A Calculus for Sequential Logic with 4 Values

Jan A. Bergstra[1,2] Jaco van de Pol[2]

[1] *University of Amsterdam, Programming Research Group*
Kruislaan 403, 1098 SJ Amsterdam, The Netherlands

[2] *Utrecht University, Department of Philosophy*
Heidelberglaan 8, 3584 CS Utrecht, The Netherlands

E-mail: `janb@fwi.uva.nl - jaco@phil.ruu.nl`

Contents

1 Introduction

This note presents a complete axiomatisation for four-valued sequential logic. 'Sequential' means that arguments are evaluated from left to right, until an answer can be obtained. Three-valued sequential logic is due to McCarthy [6]. In [1] four truth-values are introduced: *true*, *false*, *mistake* and *divergent*. Several four-valued logics arise by restricting the set of connectives. In the nomenclature of [1], four-valued sequential logic is characterized as $\Sigma_4(\neg, {}_{\circ}\!\wedge, {}^{\circ}\!\vee)$. An axiomatisation of this system has not been given before.

In [2] it is examined whether four-valued sequential logic can serve as a basis for data type specifications. That application motivates and justifies the metamathematical study of four-valued logics. Our complete axiomatisation can also be viewed as an ω-complete data type specification (see [4]). We refer to [1] for an introduction to three- and four-valued logic and also for further references.

In [3] a complete axiomatisation is given for McCarthy's system. Completeness is obtained by characterizing all algebras satisfying the axioms. The completeness proof for the axiomatisation of the four-valued system that we give is quite different. Our proof yields a systematic method to prove each valid formula from the axioms.

2 Four-valued Sequential Logic

Following [1] we extend the usual truth values t and f (for *true* and *false*) with two other constants d and m, modeling a *diverging* computation and an error-situation (i.e. a *mistake* has been made and there is no point in going on). The difference between d and m can be illustrated by the following equations for conjunction (see Table 1 for the complete definition).

$$f \wedge d = d \wedge f = f \qquad \text{but} \qquad f \wedge m = m \wedge f = m \ .$$

The first equation shows that a divergent computation may be circumvented, because the final result will be f in any case. In the second case, a mistake has been made, and this has to be reported. Conjunction behaves strict w.r.t. m, but non-strict w.r.t. d.

As the first equation reveals, conjunction needs a parallel computation. If one of the arguments can be evaluated to false, a diverging computation in the other argument must be avoided. The definedness operator ($\downarrow$) is not computable at all (in a plausible model of computation). By definition, $\downarrow X = f$ when $X = d$ or $X = m$ and $\downarrow X = t$ otherwise. Intuitively, we cannot know whether an ongoing computation of X will diverge, or result in an answer eventually.

It is also possible to study sequential connectives. Here the computation starts on the left and terminates as soon as an answer can be given. We write $\mathbin{{}_{\circ}\!\wedge}$ for the *left sequential conjunction* (also known as 'conditional and'). See Table 2 for the definition. Here a mistake is not always reported. Some typical equations are:

$$f \mathbin{{}_{\circ}\!\wedge} d = f \qquad d \mathbin{{}_{\circ}\!\wedge} f = d \qquad f \mathbin{{}_{\circ}\!\wedge} m = f \qquad m \mathbin{{}_{\circ}\!\wedge} f = m \ .$$

The system that extends the truth values $\{t, f, m, d\}$ with connectives $o_1, \ldots, o_n$ is denoted by $\Sigma_4(o_1, \ldots, o_n)$. In [1] it is proved that the system $\Sigma_4(\neg, \wedge, \downarrow, \mathbin{{}_{\circ}\!\wedge})$ is truth-functionally complete. This means that every four-valued function can be expressed in terms of the truth values, combined with negation, conjunction, definedness and the left sequential conjunction.

In this paper we focus on $\Sigma_4(\neg, \mathbin{{}_{\circ}\!\wedge}, \mathbin{{}^{\circ}\!\vee})$, the system of strongly sequential truth functions. Although $\mathbin{{}^{\circ}\!\vee}$ can be defined from $\mathbin{{}_{\circ}\!\wedge}$ and $\neg$ in the usual way, we incorporate it in the language. Table 2 contains the truth tables for the sequential connectives.

In this system, several classical principles are lacking. We mention commutativity of $\mathbin{{}_{\circ}\!\wedge}$ and right-distributivity. In Table 3 we list 10 laws that hold in $\Sigma_4(\neg, \mathbin{{}_{\circ}\!\wedge}, \mathbin{{}^{\circ}\!\vee})$. These laws are self explaining, except the last which expresses a valid variant of right-distributivity. We claim that all other valid laws can be derived from these 10. So Table 3 gives a complete axiomatisation of $\Sigma_4(\neg, \mathbin{{}_{\circ}\!\wedge}, \mathbin{{}^{\circ}\!\vee})$. This claim is proved in Section 3. In Section 4 we show that each of the laws (1)–(7) and (10) are independent of the other laws. We do not know whether (8) and (9) are independent.

$\wedge$	m	t	f	d
m	m	m	m	m
t	m	t	f	d
f	m	f	f	f
d	m	d	f	d

$\vee$	m	t	f	d
m	m	m	m	m
t	m	t	t	t
f	m	t	f	d
d	m	t	d	d

$\downarrow$	
m	f
t	t
f	t
d	f

Table 1: Parallel conjunction, parallel disjunction and definedness.

$\neg$	
m	m
t	f
f	t
d	d

$\mathbin{_\circ\!\wedge}$	m	t	f	d
m	m	m	m	m
t	m	t	f	d
f	f	f	f	f
d	d	d	d	d

$\mathbin{^\circ\!\vee}$	m	t	f	d
m	m	m	m	m
t	t	t	t	t
f	m	t	f	d
d	d	d	d	d

Table 2: The connectives of $\Sigma_4(\neg, \mathbin{_\circ\!\wedge}, \mathbin{^\circ\!\vee})$.

$$
\begin{array}{lrcl}
(1) & \neg d & = & d \\
(2) & \neg m & = & m \\
(3) & \neg t & = & f \\
(4) & \neg\neg X & = & X \\
(5) & t \mathbin{_\circ\!\wedge} X & = & X \\
(6) & f \mathbin{_\circ\!\wedge} X & = & f \\
(7) & X \mathbin{^\circ\!\vee} Y & = & \neg(\neg X \mathbin{_\circ\!\wedge} \neg Y) \\
(8) & (X \mathbin{_\circ\!\wedge} Y) \mathbin{_\circ\!\wedge} Z & = & X \mathbin{_\circ\!\wedge} (Y \mathbin{_\circ\!\wedge} Z) \\
(9) & X \mathbin{_\circ\!\wedge} (Y \mathbin{^\circ\!\vee} Z) & = & (X \mathbin{_\circ\!\wedge} Y) \mathbin{^\circ\!\vee} (X \mathbin{_\circ\!\wedge} Z) \\
(10) & (X \mathbin{^\circ\!\vee} Y) \mathbin{_\circ\!\wedge} Z & = & (\neg X \mathbin{_\circ\!\wedge} (Y \mathbin{_\circ\!\wedge} Z)) \mathbin{^\circ\!\vee} (X \mathbin{_\circ\!\wedge} Z)
\end{array}
$$

Table 3: The axiomatisation of $\Sigma_4(\neg, \mathbin{_\circ\!\wedge}, \mathbin{^\circ\!\vee})$.

3 A Complete Axiomatisation of $\Sigma_4(\neg\,, {}_{\circ}\!\wedge, {}^{\circ}\!\vee)$

We write P, Q for arbitrary open terms over $\Sigma_4(\neg\,, {}_{\circ}\!\wedge, {}^{\circ}\!\vee)$. X and Y are arbitrary variables. We write $\models P = Q$ if $P = Q$ holds in $\Sigma_4(\neg\,, {}_{\circ}\!\wedge, {}^{\circ}\!\vee)$. With $\Phi \vdash P = Q$, we denote that $P = Q$ is derivable with equational logic, using laws (1)–(10) of Table 3 and the assumptions in Φ.

Proposition 3.1. *For all P and Q, if $\vdash P = Q$ then $\models P = Q$.*

Proof: Laws (1)–(10) can be checked straightforwardly. This gives the required result. ⊠

The *dual* of a term is obtained by interchanging all occurrences of t with f, and ${}_{\circ}\!\wedge$ with ${}^{\circ}\!\vee$.

Lemma 3.2. *For all P and Q, we have $P = Q \vdash P^{\text{dual}} = Q^{\text{dual}}$.*

Proof: By (4) it satisfies to prove $\neg\neg P^{\text{dual}} = \neg\neg Q^{\text{dual}}$. Using (1)–(4) and (7), the second $\neg$ can be pushed inside step by step. Eventually, an instance of $\neg P = \neg Q$ is obtained. ⊠

In the sequel, we will denote the use of the dual of a derived law by the postfix d. E.g. (5d) is the dual of (5): $f \,{}^{\circ}\!\vee X = X$.

Lemma 3.3. *The following laws are derivable from 1–10:*

(20)	$X \,{}^{\circ}\!\vee f$	$=$	X	(*from* $3d, 5, 5d, 6, 10$)
(21)	$X \,{}_{\circ}\!\wedge X$	$=$	X	(*from* $6, 9d, 20$)
(22)	$X \,{}_{\circ}\!\wedge t$	$=$	$X \,{}_{\circ}\!\wedge X$	(*from* $20d, 21$)
(23)	$X \,{}^{\circ}\!\vee \neg X$	$=$	$\neg X \,{}^{\circ}\!\vee X$	(*from* $20d, 10, 21$)
(24)	$\neg X \,{}^{\circ}\!\vee X$	$=$	$X \,{}^{\circ}\!\vee t$	(*from* $20d, 10$)
(25)	$X \,{}_{\circ}\!\wedge \neg X$	$=$	$X \,{}_{\circ}\!\wedge f$	(*from* $23d, 24d$)
(26)	$X \,{}_{\circ}\!\wedge Y$	$=$	$X \,{}_{\circ}\!\wedge (Y \,{}_{\circ}\!\wedge X)$	(*from* $4, 9, 10, 10d, 20, 24d$)
(27)	$X \,{}_{\circ}\!\wedge (Y \,{}_{\circ}\!\wedge Z)$	$=$	$(X \,{}_{\circ}\!\wedge Y) \,{}_{\circ}\!\wedge (X \,{}_{\circ}\!\wedge Z)$	(*from* $8, 26$)
(28)	$d \,{}_{\circ}\!\wedge X$	$=$	d	(*from* $1, 6, 8, 21, 25$)
(29)	$m \,{}_{\circ}\!\wedge X$	$=$	m	(*from* $2, 6, 8, 21, 25$)

Proof:

20. Instantiate (10) with f, t and X.

21. Instantiate (9d) with X, f and f.

22. Straightforward.

23. Instantiate (10) with X, $\neg X$ and t.

24. Instantiate (10) with X, t and t.

25. Straightforward.

26.
$$\begin{aligned}
& X \mathbin{{}_\circ\!\wedge} Y \\
\stackrel{20}{=}\; & (X \mathbin{{}_\circ\!\wedge} Y) \mathbin{{}^\circ\!\vee} f \\
\stackrel{10d}{=}\; & (\neg X \mathbin{{}^\circ\!\vee} (Y \mathbin{{}^\circ\!\vee} f)) \mathbin{{}_\circ\!\wedge} (X \mathbin{{}^\circ\!\vee} f) \\
\stackrel{20}{=}\; & (\neg X \mathbin{{}^\circ\!\vee} Y) \mathbin{{}_\circ\!\wedge} X \\
\stackrel{10}{=}\; & (\neg\neg X \mathbin{{}_\circ\!\wedge} (Y \mathbin{{}_\circ\!\wedge} X)) \mathbin{{}^\circ\!\vee} (\neg X \mathbin{{}_\circ\!\wedge} X) \\
\stackrel{4}{=}\; & (X \mathbin{{}_\circ\!\wedge} (Y \mathbin{{}_\circ\!\wedge} X)) \mathbin{{}^\circ\!\vee} (\neg X \mathbin{{}_\circ\!\wedge} X) \\
\stackrel{24d}{=}\; & (X \mathbin{{}_\circ\!\wedge} (Y \mathbin{{}_\circ\!\wedge} X)) \mathbin{{}^\circ\!\vee} (X \mathbin{{}_\circ\!\wedge} f) \\
\stackrel{9}{=}\; & X \mathbin{{}_\circ\!\wedge} ((Y \mathbin{{}_\circ\!\wedge} X) \mathbin{{}^\circ\!\vee} f) \\
\stackrel{20}{=}\; & X \mathbin{{}_\circ\!\wedge} (Y \mathbin{{}_\circ\!\wedge} X)
\end{aligned}$$

27. Straightforward.

28. If $Z = \neg Z$ then $Z \stackrel{21}{=} Z \mathbin{{}_\circ\!\wedge} Z = Z \mathbin{{}_\circ\!\wedge} \neg Z \stackrel{25}{=} Z \mathbin{{}_\circ\!\wedge} f$. Hence

$$Z \mathbin{{}_\circ\!\wedge} X = (Z \mathbin{{}_\circ\!\wedge} f) \mathbin{{}_\circ\!\wedge} X \stackrel{8}{=} Z \mathbin{{}_\circ\!\wedge} (f \mathbin{{}_\circ\!\wedge} X) \stackrel{6}{=} Z \mathbin{{}_\circ\!\wedge} f = Z \;\; .$$

Now using (1), the required result follows.

29. Similar to (28)

⊠

Lemma 3.4. *Every closed term is provably equal to t, f, d or m.*

Proof: This is proved by term induction. In case of a negation, (1), (2), (3) and (3d) is used. In case of conjunction we use (5), (6), (28) and (29). Disjunction is the dual of conjunction. ⊠

Lemma 3.5. *Every term P is either provably equal to a closed term, or it is provably equal to $(X \mathbin{{}^\circ\!\vee} Q) \mathbin{{}_\circ\!\wedge} Q'$, for some variable X and terms Q and Q'. Moreover, X and the variables occurring in Q and Q' also occur in P.*

Proof: The lemma is proved with induction on P. The constants are clearly closed.

Case X: By (20) and (20d), $X = (X \mathbin{{}^\circ\!\vee} f) \mathbin{{}_\circ\!\wedge} t$.

Case $\neg P$: By induction hypothesis, either P is provably closed, (in which case $\neg P$ is provably closed too), or $\neg P = \neg((X \mathbin{{}^\circ\!\vee} Q) \mathbin{{}_\circ\!\wedge} Q')$. Applying (10), (4) and (7), we get $\neg P = (X \mathbin{{}^\circ\!\vee} (\neg Q \mathbin{{}^\circ\!\vee} \neg Q')) \mathbin{{}_\circ\!\wedge} (\neg X \mathbin{{}^\circ\!\vee} \neg Q')$, which is of the required format.

Case $P \mathbin{{}_\circ\!\wedge} Q$: If P is provably closed, then by Lemma 3.4, it is provably equal to t, f, d or m. Then $P \mathbin{{}_\circ\!\wedge} Q$ is either provably closed, or provably equal to Q. In the latter case the induction hypothesis for Q yields the required format.

If P is not provably closed, then by the induction hypothesis for P, we obtain P_1 and P_2 such that $P \mathbin{{}_{\circ}\!\wedge} Q = ((X \mathbin{{}^{\circ}\!\vee} P_1) \mathbin{{}_{\circ}\!\wedge} P_2) \mathbin{{}_{\circ}\!\wedge} Q$. Using (8) this can be brought in the required form.

Case $P \mathbin{{}^{\circ}\!\vee} Q$: The case that P is provably closed is similar to $\mathbin{{}_{\circ}\!\wedge}$. Otherwise, we find P_1 and P_2 by induction hypothesis for P, such that

$$
\begin{array}{ll}
 & P \mathbin{{}^{\circ}\!\vee} Q \\
\stackrel{IH}{=} & ((X \mathbin{{}^{\circ}\!\vee} P_1) \mathbin{{}_{\circ}\!\wedge} P_2) \mathbin{{}^{\circ}\!\vee} Q \\
\stackrel{10}{=} & ((\neg X \mathbin{{}_{\circ}\!\wedge} (P_1 \mathbin{{}_{\circ}\!\wedge} P_2)) \mathbin{{}^{\circ}\!\vee} (X \mathbin{{}_{\circ}\!\wedge} P_2)) \mathbin{{}^{\circ}\!\vee} Q \\
\stackrel{8d}{=} & (\neg X \mathbin{{}_{\circ}\!\wedge} (P_1 \mathbin{{}_{\circ}\!\wedge} P_2)) \mathbin{{}^{\circ}\!\vee} ((X \mathbin{{}_{\circ}\!\wedge} P_2) \mathbin{{}^{\circ}\!\vee} Q) \\
\stackrel{10d}{=} & (\neg\neg X \mathbin{{}^{\circ}\!\vee} ((P_1 \mathbin{{}_{\circ}\!\wedge} P_2) \mathbin{{}^{\circ}\!\vee} ((X \mathbin{{}_{\circ}\!\wedge} P_2) \mathbin{{}^{\circ}\!\vee} Q))) \mathbin{{}_{\circ}\!\wedge} (\neg X \mathbin{{}^{\circ}\!\vee} ((X \mathbin{{}_{\circ}\!\wedge} P_2) \mathbin{{}^{\circ}\!\vee} Q)) \\
\stackrel{4}{=} & (X \mathbin{{}^{\circ}\!\vee} ((P_1 \mathbin{{}_{\circ}\!\wedge} P_2) \mathbin{{}^{\circ}\!\vee} ((X \mathbin{{}_{\circ}\!\wedge} P_2) \mathbin{{}^{\circ}\!\vee} Q))) \mathbin{{}_{\circ}\!\wedge} (\neg X \mathbin{{}^{\circ}\!\vee} ((X \mathbin{{}_{\circ}\!\wedge} P_2) \mathbin{{}^{\circ}\!\vee} Q)),
\end{array}
$$

which is of the required form. ⊠

Lemma 3.6. *For any term P and variable X, we have*

(a) $\vdash X \mathbin{{}_{\circ}\!\wedge} P = X \mathbin{{}_{\circ}\!\wedge} P[X := t]$

(b) $\vdash \neg X \mathbin{{}_{\circ}\!\wedge} P = \neg X \mathbin{{}_{\circ}\!\wedge} P[X := f]$

Proof: Without loss of generality, we assume that P is built from constants, variables and negated variables, using the connectives $\mathbin{{}_{\circ}\!\wedge}$ and $\mathbin{{}^{\circ}\!\vee}$ (using (1)–(4) and (7), we can write each term in such a form). The lemma is then proved with induction on P.

Case $P = X$: use (22) for (a); use (4) and (24d) for (b).

Case $P = \neg X$: use (25) and (3) for (a); use (22) and (3d) to obtain (b).

Case $P = Y$ or $P = \neg Y$, with $Y \neq X$, is trivial.

Case $P \mathbin{{}_{\circ}\!\wedge} Q$: Use (27) and the induction hypothesis for P and Q.

Case $P \mathbin{{}^{\circ}\!\vee} Q$: Use (9) and the induction hypothesis for P and Q. ⊠

Theorem 3.7. *Axiom 1–10 is a complete axiomatisation for $\Sigma_4(\neg, \mathbin{{}_{\circ}\!\wedge}, \mathbin{{}^{\circ}\!\vee})$.*

Proof: Assume that $\models P = Q$. We will prove $\vdash P = Q$ by induction on the number of different variables occurring in this equation.

By Lemma 3.5, we have that either

(a) P is provably closed; or

(b) $\vdash P = (X \mathbin{{}^{\circ}\!\vee} P_1) \mathbin{{}_{\circ}\!\wedge} P_2$.

Similarly, we obtain that either

(c) Q is provably closed; or

(d) $\vdash Q = (Y \mathbin{{}^{\circ}\!\vee} Q_1) \mathbin{{}_{\circ}\!\wedge} Q_2$.

By Lemma 3.4, each provably closed term is provably equal to t, f, d or m. We first prove that case (a) and (d) cannot occur both, for assume (a) and (d). Then $\vdash P = c$, where c is one of the constants. By soundness, $\models P = c$ and $\models Q = (Y \mathbin{{}^{\circ}\!\vee} Q_1) \mathbin{{}_{\circ}\!\wedge} Q_2$, hence also $c = (Y \mathbin{{}^{\circ}\!\vee} Q_1) \mathbin{{}_{\circ}\!\wedge} Q_2$. Now taking $Y = d$ and $Y = m$, respectively, we get $\models d = m$, quod non. Similarly, (b) and (c) cannot occur both. Two cases remain:

- (a) and (c) hold (this includes the base of the induction). In this case $\vdash P = c$ and $\vdash Q = d$. Using soundness and the assumption that $\models P = Q$, we obtain $c \equiv d$, so $\vdash P = Q$.

- (b) and (d) hold. In this case $X \equiv Y$, for otherwise we could substitute d for X and m for Y, implying (via soundness) $\models m = d$.

  Define $P_1' \equiv (P_1 \mathbin{{}_{\circ}\!\wedge} P_2)[X := f]$, $P_2' \equiv (P_2[X := t])$. Then using (10), we have $\vdash P = (\neg X \mathbin{{}_{\circ}\!\wedge} P_1 \mathbin{{}_{\circ}\!\wedge} P_2) \mathbin{{}^{\circ}\!\vee} (X \mathbin{{}_{\circ}\!\wedge} P_2)$. By Lemma 3.6 we have $\vdash P = (\neg X \mathbin{{}_{\circ}\!\wedge} P_1') \mathbin{{}^{\circ}\!\vee} (X \mathbin{{}_{\circ}\!\wedge} P_2')$. In a similar way we can find Q_1' and Q_2' that do not contain X such that $\vdash Q = (\neg X \mathbin{{}_{\circ}\!\wedge} Q_1') \mathbin{{}^{\circ}\!\vee} (X \mathbin{{}_{\circ}\!\wedge} Q_2')$.

  Using $\models P = Q$ and soundness and taking $X = t$, we find $\models P_2' = Q_2'$. Taking $X = f$, we find $\models P_1' = Q_1'$. Now by induction hypothesis, $\vdash P_1' = Q_1'$ and $\vdash P_2' = Q_2'$. By equational logic, we find $\vdash P = Q$.

$\boxtimes$

4 Remarks

Extension. The existence of at least two error values is needed in the proof of Theorem 3.7 to make sure that $X \equiv Y$. If there is only one error value (i.e. MacCarthy's logic [6]) then the following law becomes valid; note that the leftmost variable changes:

$$((X \mathbin{{}_{\circ}\!\wedge} Y) \mathbin{{}^{\circ}\!\vee} (Y \mathbin{{}_{\circ}\!\wedge} X)) = ((Y \mathbin{{}_{\circ}\!\wedge} X) \mathbin{{}^{\circ}\!\vee} (X \mathbin{{}_{\circ}\!\wedge} Y)) \ .$$

Our proof easily generalizes to more than two error values. For a new error value, add an axiom $e = \neg e$. As in Lemma 3.3.28, we can then prove $e \mathbin{{}_{\circ}\!\wedge} X = e$; via (7) we obtain $e \mathbin{{}^{\circ}\!\vee} X = e$. With these equations, Lemma 3.4 can be extended to the new situation. Then the proof of Theorem 3.7 remains valid.

Independence of axioms. Below we list the arguments that each of the laws (1)–(7) and (10) is independent of the other laws.

1. Take as model the restriction of $\neg$, $\mathbin{{}_{\circ}\!\wedge}$ and $\mathbin{{}^{\circ}\!\vee}$ to the carrier set $\{t, f, m\}$ and interpret d by t. Then law 1 is false but laws 2–10 hold.

2. Similar to 1.

3. Without law 3, terms containing f cannot be proved equal to terms without f.

4. Without law 4, a term in which no symbols from $\{\neg, \overset{\circ}{\vee}, f, m, d\}$ occur cannot be proved equal to a term that contains some of these symbols.

5. Without law 5, terms without constants cannot be proved equal to a term containing a constant.

6. Without law 6, a closed term cannot be proved equal to an open term.

7. In the following model laws 1–6 and 8–10 hold, but 7 fails: The carrier set is $\{t, f, m, d\}$. Interpret $\neg$ as usual negation, $\overset{\circ}{\vee}$ as the constant function f and interpret $x \underset{\circ}{\wedge} y$ as y whenever $x = t$ and f otherwise.

10. In the following model, laws 1–9 hold, but 10 fails: Take $\{t, f, d\}$ as carrier set; interpret m as d. Interpret $\underset{\circ}{\wedge}$ as the restriction of $\wedge$ (Table 1) on the carrier set, and $\overset{\circ}{\vee}$ as the restriction of $\vee$. (This model is known as Kleene's three-valued logic [5]).

We have no argument for the independence of laws (8) and (9). It is easy to make a 5-valued model in which (1)–(7) and (10) are valid, but where (8) and (9) fail, so they cannot be dropped both.

References

[1] J.A. Bergstra, I. Bethke, and P. Rodenburg. A propositional logic with 4 values: true, false, divergent and meaningless. *Journal of Applied Non-Classical Logics*, 5(2):199–217, 1995.

[2] Jan Bergstra and Alex Sellink. Sequential data algebra primitives. Utrecht University, Dept. of Philosophy, 1996.

[3] F. Guzman and C.S. Squier. The algebra of conditional logic. *Algebra Universalis*, 27:88–110, 1990.

[4] J. Heering. Partial evaluation and ω-completeness of algebraic specifications. *Theoretical Computer Science*, 43:149–167, 1986.

[5] S.C. Kleene. On a notation for ordinal numbers. *Journal of Symbolic Logic*, 3:150–155, 1938.

[6] J. McCarthy. A basis for a mathematical theory of computation. In P. Braffort and D. Hirshberg, editors, *Computer Programming and Formal Systems*, pages 33–70. North-Holland, 1963.