannot be correct for all $m\in\mathbb{Z}$. The correct relation for negative values is

$$T_m(z,\mu)=\frac{\pi}{\sin\mu\pi}J_{-m-\mu}(z)J_\mu(z),\quad m\leqslant 0. \tag{2.5}$$

Observe also that (2.2) is an entire function of z, as are the right-hand sides of (2.4) and (2.5). For $m<0$ (2.4) is not entire in z; for $m>0$ (2.5) is not entire in z.

3. A RECURSION FOR $T_m(z,\mu)$

The fact that (2.4) is no longer valid for negative values of m is also revealed by a recursion for $T_m(z,\mu)$. We recall the well-known identities

$$J_{\nu-1}(z)+J_{\nu+1}(z)=(2\nu/z)J_\nu(z), \tag{3.1}$$

$$\sum_{n=-\infty}^{\infty}J_n(z)J_{n-m}(z)=\delta_{m,0}, \tag{3.2}$$

where Kronecker's symbol is used.[2] By using (3.1) we have

$$T_{m-1}(z,\mu)+T_{m+1}(z,\mu)=\frac{2}{z}\sum_{n=-\infty}^{\infty}\frac{J_n(z)(n-m)J_{n-m}(z)}{n+\mu}.$$

Writing $n-m=(n+\mu)-(\mu+m)$ we obtain with (3.2)

$$T_{m-1}(z,\mu)+T_{m+1}(z,\mu)=-\frac{2(m+\mu)}{z}T_m(z,\mu)+\frac{2}{z}\delta_{m,0}. \tag{3.3}$$

Observe that this recursion relation is an inhomogeneous version of (3.1). Without the term $(2/z)\,\delta_{m,0}$ a solution would be $(-1)^m J_{m+\mu}(z)$ $(m\in\mathbb{Z})$ times a factor not depending on m. Hence, since $T_m(z,\mu)$ satisfies (3.3) $(m\in\mathbb{Z})$ and the right-hand side of (2.4) satisfies the homogeneous version of (3.3) for $m\in\mathbb{Z}$, it follows once again that (2.4) cannot be true for all $m\in\mathbb{Z}$. For $m\neq 0$ (3.3) gives the proper recursion for both (2.4) and (2.5). On the other hand we have, using (2.4), (2.5), and (3.1),

$$T_{-1}(z,\mu)+T_1(z,\mu)=\frac{-\pi}{\sin\mu\pi}\left[J_\mu(z)J_{-\mu-1}(z)+J_{-\mu}(z)J_{\mu+1}(z)\right]-\frac{2\mu\pi}{z\sin\mu\pi}J_\mu(z)J_{-\mu}(z).$$

Interpreting the cross product of Bessel functions as a well-known Wronskian relation for these functions[3] we obtain

$$T_{-1}(z,\mu)+T_1(z,\mu)=2/z-(2\mu/z)T_0(z,\mu),$$

which confirms (3.3) for $m=0$.

4. CONVERGENCE OF THE SERIES (1.1)

The convergence of the series (1.1) follows from the asymptotic expansions

$$J_\nu(z)\sim(\tfrac{1}{2}z)^\nu/\Gamma(\nu+1),\qquad \operatorname{Re}\nu\to\infty. \quad (4.1)$$
$$J_{-\nu}(z)\sim(1/\pi)(\tfrac{1}{2}z)^{-\nu}\Gamma(\nu)\sin\nu\pi,$$

The second line holds for noninteger values of ν; otherwise we use $J_{-n}(z)=(-1)^nJ_n(z)$, $n\in\mathbb{Z}$. Using (4.1) for the terms of (1.1) we obtain

$$\frac{n^j}{n+\mu}J_{\alpha-\gamma n}(z)J_{\beta+\gamma n}(z)\sim n^{j-1}(\tfrac{1}{2}z)^{\alpha+\beta}\times\sin[(\gamma n-\alpha)\pi]\,\frac{\Gamma(\gamma n-\alpha)}{\Gamma(\gamma n+\beta+1)}.$$

Using $\Gamma(z+a)/\Gamma(z+b)\sim z^{a-b}$, $\operatorname{Re} z\to\infty$, we conclude that the series diverges when

$$\operatorname{Re}(\alpha+\beta)<j-2,$$

unless $\alpha,\beta\in\mathbb{Z}$, $\gamma=1$. In general, large terms for $n\to\pm\infty$ will not cancel each other. Thence there is no chance that the divergence at $n=+\infty$ combined with that at $n=-\infty$ is removed.

The series (1.1) is absolutely convergent when $\operatorname{Re}(\alpha+\beta)>j-1$. This condition is sufficient to make the sum holomorphic with respect to α and β in this domain.

It follows that S, as a function of the complex parameters α and β, is defined and holomorphic for $\operatorname{Re}(\alpha+\beta)>j-1$. Possibly there is an analytic continuation of $S(\alpha,\beta,\mu)$ with respect to $\operatorname{Re}(\alpha,\beta)\leqslant j-1$, but it is not clear what this continuation looks like. For $\alpha,\beta\in\mathbb{Z}$, $\gamma=1$, the symmetry rule (2.1) gives the value for negative α and β.

Newberger used the splitting

$$S=(-\mu)^jS_1+S_2 \quad (4.2)$$

with S_1 equal to (1.1) with $j=0$. He evaluated this expression in the form [see his formula (2.8)]

$$S_1=\frac{\pi}{\sin\mu\pi}J_{\alpha+\gamma\mu}(z)J_{\beta-\gamma\mu}(z). \quad (4.3)$$

His proof is correct for the range $\operatorname{Re}(\alpha+\beta)>-1$. The right-hand side is entire in α and β, whereas from the above remarks it follows that some combinations of α and β yield a divergent series. Extension of (4.3) to all complex α and β (and γ) is therefore not allowed.

5. A FINAL REMARK

The second part of (4.2), i.e., S_2, is also evaluated in terms of derivatives of Bessel functions. Starting point is the evaluation of

$$\hat{S}_2=\sum_{n=-\infty}^{\infty}(-1)^n n^pJ_{\alpha-\gamma n}(z)J_{\beta+\gamma n}(z), \quad (5.1)$$

where p is an integer, $0<p\leqslant j-1$.

As admitted by Newberger, the analysis for deriving the sum rule for S_2 is quite formal, with an appeal to the theory of generalized functions. However, an approach without distributions is possible here. For instance, application of (3.1) gives a recursion relation [denote (5.1) by $\hat{S}_2(\alpha,\beta,p)$]

$$\gamma\hat{S}_2(\alpha,\beta,p)=\alpha\hat{S}_2(\alpha,\beta,p-1)-\tfrac{1}{2}z[\hat{S}_2(\alpha+1,\beta,p-1)+\hat{S}_2(\alpha-1,\beta,p-1)].$$

Repeated application reduces (5.1) to the case $p=0$. One further step makes the Fourier series in Newberger's formulas (2.12) and (2.13) convergent.

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[1]B. S. Newberger, "New sum rule for products of Bessel functions with application to Plasma Physics," J. Math. Phys. 23, 1278–1281 (1982).
[2]G. N. Watson, *A treatise on the theory of Bessel functions*, 2nd ed. (Cambridge U. P., London, 1952), Sec. 5.3.
[3]Reference 2, p. 46.