Note

On the Numerical Evaluation of the Ordinary Bessel Function of the Second Kind

1. Introduction

1.1. Definitions and Relevant Properties

The ordinary Bessel function of the first kind

\[ J_\nu(z) = (z/2)^\nu \sum_{k=0}^{\infty} \frac{(-z^2/4)^k}{k!(\nu + k + 1)} \]  

(1.1)

and the ordinary Bessel function of the second kind

\[ Y_\nu(z) = \frac{\cos \nu\pi J_\nu(z) - J_{-\nu}(z)}{\sin \nu\pi} \]  

(1.2)

are two linearly independent solutions of the difference equation

\[ f_{\nu+1} - (2\nu/z)f_{\nu} + f_{\nu-1} = 0. \]  

(1.3)

This equation can be used to compute \( Y_{\nu+n} \) for \( n = 2, 3, \ldots \) when \( Y_{\nu} \) and \( Y_{\nu+1} \) are given. In the forward direction the recurrence formula (1.3) for \( Y_{\nu} \) is numerically stable, whereas it is unstable for \( J_{\nu} \) (see Gautschi [1]).

The ordinary Bessel functions of the third kind are the Hankel functions

\[ H_v^{(1)}(z) = J_v(z) + iY_v(z), \quad H_v^{(0)}(z) = J_v(z) - iY_v(z). \]  

(1.4)

Important for the representation of the Hankel functions for large \( |z| \) are the functions \( P(\nu, z) \) and \( Q(\nu, z) \) defined by

\[ H_v^{(1,2)}(z) = [2(\pi z)]^{1/2} e^{\pm i\chi} [P(\nu, z) \pm iQ(\nu, z)], \]  

(1.5)

where the + sign is used for \( H_v^{(1)} \), the − sign is used for \( H_v^{(2)} \) and

\[ \chi = z - \pi(2\nu + 1)/4. \]  

(1.6)

For large \( |z| \), \( P \) and \( Q \) are slowly varying and the oscillatory behavior of \( H_v^{(1)} \) and \( H_v^{(2)} \).
$H^{(2)}_\nu$ is contained in the exponential function in (1.5). From (1.4) and (1.5) we obtain

\[
Y_\nu(z) = \frac{2}{(\pi z)^{1/2}} \left[ P(\nu, z) \sin \chi + Q(\nu, z) \cos \chi \right]
\]

\[
J_\nu(z) = \frac{2}{(\pi z)^{1/2}} \left[ P(\nu, z) \cos \chi - Q(\nu, z) \sin \chi \right].
\]

(1.7)

Again, the oscillatory behavior of $J_\nu$ and $Y_\nu$ is fully described by the circular functions in (1.7).

The connection between the ordinary Bessel functions and the modified Bessel functions follows from

\[
H^{(1)}_\nu(z) = -2i\pi^{-1}e^{-\nu \pi i/2}K_\nu(ze^{-\nu \pi i/2}) \quad (-\frac{1}{2}\pi < \arg z \leq \pi),
\]

\[
H^{(2)}_\nu(z) = 2i\pi^{-1}e^{\nu \pi i/2}K_\nu(ze^{\nu \pi i/2}) \quad (-\pi < \arg z \leq \frac{3}{2}\pi).
\]

(1.8)

From the Wronskian

\[
J_{\nu+1}(z) Y_\nu(z) - J_\nu(z) Y_{\nu+1}(z) = \frac{2}{\pi z}
\]

and (1.7) it easily follows that

\[
P(\nu, z) P(\nu + 1, z) + Q(\nu, z) Q(\nu + 1, z) = 1.
\]

(1.9)

1.2. Contents of the Paper

We give algorithms for the computation of $Y_\nu$ and $Y_{\nu+1}$ and we use the methods of our previous paper on the computation of $K_\nu$ and $K_{\nu+1}$ (see Temme [6]). Our results in [6] can be used for complex values of $z$. Here we give the explicit results for $Y_\nu$ and $Y_{\nu+1}$ and these results follow immediately from [6] by using (1.8).

For the computation of $J_\nu$, the reader is referred to Gautschi [1], where an algorithm is given for the computation of $J_{\nu+n}(z)$, $n = 0, 1, 2, \ldots, N$. See also Gautschi [2]. In Luke [4] rational approximations for $J_\nu$ and $Y_\nu$ are given based on Padé-representations for large $|z|$. In Luke [5] a double series of Chebyshev polynomials and values of the coefficients are given for both $Y_\nu$, $J_\nu$ for $z \geq 5$. In Goldstein and Thaler [3] the computation of $Y_\nu$ is based on series expansions in ordinary Bessel functions of the first kind, but the treatment of small $|\nu|$-values is not satisfactory.

2. The Computation for Small $|z|$

In order to obtain a more symmetric representation in (1.2) we write

\[
\cos \nu \pi J_\nu(z) - J_{-\nu}(z) = J_\nu(z) - J_{-\nu}(z) - 2 \sin^2(\nu \pi /2) J_\nu(z).
\]

(2.1)
Furthermore we introduce the following notation
\[ c_k = (-z^2/4)^k/k!, \]
\[ p_k = (v/\sin \nu \pi)(z/2)^{-v}/\Gamma(k + 1 - \nu), \]
\[ q_k = (v/\sin \nu \pi)(z/2)^v/\Gamma(k + 1 + \nu), \]
\[ f_k = (p_k - q_k)/v, \]
\[ g_k = f_k + 2v^{-1} \sin^2(\nu \pi/2) q_k, \]
\[ h_k = -kg_k + p_k, \]
where \( k = 0, 1, \ldots \). We have for \( k = 1, 2, \ldots \) the recurrence relations
\[ p_k = p_{k-1}/(k - \nu), \quad q_k = q_{k-1}/(k + \nu), \]
\[ f_k = (kf_{k-1} + p_{k-1} + q_{k-1})/(k^2 - \nu^2). \]

Substitution of (1.1) in (1.2) and using (2.1) yields
\[ Y_v(z) = - \sum_{k=0}^{\infty} c_k g_k. \] (2.2)

Considering (2.1) with \( \nu \) replaced by \( \nu + 1 \) and using (1.3) we have
\[ \cos(\nu + 1) \pi J_{\nu+1}(z) - J_{\nu-1}(z) = -[J_{\nu+1}(z) - J_{\nu+2}(z)] + (2/\nu) J_{\nu}(z) + 2 \sin^2(\nu \pi/2) J_{\nu+1}(z). \]

We obtain by substitution of (1.1)
\[ Y_{\nu+1}(z) = - (2/\nu) \sum_{k=0}^{\infty} c_k h_k. \] (2.3)

As in [6], \( f_0 \) can be represented in such a way that it can be computed with a satisfactorily small relative error.

For small values of \(|z|\) the series in (2.2) and (2.3) converge rapidly. But cancellation may occur in summing the series numerically. A strict error analysis, as for the modified Bessel function, can not easily be given, but from numerical experiments it turns out that for \(|z| < 3\) the computation is stable.

3. The Computation for \(|z| \geq 3\)

For \(|z| \geq 3\) we compute \( P(v, z), P(v + 1, z), Q(v, z) \) and \( Q(v + 1, z) \), by using the functions \( k_n(z) \) introduced in our previous paper [6]. For \( K_v \) and \( K_{v+1} \) we needed \( k_0(z) \) and \( k_1(z) \). From (1.8) it turns out that for the \( P- \) and \( Q- \) functions the functions \( k_0(\pm iz) \) and \( k_1(\pm iz) \) can be used. The application of the method in [6] is straightforward. However, the determination of the starting index \( N \) for the Miller
algorithm caused some trouble, since our error analysis in [6] was based on the case of real variables. But trying out the results of [6] for the $P$- and $Q$-functions we noticed that the determination of the starting index $N$ can indeed be based upon the estimations given in [6].

4. ALGOL 60 PROCEDURES

The algorithms for the computation of $Y_v(z)$ and $Y_{v+1}(z)$ are given as an ALGOL 60 procedure for the case of real values of $v$ and $z$, $z > 0$. For convenience we write $v = a$ and $z = x$.

The procedure `bessya` computes for $x > 0$ and $a \in \mathbb{R}$ the functions $Y_a(x)$ and $Y_{a+1}(x)$; `bessya` calls for three nonlocal procedures `sinh`, `recip gamma`, and `besspqa`. For the text of `sinh`, and `recip gamma` the reader is referred to [6]. In `besspqa` the functions $P(a, x)$, $P(a + 1, x)$, $Q(a, x)$ and $Q(a + 1, x)$ are computed. We supply `besspqa` as a separate procedure since it can also be used for the computation of the Bessel functions $J_a(x)$ and $J_{a+1}(x)$ (see (1.7)). In `bessya` the procedure `besspqa` is called for $x \sim 3$ and $|a| < .5$, but the algorithm in `besspqa` converges for all $x$ and $a$ ($x > 0$). It is recommended, however, to take $x > \max(|a|, 3)$. For $|a| > x$ the recurrence relations

$$P(a + 1, x) = P(a - 1, x) - 2a/x Q(a, x)$$
$$Q(a + 1, x) = Q(a - 1, x) + 2a/x P(a, x)$$

can be used. These relations are valid for real $a$ and $x$. They can be derived by substitution of (1.5) in (1.3). However, for $|a| + 1 > x$, computation of $J_a(x)$ and $J_{a+1}(x)$ by using (1.7) will cause a loss of correct significant digits.

The precision in the procedures `bessya` and `besspqa` can be controlled by using the variable `eps`. For `besspqa` its entry value corresponds to the desired relative accuracy in $pa$, $pa_1$, $qa$ and $qa_1$. Also in `bessya` it corresponds to relative accuracy, except in the neighborhoods of zeros of $Y_a(x)$ or $Y_{a+1}(x)$. In that case $ya$ or $ya_1$ are given with absolute accuracy `eps`.

The procedures `bessya` and `besspqa` were tested on the CD CYBER 73 of SARA, Amsterdam. For $a = 0, 0.2, 0.4, x = .5, 1, 2, 3, 5, 7, 10, 20, 50, 100$ and $eps = 10^{-10}$ we checked relation (1.9). The output of $|pa, pa_1 + qa, qa_1 - 1|$ is given in Table I. The procedure `bessya` was also tested in the neighborhood of $x = 3$. For $x^\pm = 3 \pm 2^{-48}$ we computed the numerical values of the expressions

$$d_0 = \{Y_a(x^-) - Y_a(x^+)\},$$
$$d_1 = \{Y_{a+1}(x^-) - Y_{a+1}(x^+)\}.$$  

In Table II we give $d_0$, $d_1$, the maximum number of terms ($n$) used in (2.1), and the starting index $N$ for the Miller algorithm.
### TABLE I

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<td>$7.1_{10} - 15$</td>
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### TABLE II

<table>
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<td></td>
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</tr>
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<tr>
<td></td>
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<td>$3.6_{10} - 14$</td>
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<tr>
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<td>$(n, N)$</td>
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<td>$(n, N)$</td>
<td>$(9, 17)$</td>
<td>$(11, 37)$</td>
<td>$(13, 64)$</td>
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</tbody>
</table>
procedure bessya(a,x,eps,ya,yal); value a,x,eps; real a,x,eps,ya,yal;
begin real b,c,d,e,f,g,h,p,q,r,s;
integer n,na; Boolean rec, rev;
pi:= 4 \times arctan(1); na:= entier(a+.5); rec:= a \geq .5;
rev:= a < -.5; if rev \lor rec then a:= a-na;
if a = -.5 then
begin p:= sqrt(2/pi/x); f:= p \times sin(x); g:= -p \times cos(x) end else
if x < 3 then
begin b:= x/2; d:= -ln(b); e:= a \times d;
c:= if abs(a) < 10^{-15} then 1/pi else a/sin(a \times pi);
s:= if abs(e) < 10^{-15} then sinh(e)/e; e:= (e+1/e)/2;
f:= 2 \times c \times (p \times e + q \times s \times d); e:= a \times a;
p:= g \times c; q:= 1/g/pi; c:= a \times pi/2;
r:= if abs(e) < 10^{-15} then 1 else sin(c)/c; r:= pi \times c \times r \times r;
c:= 1; d:= -b \times b; ya:= f + r \times q; yal:= p;
for n:= 1, n+1 while
abs(g/(1+abs(ya)))+abs(h/(1+abs(yal))) > eps do
begin f:= (f \times n + p + q)/(n \times n - e); c:= c \times d/n;
p:= p/(n-a); q:= q/(n+a);
g:= c \times (f + r \times q); h:= c \times p - n \times g;
ya:= ya + g; yal:= yal + h
end;
end else
begin b:= x - pi \times (a + .5)/2; c:= cos(b); s:= sin(b);
d:= sqrt(2|x|pi);
besspqa(a,x,eps,p,q,b,h);
f:= d \times (p \times s + q \times c); g:= d \times (h \times s - b \times c)
end;
end if
begin x:= 2|x|;
for n:= 0 step 1 until na do
begin h:= x \times (a - n) \times f - g; g:= f; f:= h end
end else if rec then
begin x:= 2|x|;
for n:= 1 step 1 until na do
begin h:= x \times (a + n) \times g - f; f:= g; g:= h end
end;
yaa:= f; yal:= g
end bessya;
procedure besspqa(a,x,eps,pa,qa,pa1,qa1); value a,x,eps;
real a,x,eps,pa,qa,pa1,qa1;
begin real b,c,d,e,f,g,p,p0,q,q0,r,s;
integer n,na; Boolean rec,rev;
rev:= a < -.5; if rev then a:= -a-1;
rec:= a ≥ .5; if rec then
begin na:= entier(a+.5); a:= a - na end;
if a = -.5 then
begin pa:= pa1:= 1; qa:= qa1:= 0 end else
begin c:= .25 - a × a; b:= x + x; p:= 4 × arctan(1);
e:= (x × cos(a × p)/p/eps)‡2; p:= 1; q:= -x; r:= s:= 1 + x × x;
for n:= 2, n + 1 while r × n × n < e do
begin d:= (n + 1 × c/n)/s; p:= (2 × n - p × d)/(n + 1);
qu:= (-b + q × d)/(n + 1); s:= p × p + q × q; r:= r × s;
end;
f:= p:= p/s; g:= q:= -q/s;
for n:= n, n - 1 while n > 0 do
begin r:= (n+1) × (2-p) - 2; s:= b + (n+1) × q; d:= (n - 1 + c/n)/
(r × r + s × s); p:= d × r; q:= d × s; e:= f;
f:= p × (e + 1) - g × q; g:= q × (e + 1) + p × g
end;
f:= 1 + f; d:= f × f + g × g;
pa:= f/d; qa:= -g/d; d:= a + .5 - p; q:= a + x;
pa1:= (pa × q - qa × d)/x;
qa1:= (qa × q + pa × d)/x;
if rec then
begin x:= 2/x; b:= (a + 1) × x;
for n:= 1 step 1 until na do
begin p0:= pa - qa1 × b; q0:= qa + pa1 × b;
    pa:= pa1; pa1:= p0; qa:= qa1; qa1:= q0; b:= b + x
end
end;
if rev then
begin p0:= pa1; pa1:= pa; pa:= p0;
    q0:= qa1; qa1:= qa; qa:= q0
end
end besspqa;
REFERENCES


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