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REMARKS ON A PAPER OF A. ERDÉLYI*

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Abstract. An alternative asymptotic expansion is given for an integral, which was recently considered by Erdélyi by means of fractional derivatives. The new expansion is simpler and the bounds of the remainder terms are of the same kind.

1. Introduction. In a recent paper [3], Professor Erdélyi considered integrals of the form

$$F(z, a) = \int_a^\infty e^{-z(t-a)} t^{\lambda-1} g(t)\, dt, \tag{1.1}$$

where $a \geqq 0$, $0 < \lambda < 1$, and z is a large parameter. In order to obtain an asymptotic expansion for $z \to \infty$, uniformly valid for $a \geqq 0$, he replaced the function $t^{\lambda-1}g(t)$ by a fractional integral $I^{\lambda-1}f(t)$, the operator I^α being defined by

$$I^\alpha f(t) = \frac{1}{\Gamma(\alpha)} \int_0^t (t-s)^{\alpha-1} f(s)\, ds.$$

By an integration by parts procedure, Erdélyi obtained the uniform expansion

$$F(z, a) = Q \sum_{k=0}^{n-1} \Gamma(k+\lambda) g^k(0) z^{-k}/k! + \sum_{k=1}^{n-1} z^{-k} I^\lambda f^{(k)}(a) + R_n, \tag{1.2}$$

where Q is related to the incomplete gamma function and is given by

$$Q = z^{-\lambda} e^{az} \Gamma(\lambda, az)/\Gamma(\lambda). \tag{1.3}$$

The remainder R_n is estimated uniformly in a for $a \geqq 0$. The expression $I^\lambda f^{(k)}(a)$ is explicitly given in terms of derivatives of the function $g(t)$ at $t = 0$ and $t = a$ as

$$\begin{aligned} I^\lambda f^{(k)}(a) = \sum_{m=1}^{k} \frac{a^{\lambda-m}}{(k-m)!} \Bigg[(-1)^{m-1} \frac{\Gamma(k)\Gamma(m-\lambda)}{\Gamma(m)\Gamma(1-\lambda)} g^{(k-m)}(a) \\ - \frac{\Gamma(k+\lambda-m)}{\Gamma(\lambda-m+1)} g^{(k-m)}(0) \Bigg], \qquad k = 1, 2, \cdots. \end{aligned} \tag{1.4}$$

As remarked by Erdélyi, the expansion (1.2) could have been obtained via integration by parts of (1.1), but the explicit form (1.4) in (1.2) is not easily obtained in that way.

In this note we give an alternative expansion of $F(z, a)$, which is simpler than (1.2), and in which the bounds of the remainder terms are of the same kind. Both expansions may be derived from each other by formal rearrangement of infinite series.

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2. From a numerical point of view, (1.2) is not attractive because of the $I^\lambda f^{(k)}(a)$ in the second series. Recurrence relations for these factors based on (cf. [3, (2.3)])

$$I^{\lambda-1}f(t) = \frac{d}{dt}I^\lambda f(t) = \frac{f(0)}{\Gamma(\lambda)}t^{\lambda-1} + I^\lambda f'(t)$$

are not suitable for numerical evaluation of a sequence of $I^\lambda f^{(k)}(a)$, $k = 0, 1, \cdots, n$.

Furthermore, the terms $g^{(k)}(0)$ in (1.2) are somewhat surprising. Of course, the singularity at $t = 0$ due to $t^{\lambda-1}$ gives a hint that this point may significantly contribute to the asymptotic expansion, especially when a is small. But for moderate and large values of a, we cannot expect relevant information from the function values at $t = 0$.

In our opinion, the expansion (1.2) can be considerably simplified. Let us suppose that g and its first n derivatives are continuous and bounded on $[0, \infty)$. We write

$$g(t) = \sum_{k=0}^{n-1} c_k(t-a)^k + r_n(t), \qquad c_k = g^{(k)}(a)/k!.$$

Then we have

$$F(z, a) = \sum_{k=0}^{n-1} c_k F_k + R_n \tag{2.1}$$

with

$$F_k = \int_a^\infty e^{-z(t-a)}t^{\lambda-1}(t-a)^k\,dt, \tag{2.2}$$

$$R_n = \int_a^\infty e^{-z(t-a)}t^{\lambda-1}r_n(t)\,dt. \tag{2.3}$$

The first few functions F_k are easily computed. It turns out that

$$F_0 = \Gamma(\lambda)Q, \qquad F_1 = (\lambda z^{-1} - a)F_0 + a^\lambda z^{-1}, \tag{2.4}$$

where Q is essentially an incomplete gamma function and is defined in (1.3). By partial integration of (2.2) we obtain

$$F_{k+1} = z^{-1}[(k + \lambda - az)F_k + akF_{k-1}], \qquad k \geqq 1. \tag{2.5}$$

Hence, if F_0 is computed, the remaining F_k can be generated by (2.5).

The functions F_k are confluent hypergeometric functions. In the notation of [1], we have

$$\begin{aligned} F_k &= k!a^{k+\lambda}U(k+1, k+1+\lambda, az) \\ &= k!z^{-k-\lambda}U(1-\lambda, 1-\lambda-k, az). \end{aligned} \tag{2.6}$$

The second representation enables us to write for $0 < \lambda < 1$,

$$F_k = \frac{k!\,z^{-k-\lambda}}{\Gamma(1-\lambda)}\int_0^\infty e^{-azt}t^{-\lambda}(1+t)^{-k-1}\,dt, \tag{2.7}$$

from which follows, by majorizing the exponential function in the integrand by 1,

$$F_k \leqq z^{-k-\lambda}\Gamma(k+\lambda). \tag{2.8}$$

As follows from (2.2), this bound is also valid for $\lambda = 1$.

If on $[0, \infty)$ an estimate is known for $g^{(k)}$, say $|g^{(k)}(t)| \leqq a_k$, and a, λ and z are real, then R_n in (2.3) may be majorized by $|R_n| \leqq a_n F_n/n!$. Using (2.8), we obtain uniformly in a for $a \geqq 0$,

$$|R_n| \leqq a_n z^{-n-\lambda}\Gamma(n+\lambda)/n!.$$

Consequently, in the notation of [2], we have

$$F(z,a) \sim \sum c_k F_k \quad \{z^{-k-\lambda}\} \quad \text{as } z \to \infty.$$

This shows that (2.1) is an asymptotic expansion, holding uniformly in a for $a \geqq 0$, with respect to the asymptotic sequence $\{z^{-n-\lambda}\}$, which does not depend on a.

From a practical point of view, the expansion in (2.1) is more suitable than (1.2), since the coefficients c_k are simply expressed in terms of $g^{(k)}(a)$. Both expansions have the same bounds for the remainders. As a minor improvement, our expansion is also uniformly valid with respect to λ on compact subintervals of $(0, 1]$.

3. The numerical analyst may wonder if the sequence $\{F_k\}$ can be generated in a stable way by using (2.5). The answer is affirmative, as one easily deduces from the qualitative behavior of the linearly independent solutions of the second order difference equation (2.5). With

$$G_k = \int_0^a e^{-zt}(t-a)^k t^{\lambda-1}\,dt = a^{\lambda+k}(-1)^k \frac{\Gamma(\lambda)\Gamma(k+1)}{\Gamma(k+\lambda+1)} M(\lambda, k+\lambda+1, -az), \tag{3.1}$$

the functions F_n, G_n constitute a linearly independent pair of solutions of (2.5), as follows from the asymptotic behavior

$$F_n \sim n!\,z^{-n-\lambda}(1+a/n)^{\lambda+1}n^{\lambda-1}, \quad n \to \infty, \quad \text{uniformly in } a \geqq 0, \tag{3.2}$$

and from the inequality,

$$|G_n| \leqq a^{n+\lambda}\Gamma(\lambda)\Gamma(n+1)/\Gamma(n+\lambda+1), \qquad n = 0, 1, \cdots. \tag{3.3}$$

Formula (3.2) is easily derived with saddle point techniques from (2.7), and (3.3) follows from (3.1) by majorizing the exponential function by 1.

The relations (3.2) and (3.3) show that, in the sense of [4], the solution G_n is a minimal solution of (2.5) and F_n a dominant solution.

4. The relation between Erdélyi's expansion (1.2) and our expansion (2.1) can be illustrated by writing

$$F_k = P_k F_0 + Q_k a^\lambda z^{-1}, \qquad k = 0, 1, \cdots.$$

P_k and Q_k are polynomials in z^{-1} satisfying (2.5) with initial values $P_0 = 1, Q_0 = 0$, $P_1 = \lambda z^{-1} - a$, $Q_1 = 1$. By using the recurrence relation it can be proved that

$$P_k = z^{-k}\sum_{j=0}^{k} (-az)^{k-j}\binom{k}{j}\Gamma(\lambda+j)/\Gamma(\lambda), \qquad k = 0, 1, \cdots. \tag{4.1}$$

Hence, in a formal way, our expansion (2.1) can be written as

$$F(z, a) \sim F_0 \sum c_k P_k + a^\lambda z^{-1} \sum c_k Q_k. \tag{4.2}$$

With the substitution of (4.1) and using the (formal) expansion

$$g^{(j)}(t) = \sum_{k=j}^{\infty} c_k \frac{k!}{(k-j)!}(t-a)^{k-j}$$

at $t = 0$, we obtain, by interchanging the order of summation,

$$F(z, a) \sim Q \sum z^{-k}\Gamma(k+\lambda)g^{(k)}(0)/k! + a^\lambda z^{-1} \sum c_k Q_k.$$

The first series in this expression is exactly the first series of Erdélyi in (1.2). The second series is much more complicated, but probably it can be identified with the corresponding series of Erdélyi.

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