INDICES OF DEPENDENCE BETWEEN TYPES IN MULTIVARIATE POINT PATTERNS

M.N.M VAN LIESHOUT
CENTRE FOR MATHEMATICS AND COMPUTER SCIENCE
PO BOX 94079, 1090 GB AMSTERDAM
THE NETHERLANDS

A.J. BADDELEY
DEPARTMENT OF MATHEMATICS
UNIVERSITY OF WESTERN AUSTRALIA
NEDLANDS WA 6907
AUSTRALIA

Indices of dependence between types in multivariate point patterns

M N M van Lieshout

Centre for Mathematics and Computer Science, PO Box 94079, 1090 GB Amsterdam, The Netherlands

A J Baddeley

Department of Mathematics, University of Western Australia, Nedlands WA 6907, Australia

Abstract

We propose new summary statistics quantifying several forms of dependence between types in a spatial pattern of points classified into distinct types. These statistics are the multivariate counterparts of the J-function for point processes of a single type, introduced in [18]. They are based on comparing distances from a type i point to *either* the nearest type j point *or* to the nearest point in the pattern regardless of type to these distances seen from an arbitrary point in space. Information about the range of interaction can also be inferred. Our statistics can be computed explicitly for a range of well-known multivariate point process models. Some applications to bivariate data sets are presented as well.

Keywords & Phrases: ants' nests, beta cells, empty space function, hamster tumour, J-function, multi-type point patterns, myrtle disease, nearest-neighbour distance distribution function, point processes, random labelling, spatial interaction, spatial statistics.

AMS Mathematics Subject Classification (Revision 1991): 60G55, 62M30, 62M11, 62G99.

1 Introduction

A *multivariate point pattern* is a spatial pattern of points, each point belonging to one of a finite number of distinct types [8]. Bivariate or two-type patterns in particular have often been reported and analysed. Examples considered in section 5 are a map of two species of ants' nests, a map of trees identified as healthy or diseased, a microscope image of retinal ganglion cells identified as 'on' or 'off', and a microscope image of cell nuclei classified into two types.

To investigate dependence between the different types of points, the usual approach [3, 9, 11, 12, 15, 19, 23, 28] begins by estimating 'cross-type' versions of the standard summary functions G and K. The purpose of this paper is to pursue an alternative.

In [18] we introduced a new summary function $J(t)$ for univariate (single-type) point patterns X:

$$J(t) = \frac{1 - G(t)}{1 - F(t)} \tag{1}$$

defined for all $t \geq 0$ with $F(t) \neq 1$, where the 'empty space function' F is the distribution function of the distance from an arbitrary fixed point 0 to the nearest point of the pattern X,

and the 'nearest neighbour distance function' G is the distribution function of the distance from a typical point of X to the nearest other point of X. The J function is an index of spatial interaction, identically equal to 1 for a Poisson process, and generally takes values less than 1 for clustered patterns and greater than 1 for ordered patterns. An appealing property is that the superposition $X_\bullet = X_1 \cup X_2$ of two *independent* point processes X_1, X_2 has J-function

$$J(t) = \frac{\lambda_1}{\lambda_1 + \lambda_2} J_1(t) + \frac{\lambda_2}{\lambda_1 + \lambda_2} J_2(t) \tag{2}$$

where J_1, J_2 are the J-functions of X_1, X_2 respectively and λ_1, λ_2 are their intensities. A similar statement holds for the superposition of m independent point processes.

In the present paper we extend these ideas to multivariate point patterns. Let X_i be the process of type i points and $X_\bullet = X_1 \cup \cdots \cup X_m$ the process of all points regardless of type. Three approaches are proposed, which correspond to investigating three different forms of independence between types. First we may compare the left and right hand sides of (2) or its analogue for m types. These two expressions are equal if $X_1, \ldots, X_m$ are independent. Secondly we may construct an 'inter-type' J function $J_{ij}(t)$ for each pair of types i and j. This is identically equal to 1 if X_i and X_j are (marginally) independent. Thirdly we may construct a function $J_{i\bullet}(t)$ for each i summarising the dependence of $X_\bullet$ on X_i. This reduces to a simple form if X_i is independent of $(X_j, j \neq i)$, and to another simple form if the 'independent random labelling' model holds.

Section 2 contains preliminaries and the main definitions of the J functions; in section 3 we calculate them for a wide variety of stochastic models and in section 4 we exhibit some applications to bivariate point pattern data.

2 Definitions and notation

2.1 Univariate J-function

First we recall some definitions from [18]. Suppose X is a stationary point process in $\mathbb{R}^d$ with finite positive intensity λ. The *empty space function* F of X is the cumulative distribution function of the distance from a fixed point (say, the origin) to the nearest point of X. Thus for $t \geq 0$

$$F(t) = \mathbb{P}\{X \cap B(0,t) \neq \emptyset\}$$

where $B(0,t)$ is the closed ball of radius t centred at the origin 0. The *nearest neighbour distance function* G is the distribution of the distance from a typical point of X to the nearest other point of X. For $t \geq 0$

$$\begin{aligned} G(t) &= P^0\{X \cap B(0,t) \setminus \{0\} \neq \emptyset\} \\ &= P^{!0}\{X \cap B(0,t) \neq \emptyset\} \end{aligned}$$

where P^0 is the Palm distribution [10, 17, 28] of X at 0, which can be interpreted as the conditional distribution of X given that there is a point of X at 0. On the last line $P^{!0}$ is the reduced Palm distribution, defined as the Palm distribution of the process with the point at 0 removed, i.e. $P^{!0}$ is the distribution of $X \setminus \{0\}$ under P^0.

In [18] we introduced

$$J(t) = \frac{1 - G(t)}{1 - F(t)}$$

defined for all $t \geq 0$ with $F(t) < 1$. If X is a Poisson process then $J(t) \equiv 1$. We found several representations of $J(t)$ derived from the *Nguyen-Zessin formula* [22]

$$\lambda \, \mathbf{E}^{!0}[f(X)] = \mathbf{E}[\lambda(0; X) \, f(X)] \tag{3}$$

where f is any nonnegative measurable function on the space of realisations of X, $\mathbf{E}^{!0}$ denotes the expectation with respect to $P^{!0}$, and $\lambda(0; X)$ is the conditional intensity of X at 0 assuming this exists.

2.2 Multivariate J-functions

Throughout the paper we consider a stationary multivariate point pattern Y in $\mathbb{R}^d$, each point belonging to one of m types. Formally Y is a stationary marked point process in $\mathbb{R}^d$ with marks in $\{1, 2, \ldots, m\}$. Equivalently $Y = (X_1, \ldots, X_m)$ is an m-tuple of jointly stationary point processes in $\mathbb{R}^d$, where X_i is the process consisting of points of type i. Write λ_i for the intensity of X_i. Define

$$X_\bullet = \bigcup_{i=1}^{m} X_i \, ,$$

the point process consisting of all random points regardless of type.

Henceforth i and j denote indices from the set $\{1, \ldots, m\}$. Let F_i and $F_\bullet$ be the empty space functions of X_i and $X_\bullet$ respectively; thus for $t \geq 0$

$$\begin{aligned} F_i(t) &= \mathbf{P}\{X_i \cap B(0,t) \neq \emptyset\}, \\ F_\bullet(t) &= \mathbf{P}\{X_\bullet \cap B(0,t) \neq \emptyset\}. \end{aligned}$$

Let $P^{(0,i)}$ be the Palm distribution of Y conditional on a point at 0 with mark i, and $P^{!(0,i)}$ the corresponding reduced Palm distribution, i.e. the distribution of $Y \setminus \{(0,i)\}$ under $P^{(0,i)}$. Then define the 'i-to-j' nearest neighbour distance function

$$G_{ij}(t) = P^{!(0,i)}\{X_j \cap B(0,t) \neq \emptyset\}$$

and the 'i-to-any' nearest neighbour distance function

$$G_{i\bullet}(t) = P^{!(0,i)}\{X_\bullet \cap B(0,t) \neq \emptyset\}$$

Thus G_{ij} is the distribution function of the distance from a typical point of type i to the nearest point of type j, and $G_{i\bullet}$ from a typical point of type i to the nearest point of any type. To keep notation uniform we write $G_{\bullet\bullet}$ for the ordinary G function of $X_\bullet$.

Definition 1 *For a stationary multivariate point process $(X_1, \ldots, X_m)$ on $\mathbb{R}^d$ define (for $i, j = 1, \ldots, m$)*

$$J_{ij}(t) = \frac{1 - G_{ij}(t)}{1 - F_j(t)} \tag{4}$$

$$J_{i\bullet}(t) = \frac{1 - G_{i\bullet}(t)}{1 - F_\bullet(t)} \tag{5}$$

for all $t \geq 0$ *for which* $F_j(t) < 1$ *or* $F_\bullet(t) < 1$ *respectively. For uniformity of notation we will also write*

$$J_{\bullet\bullet}(t) = \frac{1 - G_{\bullet\bullet}(t)}{1 - F_\bullet(t)}$$

for the J-function of $X_\bullet$.

In particular J_{ii} is the J-function of the univariate process X_i. Note that the definition of J_{ij} is not symmetric in i and j. While this may be undesirable for inference on processes appearing on an equal footing, it may be easier to interpret, especially when considering qualitatively different patterns.

Intuitively J_{ij} is a comparison between the distributions of the distances to the nearest type j point, measured from (a) an arbitrary fixed point in $\mathbb{R}^d$, (b) a typical type i point. The denominator of (4) is the unconditional probability of the event that there is no type j point within a distance t of 0. The numerator is the "conditional probability" of the same event given that there is a type i point at 0.

As in the univariate case [18], the value 1 is obtained when there is no spatial interaction: if X_i, X_j are independent processes, then standard calculations give $J_{ij} \equiv 1$. However, having a J_{ij}-function taking value 1 everywhere should not be seen as a characterisation of independence. In particular $J_{ii} \equiv 1$ is not a sufficient condition for X_i to be a Poisson process (cf. [4]). Similar remarks apply to the other statistics introduced in Definition 1.

Values $J_{ij} > 1$ can be interpreted as indicating inhibition (of type j points by type i points) since this is equivalent to $G_{ij}(t) < F_j(t)$, i.e. the presence of a type i point decreases the probability of finding a type j point nearby. Similarly, values less than 1 suggest positive association.

The statistic $J_{i\bullet}$ is a comparison between the distributions of the distances to the nearest random point of *any* type, measured from the origin and from a type i point. An interpretation of the values of $J_{i\bullet}$ analogous to that of J_{ij} applies. If $(X_1, \ldots, X_m)$ are independent then $J_{i\bullet}(t) = J_{ii}(t)$, the marginal J-function. If furthermore X_i is a Poisson process, then $J_{i\bullet}(t) \equiv 1$.

With equation (2) in mind we introduce the following function I.

Definition 2 *For a stationary multivariate point process* $(X_1, \ldots, X_m)$ *on* $\mathbb{R}^d$ *define*

$$I(t) = \sum_{i=1}^{m} \frac{\lambda_i}{\lambda_\bullet} J_{ii}(t) - J_{\bullet\bullet}(t) \tag{6}$$

where λ_i *is the intensity of* X_i *and* $\lambda_\bullet = \sum_{i=1}^m \lambda_i$ *is the intensity of* $X_\bullet$.

If $X_1, \ldots, X_m$ are independent then $I \equiv 0$ by [18, Theorem 2].

At least for bivariate processes $Y = (X_1, X_2)$, the sign of $I(t)$ should indicate the type of association between the two components X_1, X_2 with a positive value being suggestive of positive dependence. This definition is similar to Lotwick and Silverman's [19] suggestion of studying the sign of

$$T = \log(1 - F_\bullet) - \log(1 - F_1) - \log(1 - F_2)$$

for a bivariate point process, since T is zero when X_1 and X_2 are independent point processes.

3 Basic properties

3.1 Mixture formulas

Lemma 1 *For any stationary multivariate process* $Y = (X_1, \ldots, X_m)$

$$G_{\bullet\bullet}(t) = \sum_{i=1}^{m} \frac{\lambda_i}{\lambda_\bullet} G_{i\bullet}(t) \tag{7}$$

$$J_{\bullet\bullet}(t) = \sum_{i=1}^{m} \frac{\lambda_i}{\lambda_\bullet} J_{i\bullet}(t) \tag{8}$$

$$I(t) = \sum_{i=1}^{m} \frac{\lambda_i}{\lambda_\bullet} [J_{ii}(t) - J_{i\bullet}] \tag{9}$$

for all $t \geq 0$ *with* $F_\bullet(t) < 1$.

Proof : The reduced Palm distribution at 0 of Y with respect to $X_\bullet$ is

$$\sum_{i=1}^{m} \frac{\lambda_i}{\lambda_\bullet} P^{!(0,i)}. \tag{10}$$

This yields the expression for $G_{\bullet\bullet}$. The remaining identities follow by substitution. □

3.2 Case of independence

Here we calculate the multivariate J functions when some form of independence holds between types.

Lemma 2 *Let* $i \neq j$. *If* X_i *and* X_j *are (marginally) independent then* $J_{ij} \equiv 1$ *where defined.*

Proof : Clearly J_{ij} depends only on the marginal joint distribution of (X_i, X_j). If X_i, X_j are independent then the distribution of X_j under $P^{!(0,i)}$ is the same as its ordinary marginal distribution, so $G_{ij} \equiv F_j$ (see e.g. [11, p. 92] or [9, p. 700]) hence $J_{ij} \equiv 1$. □

Lemma 3 *If* X_i *is independent of* $(X_j, j \neq i)$, *then* $J_{i\bullet} \equiv J_{ii}$ *where defined. More generally this holds if* X_i *is independent of* $X_{-i} = \bigcup_{j \neq i} X_j$, *the univariate process consisting of points of all types except* i.

Proof : Let F_{-i} be the empty space function of X_{-i}. If X_i and X_{-i} are independent, then

$$1 - F_\bullet(t) = (1 - F_i(t))\,(1 - F_{-i}(t))$$

and under $P^{!(0,i)}$, X_i and X_{-i} are also independent with X_i governed by its reduced Palm distribution at 0, and X_{-i} by its ordinary marginal distribution. Thus

$$1 - G_{i\bullet}(t) = (1 - G_{ii}(t))\,(1 - F_{-i}(t))$$

and division by $1 - F_\bullet(t)$ yields the result. □

Definition 3 *The marked point process Y has the* random labelling property *if the marks (types) of the points are conditionally i.i.d. given the locations of the points.*

For example, if $Y = (X_1, \ldots, X_m)$ for independent Poisson processes X_i, $i = 1, \ldots, m$, then Y has the random labelling property.

Lemma 4 *Under the random labelling assumption with label probabilities p_i $(i = 1, \ldots, m)$, for all $t \geq 0$ with $F_\bullet(t) < 1$*

$$J_{i\bullet}(t) = J_{\bullet\bullet}(t) \tag{11}$$

while whenever $F_j(t) < 1$

$$J_{ij}(t) = \frac{\mathbf{E}^{!0}\left[(1-p_j)^{N_t}\right]}{\mathbf{E}[(1-p_j)^{N_t}]} \tag{12}$$

where $N_t = N(X_\bullet \cap B(0,t))$ is the number of points of any type in $B(0,t)$, and $\mathbf{E}^{!0}$ denotes expectation with respect to the reduced Palm distribution of $X_\bullet$.

Proof : If Q is the distribution of any univariate point process Z, let $\rho(Q)$ denote the distribution of the multivariate point process obtained by assigning i.i.d. random marks to the points of Z.

The random labelling assumption is that $P = \rho(P_\bullet)$ where P is the distribution of $(X_1, \ldots, X_m)$ and $P_\bullet$ is the distribution of $X_\bullet$. It can easily be shown that under random labelling,

$$P^{!(0,i)} = \rho\left(P_\bullet^{!0}\right) \tag{13}$$

i.e. the reduced Palm distribution of $(X_1, \ldots, X_m)$ given a point of type i at 0 is equivalent to applying random labelling to the reduced Palm distribution of $X_\bullet$. Thus

$$\begin{aligned} G_{i\bullet}(t) &= P^{!(0,i)}\{X_\bullet \cap B(0,t) \neq \emptyset\} \\ &= \rho\left(P_\bullet^{!0}\right)\{X_\bullet \cap B(0,t) \neq \emptyset\} \\ &= P_\bullet^{!0}\{X_\bullet \cap B(0,t) \neq \emptyset\} \\ &= G_{\bullet\bullet}(t) \end{aligned}$$

and substituting in the definition of $J_{i\bullet}$ yields (11) The second result (12) follows from (13) and the representation of the J-function for an independent thinning in [18, Theorem 3]. □

3.3 Representations

Here we investigate explicit representations for the various J and I functions in terms of conditional intensities, analogous to the univariate case [18, Theorem 1].

Write $\lambda_i(0; X_i)$ for the conditional intensity of X_i at 0, and $\lambda_\bullet(0; X_\bullet)$ for that of $X_\bullet$, defined to satisfy the analogues of the Nguyen-Zessin formula (3), if they exist. Let $\lambda((0,i); Y)$ be the

conditional intensity (if it exists) of the multivariate process $Y = (X_1, \ldots, X_m)$ for a point at 0 with type i, defined to satisfy the multivariate counterpart of (3)

$$\lambda_i \, \mathbf{E}^{!(0,i)} f(Y) = \mathbf{E}[\lambda((0,i);Y) \, f(Y)] \tag{14}$$

for any nonnegative measurable function f on the space of realisations of Y. In particular taking $f \equiv 1$

$$\lambda_i = \mathbf{E}\lambda((0,i);Y).$$

The existence of $\lambda((0,i);Y)$ implies that of $\lambda_i(0;X_i)$ and $\lambda_\bullet(0;X_\bullet)$ and indeed

$$\mathbf{E}[\lambda((0,i);Y)|X_i] = \lambda_i(0;X_i) \quad \text{a.s.} \tag{15}$$

$$\mathbf{E}\left[\sum_{i=1}^{m} \lambda((0,i);Y) \middle| X_\bullet\right] = \lambda_\bullet(0;X_\bullet) \quad \text{a.s.} \tag{16}$$

by (3) and (14).

Lemma 5 *Let $Y = (X_1, \ldots, X_m)$ be any stationary multivariate process for which the conditional intensity $\lambda((0,i);Y)$ exists and satisfies (14).*

Then $G_{ij}(t) < 1$ implies $F_j(t) < 1$ and

$$J_{ij}(t) = \mathbf{E}\left[\frac{\lambda((0,i);Y)}{\lambda_i} \middle| X_j \cap B(0,t) = \emptyset\right] \tag{17}$$

$$= \left(\mathbf{E}^{!(0,i)}\left[\frac{\lambda_i}{\lambda((0,i);Y)} \middle| X_j \cap B(0,t) = \emptyset\right]\right)^{-1} \tag{18}$$

Similarly $G_{i\bullet}(t) < 1$ implies $F_\bullet(t) < 1$ and

$$J_{i\bullet}(t) = \mathbf{E}\left[\frac{\lambda((0,i);Y)}{\lambda_i} \middle| X_\bullet \cap B(0,t) = \emptyset\right] \tag{19}$$

$$= \left(\mathbf{E}^{(!(0,i)}\left[\frac{\lambda_i}{\lambda((0,i);Y)} \middle| X_\bullet \cap B(0,t) = \emptyset\right]\right)^{-1} \tag{20}$$

Expressions for $I(t)$ can be obtained by substituting (17)-(20) in (9).

These results should be compared to similar expressions in the univariate case, see [18, Theorem 1].

Proof : For (17)–(18), use the Nguyen-Zessin formula (14) taking $f(Y) = \mathbf{1}\{X_j \cap B(0,t) = \emptyset\}$ or $f(Y) = \mathbf{1}\{X_j \cap B(0,t) = \emptyset\}/\lambda((0,i);Y)$. For (19)–(20) take $f(Y) = \mathbf{1}\{X_\bullet \cap B(0,t) = \emptyset\}$ or $f(Y) = \mathbf{1}\{X_\bullet \cap B(0,t) = \emptyset\}/\lambda((0,i);Y)$. □

The following corollary describes how J_{ij} and $J_{i\bullet}$ may be interpreted as indicating positive or negative association between types, cf. [18, Cor. 1, eq. (3.7)].

Lemma 6 *Let $Y = (X_1, \ldots, X_m)$ be any stationary multivariate process for which the conditional intensity $\lambda((0,i);Y)$ exists and satisfies (14). Then $J_{ij}(t) \geq 1$ iff*

$$\mathrm{Cov}\,(\lambda((0,i);Y), \mathbf{1}\{X_j \cap B(0,t) = \emptyset\}) \geq 0$$

and $J_{i\bullet}(t) \geq 1$ iff

$$\mathrm{Cov}\,(\lambda((0,i);Y), \mathbf{1}\{X_\bullet \cap B(0,t) = \emptyset\}) \geq 0.$$

Proof : Rewriting (17) as

$$J_{ij}(t) = \frac{\mathrm{Cov}\,(\lambda((0,i);Y), \mathbf{1}\{X_j \cap B(0,t) = \emptyset\})}{\lambda_i \mathbb{P}(X_j \cap B(0,t) = \emptyset)} + 1$$

we obtain the first result. Similarly for the second. □

Using Lemma 5 and the decomposition (9) we find that a *sufficient* condition for $I(t) \geq 0$ is that for all i,

$$\mathbb{E}[\lambda_i(0; X_i) | X_i \cap B(0,t) = \emptyset] \geq \mathbb{E}[\lambda((0,i);Y) | X_\bullet \cap B(0,t) = \emptyset].$$

Reversing the signs gives a similar, sufficient condition for $I(t) \leq 0$.

3.4 Finite interaction range

Here we derive multivariate versions of the result in [18, Theorem 1(b)] that the J-function is constant for all $t \geq s$ if the process has finite interaction range s.

Following [18] a univariate point process X has *interaction range* s, $0 < s < \infty$, if its conditional intensity $\lambda_X(0;X)$ is constant for all patterns X which contain no points in $B(0,s)$. That is, $X \cap B(0,s) = \emptyset$ implies $\lambda_X(0;X) = \lambda_X(0;\emptyset)$.

Definition 4 *A multivariate point process $Y = (X_1, \ldots, X_m)$ has* joint interaction range s *if for each i, its multivariate conditional intensity $\lambda((0,i);Y)$ is constant for all realisations which contain no points in $B(0,s)$. That is, $X_i \cap B(0,s) = \emptyset$ for all i implies $\lambda((0,i);Y) = \lambda((0,i);\emptyset)$.*

A sufficient condition is that $\lambda((0,i);Y)$ depend only on $Y \cap B(0,s)$, the restriction of Y to $B(0,s)$.

Lemma 7 *If Y has joint interaction range s, $0 < s < \infty$, then $J_{i\bullet}(t)$ is constant for $t \geq s$,*

$$J_{i\bullet}(t) \equiv \frac{\lambda((0,i);\emptyset)}{\lambda_i} \quad , t \geq s. \tag{21}$$

If additionally the marginal processes X_i each have interaction range s, then $I(t)$ is constant for $t \geq s$,

$$I(t) \equiv \frac{1}{\lambda_\bullet} \sum_{i=1}^{m} [\lambda_i(0;\emptyset) - \lambda((0,i);\emptyset)] \quad , t \geq s \tag{22}$$

or equivalently

$$I(t) \equiv \frac{\sum_{i=1}^m \lambda_i(0;\emptyset)}{\sum_{i=1}^m \lambda_i} - \frac{\lambda_\bullet(0;\emptyset)}{\lambda_\bullet} \quad , t \geq s. \tag{23}$$

An analogous statement for J_{ij} does not hold in general.

Proof : In (19) observe that $X_\bullet \cap B(0,t) = \emptyset$ for $t \geq s$ implies $X_i \cap B(0,s) = \emptyset$ for all i so that $\lambda((0,i);Y)$ is conditionally constant and equal to $\lambda((0,i);\emptyset)$. The first result (21) follows. The second result is proved by combining (21) with (9) and [18, Theorem 1(b)]. The third result follows using (16) and the fact that $\lambda((0,i);Y)$ is conditionally constant. □

4 Theoretical examples

In this section we calculate the multivariate J functions for a variety of multivariate point pattern models.

4.1 Multitype cluster processes

By a multivariate cluster process in $\mathbb{R}^d$ we mean a general cluster process [10] constructed from a univariate point process in $\mathbb{R}^d$ (of 'parent' points) by associating with each parent a cluster (of 'offspring' points) which is a finite multivariate point process, i.e. a finite point process in $\mathbb{R}^d \times \{1,\ldots,m\}$. Only the offspring points are observed.

We shall consider only the stationary multivariate Poisson cluster process in which the parents are a stationary Poisson process in $\mathbb{R}^d$ and, for a parent located at $x \in \mathbb{R}^d$, the cluster Z_x of offspring of x is distributed as $Z + x$, the vector translation by x of a given, a.s. finite, multivariate point process Z. Offspring clusters from different parents x_i are independent. Thus $Y = \cup_i Z_{x_i}$.

4.2 General result

We need the multivariate version of a basic identity for cluster processes [10, §8.3, p. 243 ff.], [28, p. 143], see [3, p. 8–9]. It is a trivial rephrasing of the univariate result. The functional form of the Palm distribution involved for a variety of univariate cluster processes is studied in [27].

Lemma 8 *Let $Y = (X_1,\ldots,X_m)$ be a stationary multivariate Poisson cluster process in $\mathbb{R}^d$. Then the Palm distribution of Y with respect to a point of type i at 0 can be written*

$$P^{(0,i)} = C^{(0,i)} * P \tag{24}$$

where P is the distribution of Y and $C^{(0,i)}$ is the Palm distribution of the typical cluster with respect to a point of type i at 0. Similarly for the reduced Palm distributions,

$$P^{!(0,i)} = C^{!(0,i)} * P. \tag{25}$$

The following is a trivial corollary.

Lemma 9 *For a stationary multivariate Poisson cluster process,*

$$J_{ij}(t) = C^{!(0,i)}\{Z_j \cap B(0,t) = \emptyset\} \quad (26)$$

$$J_{i\bullet}(t) = C^{!(0,i)}\{Z_\bullet \cap B(0,t) = \emptyset\} \quad (27)$$

where Z_j denotes the finite point process of points of type j in the cluster, and $Z_\bullet$ the process of all points in the cluster regardless of type.

Thus, all J_{ij} and $J_{i\bullet}$ functions are decreasing and bounded above by 1. Moreover $J_{i\bullet}(t) \leq J_{ii}(t)$, suggesting clustered behaviour.

Regarding the range of interaction, if all clusters have maximum diameter s then $J_{ij}(t)$ and $J_{i\bullet}(t)$ are constant for all $t \geq s$.

4.2.1 Two–type Gauss–Poisson process

A Gauss–Poisson process [5, 20, 21] is a (univariate) Poisson cluster process in which each cluster consists either of one point (with probability $1-p$) or two points (with probability p). If a cluster has two points, they are separated by a random vector displacement V which has probability density h on $\mathbb{R}^d$.

Here we study the associated bivariate point process in which parent points are labelled as being of type 1 and daughter points, type 2. This is a multivariate Poisson cluster process.

Lemma 10 *For the two–type Gauss–Poisson process as described above,*

$$\begin{aligned} J_{11}(t) &= J_{22}(t) = 1 \\ J_{12}(t) &= J_{1\bullet}(t) = 1 - pH(t) \\ J_{21}(t) &= J_{2\bullet}(t) = 1 - H(t) \\ J_{\bullet\bullet}(t) &= 1 - \frac{2p}{1+p}H(t) \\ I(t) &= \frac{2p}{1+p}H(t) \end{aligned}$$

where

$$H(t) = \int_{B(0,t)} h(x)\,dx.$$

Thus, the types 1 and 2 in this process are positively associated, in the senses measured by $J_{ii}, J_{ij}, J_{\bullet\bullet}$ and I functions. All the J functions are decreasing and less than or equal to 1 (since h is a probability density); I is increasing and non-negative.

Furthermore, $J_{i\bullet} \leq J_{ii}$, suggesting positive association as well. However, regarding association between labels, for $p \notin \{0,1\}$, $J_{2\bullet} - J_{\bullet\bullet} = \frac{p-1}{p+1}H(t) < 0$ and decreasing, suggesting that the "2 to any" distances appear more clustered than if the type is disregarded, while $J_{1\bullet} - J_{\bullet\bullet} = \frac{p(1-p)}{p+1}H(t) > 0$ and increasing, indicating that distances from a type 1 point

appear more regular. A possible explanation is that any type 2 point will have a parent type 1 point, while a type 1 point may not have an associated type 2 point.

If we assume that h is concentrated on a ball $B(0,R)$, then $J_{ij}, J_{i\bullet}(t), J_{\bullet\bullet}$ and I are constant for all $t \geq R$, mirroring the results for a univariate Poisson cluster process [18].

Proof : Both X_1 and X_2 are stationary Poisson processes, so $J_{ii} \equiv 1$.

Under the Palm distribution $C^{(0,1)}$, with probability $1-p$ the cluster Z has only a single point (of type 1 at 0), while with probability p it has two points (one of type 1 at 0 and one of type 2 at V, where V is random with density h). Applying Lemma 9,

$$\begin{aligned} J_{12}(t) &= C^{!(0,1)}\{Z_2 \cap B(0,t) = \emptyset\} \\ &= (1-p) + p\,\mathbb{P}\{V \notin B(0,t)\} \\ &= (1-p) + p\,(1-H(t)) \\ &= 1 - pH(t). \end{aligned}$$

The calculation for $J_{1\bullet}$ produces the same result since $Z_2 \equiv Z_\bullet$ under $C^{!(0,1)}$.

Under $C^{(0,2)}$, with probability 1 the cluster has two points (of type 2 at 0 and of type 1 at $-V$). Applying Lemma 9,

$$\begin{aligned} J_{21}(t) &= C^{!(0,2)}\{Z_1 \cap B(0,t) = \emptyset\} \\ &= \mathbb{P}\{-V \notin B(0,t)\} \\ &= 1 - H(t). \end{aligned}$$

Again the calculation for $J_{2\bullet}$ is identical.

Regarding $J_{\bullet\bullet}$, we apply the univariate version of Lemma 8. Now C^0 is number weighted, hence

$$\begin{aligned} J_{\bullet\bullet} &= C^0\{Z_\bullet \cap B(0,t) = \{0\}\} \\ &= \frac{1-p}{1+p} + \frac{2p}{1+p}\,(1-H(t)). \end{aligned}$$

The result for I follows using Lemma 1. □

4.3 Bivariate Poisson processes

A *bivariate Poisson process* is a two-type process in which the marginal distribution of each of the components is that of a stationary Poisson process. We treat several standard examples.

4.3.1 Linked Poisson

A *linked Poisson process* [12] is a bivariate Poisson cluster process in which every cluster consists of exactly two points, a type 1 point and a type 2 point, separated by a random displacement V where V has density h on $\mathbb{R}^d$.

This is the special case $p = 1$ of our two-type Gauss-Poisson process, so Lemma 10 can be applied and we obtain $J_{11} \equiv J_{22} \equiv 1$, $J_{12}(t) = J_{21}(t) = J_{1\bullet}(t) = J_{2\bullet}(t) = J_{\bullet\bullet}(t) = 1 - H(t)$

and $I(t) = H(t)$. In particular, $J_{i\bullet} = J_{\bullet\bullet}$ $(i = 1, 2)$, so the model is consistent with a random labelling assumption. However, in general, a randomly labelled linked Poisson process is not a linked Poisson process which can be seen easily if the displacement V is taken to be deterministic.

4.4 Bivariate Cox processes

A bivariate Cox process [7, 8, 13] is formed as follows. We start with two random measures Λ_1, Λ_2 on $\mathbb{R}^d$ which are typically dependent. Conditional on $(\Lambda_1, \Lambda_2) = (\lambda_1, \lambda_2)$, let X_1 and X_2 be independent inhomogeneous Poisson processes with intensity measures λ_1 and λ_2 respectively. Then the unconditional model $Y = (X_1, X_2)$ is a bivariate Cox process.

4.4.1 Linked bivariate Cox process

As an example of positive dependence consider the case where $\Lambda_1 = \nu\Lambda_2$ for some fixed positive constant ν; the resulting Y is called a linked bivariate Cox process [8].

Lemma 11 *Let (X_1, X_2) be a linked Cox process where the intensity measures are 'mixed Poisson', $\Lambda_2 = Am$, for some non-negative random variable A with finite positive expectation and where m is Lebesgue measure. Write $L(s) = \mathbb{E}e^{-sA}$ for the moment generating function of A. Then $\mathbb{E}\left[Ae^{-sA}\right] = -L'(s)$ and*

$$\begin{aligned}
J_{11}(t) &= J_{21}(t) = L'(\nu\kappa_d t^d)/(L'(0)L(\nu\kappa_d t^d)) \\
J_{22}(t) &= J_{12}(t) = L'(\kappa_d t^d)/(L'(0)L(\kappa_d t^d)) \\
J_{\bullet\bullet}(t) &= J_{1\bullet}(t) = J_{2\bullet} = L'((1+\nu)\kappa_d t^d)/(L'(0)L((1+\nu)\kappa_d t^d)) \\
I(t) &= \frac{-\nu}{L'(0)(1+\nu)}\left[\frac{-L'(\nu\kappa_d t^d)}{L(\nu\kappa_d t^d)} + \frac{L'((1+\nu)\kappa_d t^d)}{L((1+\nu)\kappa_d t^d)}\right] \\
&+ \frac{-1}{L'(0)(1+\nu)}\left[\frac{-L'(\kappa_d t^d)}{L(\kappa_d t^d)} + \frac{L'((1+\nu)\kappa_d t^d)}{L((1+\nu)\kappa_d t^d)}\right]
\end{aligned}$$

where $\kappa_d = m(B(0,1))$ is the volume of the unit ball in $\mathbb{R}^d$.

In the general case where (Λ_1, Λ_2) are stationary random measures, this result holds true with A-weighted means replaced by expectations under the Palm distribution at 0 of Λ_2.

Proof : Since X_1, X_2 and $X_\bullet$ are mixed Poisson processes with random intensity measures νAm, Am and $(1+\nu)Am$ respectively, the equations for J_{11}, J_{22} and $J_{\bullet\bullet}$ follow from Theorem 6 in [18] and the discussion therein. The expression for I follows easily.

The reduced Palm distributions $P^{!(0,1)}, P^{!(0,2)}$ are both bivariate Cox processes with Λ_1 (respectively Λ_2) replaced by its A-weighted distribution, $\mathbf{E}_{\text{weighted}} f(A) = \mathbb{E}[Af(A)]/\mathbb{E}[A]$. The remaining identities follow. □

By [18, Theorem 6], $J_{ij} \leq 1$ (with equality only if A is constant a.s.), suggesting positive correlation between the component processes. Moreover, J_{ij} is decreasing with $\lim_{t\to\infty} J_{ij}(t) =$

$\frac{\text{essinf } A}{\mathbf{E}A}$. By the same argument, $J_{i\bullet}$ and $J_{\bullet\bullet}$ are bounded above by 1, decreasing to $\frac{\text{essinf } A}{\mathbf{E}A}$. Hence, I converges to 0 as $t \to \infty$. Since the function

$$s \mapsto \frac{L'(s)}{L'(0)L(s)}$$

is monotonically decreasing, the terms in brackets in the expression for I are both non-negative. Thus I is non-negative, confirming the positive dependence between the components. Finally, $J_{i\bullet}(t) \leq J_{ii}(t)$, confirming the positive dependence. Note that given A, the conditional distribution of (X_1, X_2) given $X_\bullet$ is that of a random labelling with $(p_1, p_2) = \frac{1}{\nu+1}(\nu, 1)$. Since this distribution does not depend on A, Y has the random labelling property. This is reflected in the fact that $J_{i\bullet}(t) = J_{\bullet\bullet}(t)$.

4.4.2 Balanced Cox

An example of negative dependence is the class of balanced Cox processes [13] where

$$\Lambda_1 + \Lambda_2 = \nu m,$$

m again denoting Lebesgue measure. Note that the superposition is always distributed as a Poisson process with intensity ν.

Lemma 12 *Let (X_1, X_2) be a balanced Cox process on $\mathbb{R}^d$ with $\Lambda_2 = Am$, for a random variable A concentrated on $(0, \nu)$ with $0 < \mathbf{E}A < \nu$. Then, writing $\tilde{L}$ for the moment generating function of $\nu - A$,*

$$\begin{aligned}
J_{11}(t) &= \tilde{L}'(\kappa_d t^d)/(\tilde{L}'(0)\tilde{L}(\kappa_d t^d)) \\
J_{22}(t) &= L'(\kappa_d t^d)/(L'(0)L(\kappa_d t^d)) \\
J_{12}(t) &= \frac{\mathbf{E}\left[(\nu - A)e^{-A\kappa_d t^d}\right]}{\mathbf{E}[\nu - A]\,\mathbf{E}e^{-A\kappa_d t^d}} \\
J_{21}(t) &= \frac{\mathbf{E}\left[Ae^{-(\nu-A)\kappa_d t^d}\right]}{\mathbf{E}[A]\,\mathbf{E}e^{-(\nu-A)\kappa_d t^d}} \\
J_{\bullet\bullet}(t) &= J_{1\bullet}(t) = J_{2\bullet}(t) = 1 \\
I(t) &= \frac{1}{\nu}\left\{\frac{-L'(\kappa_d t^d)}{L(\kappa_d t^d)} + \frac{\tilde{L}'(\kappa_d t^d)}{\tilde{L}(\kappa_d t^d)}\right\}
\end{aligned}$$

as before writing $\kappa_d = m(B(0,1))$ for the volume of the unit ball in $\mathbb{R}^d$.

Proof : As the superposition is a Poisson process, $J_{\bullet\bullet}(t) = 1$. Again applying [18, Theorem 6], the formulae for J_{11} and J_{22} follow, yielding the expression for I.

The reduced Palm distribution $P^{!(0,2)}$ is that of a bivariate Cox process with Λ_2 replaced by its A-weighted distribution, while $P^{!(0,1)}$ is similar with Λ_1 governed by its $(\nu - A)$-weighted distribution. The other results follow. □

Using

$$-L'(\kappa_d t^d) \leq -L'(0)L(\kappa_d t^d)$$

and a similar inequality with A replaced by $(\nu - A)$, it is easily seen that $I(t) \leq 0$, indicating negative dependence. By the discussion following [18, Theorem 6], $I(t)$ decreases to (essinf A - esssup A) / ν as $t \to \infty$.

We can verify that $J_{ij} \geq 1$, suggesting negative correlation between the component processes. For example consider J_{21}. Then

$$\begin{aligned}\frac{\mathbf{E}\left[Ae^{-(\nu-A)\kappa_d t^d}\right]}{\mathbf{E}A\,\mathbf{E}e^{-(\nu-A)\kappa_d t^d}} &= -\frac{\mathbf{E}\left[(\nu-A)e^{-(\nu-A)\kappa_d t^d}\right]}{\mathbf{E}A\,\mathbf{E}e^{-(\nu-A)\kappa_d t^d}} + \frac{\nu}{\mathbf{E}A} \\ &\geq -\frac{(\nu-\mathbf{E}A)\,\mathbf{E}e^{-(\nu-A)\kappa_d t^d}}{\mathbf{E}A\,\mathbf{E}e^{-(\nu-A)\kappa_d t^d}} + \frac{\nu}{\mathbf{E}A} = 1.\end{aligned}$$

Moreover, both $J_{12}(t)$ and $J_{21}(t)$ are monotonically increasing with limits $\frac{\nu\text{-essinf } A}{\mathbf{E}(\nu-A)}$ and $\frac{\text{esssup } A}{\mathbf{E}A}$ respectively ($t \to \infty$). From [18], $1 = J_{i\bullet}(t) \geq J_{ii}(t)$, again suggesting negative dependence between the component processes.

Turning attention to the label association, note that given A, the conditional distribution of (X_1, X_2) given $X_\bullet$ is that of a random labelling with probabilities $(p_1, p_2) = (\frac{\nu-A}{\nu}, \frac{A}{\nu})$. Thus the allocation probabilities are random, and in general Y does not satisfy the random labelling property. (This could also have been seen more directly by noting that $X_\bullet$ is a Poisson process and therefore its random labelling is a bivariate Poisson process with independent components rather than the balanced Cox model we started with.) However $J_{i\bullet} - J_{\bullet\bullet} \equiv 0$, which is in accordance with a random label allocation.

4.5 Pairwise interaction Gibbs processes

Consider a multitype point process Y with conditional intensity of the form

$$\lambda((u,i);Y) = \beta_i \prod_{(x_j,j)\in Y} \gamma_{ij}(||x_j - u||)$$

where $\beta_1, \ldots, \beta_m$ are nonnegative constants and γ_{ij} are nonnegative real functions. Without loss of generality $\gamma_{ij} \equiv \gamma_{ji}$. In general, terms γ_{ii} appear. This might be called a stationary pairwise interaction Gibbs process, cf. [2, 25, 24].

Lemma 5 and equation (8) immediately yield the following.

Lemma 13 *For a pairwise-interaction Gibbs process as above,*

$$J_{\bullet\bullet}(t) = \sum_{i=1}^{m} \frac{\beta_i}{\lambda_\bullet} \mathbf{E}\left[\prod_{(x,k)\in Y} \gamma_{ik}(||x||) \,\middle|\, X_\bullet \cap B(0,t) = \emptyset\right] \tag{28}$$

$$J_{ij}(t) = \frac{\beta_i}{\lambda_i} \mathbf{E}\left[\prod_{(x,k)\in Y} \gamma_{ik}(||x||) \,\middle|\, X_j \cap B(0,t) = \emptyset\right] \tag{29}$$

$$J_{i\bullet}(t) = \frac{\beta_i}{\lambda_i} \mathbf{E}\left[\prod_{(x,k)\in Y} \gamma_{ik}(||x||) \,\middle|\, X_\bullet \cap B(0,t) = \emptyset\right] \tag{30}$$

wherever defined. If there is finite range interaction in the sense that $\gamma_{ij}(||x||) = 1$ *for* $||x|| > r_{ij}$*, the formulae above reduce to* $J_{\bullet\bullet}(t) = \sum \frac{\beta_i}{\lambda}$ *for* $t \geq r = \max r_{ij}$ *and* $J_{i\bullet}(t) = \frac{\beta_i}{\lambda_i}$ *for* $t \geq r_i = \max_j r_{ij}$*. Since in (29) the conditioning is only on no point of type* j *in a ball around the origin, a similar reduction for* J_{ij} *in general will not hold.*

5 Applications

In this Section we analyse four bivariate data sets with a range of correlation structures between the component processes, using empirical I and J-functions and Monte Carlo inference.

We consider two different null hypotheses: random labelling as described in Definition 3, and independence of the components X_1 and X_2. To test the random labelling null hypothesis we condition on the locations and the relative frequency of the types, more specifically given a data set consisting of n_i type i events ($i = 1, 2$), the labels are permuted randomly, leaving the locations unchanged. Alternatively, we could condition on the location of the events only and sample the labels with replacement. A disadvantage of the latter is that the label probabilities are unknown – although they can be estimated by $p_i = n_i/n$ where n_i is the number of observed i-events and n the total number of events – and that the relative frequency of the labels is variable.

In general, non-parametric sampling from the unconditional null-hypothesis of independent components is hard. For rectangular windows however. Lotwick and Silverman proposed identifying opposite sides of the window to obtain a torus, and then translating the type 1 pattern randomly over the torus [19]. Hence this approach is conditional on the within-component patterns rather than on the superposition locations.

Estimates of the various F- and G-functions were computed using the Kaplan-Meier estimators [1]. The corresponding J-functions were derived by substitution. In the case of F, the windows were discretised into (subsets of) rectangular pixel arrays and the distances from each pixel to the nearest data point were computed using the distance transform algorithm [6]. The algorithms were implemented in Splus and C.

5.1 Beta cells in the cat retina

Figure 5.1 depicts a pattern of beta-type ganglion cells in the retina of a cat recorded by Wässle et al. [32] and kindly provided by P. Diggle. The window is a rectangle $(0, 1)\times(0, .7533)$ in units of approximately 1000μm. Beta cells are associated with the resolution of fine detail in the cat's visual system. They can be classified as 'on' or 'off', depending on the branching level of their dendritic tree in the inner plexiform layer. Analysis of the spatial pattern provides information on the cat's visual discrimination. In particular, independence of the 'on'- and 'off'-components would strengthen the assumption that there are two separate channels for 'brightness' and 'darkness' as postulated by Hering in 1874. For details see [32].

Wässle et al. [32] investigated this pattern using histograms of nearest-neighbour distances (ignoring edge effects). To test independence of the 'on' and 'off' patterns, a random translation of the 'off'-component was superimposed on the 'on'-component, and the resulting nearest-neighbour histogram compared with the original one by a sign reversal test.

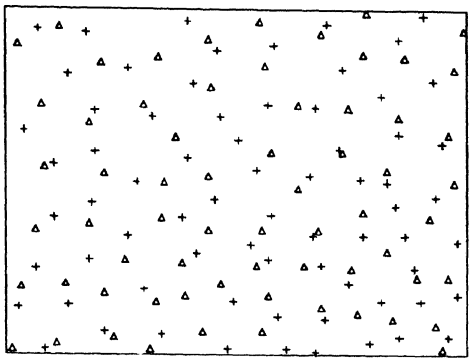

Figure 5.1: 65 'on' (Δ) and 70 'off' (+) beta cells in a cat retina.

They concluded that both types of beta cells form a regular lattice, which are superimposed independently.

Our analysis begins by computing estimates of the summary statistics. The marginals G_{11} and G_{22} lie below the graph of $G_{\bullet\bullet}$ (and similarly for F) and the cross G-functions are similar to $G_{\bullet\bullet}$, due to the fact that most cells have a nearest neighbour of the opposite type. The marginal J-functions are increasing and exceed 1, suggesting repulsion between the cells.

To investigate independence of the components we took the Lotwick-Silverman approach and repeatedly translated the first component over the torus. For each simulation, J_{12} was computed. The envelopes for 99 translations and the empirical estimate of J_{12} are depicted in Figure 5.2. For almost all t, the null hypothesis is accepted as the data lies between the envelopes. Using the statistics $J_{2\bullet} - J_{22}$ or I leads to acceptance of the null hypothesis as well. This confirms conclusions of [32].

A random label allocation of types to beta cells on the other hand does not seem appropriate as a null hypothesis for testing the Hering postulate. However, for illustrative purposes we did perform the test. Since most points have a nearest neighbour of the opposite type, $J_{i\bullet} - J_{\bullet\bullet}$ is not a suitable test statistic, but both J_{12} and I lead to rejection of the random labelling hypothesis. See Figure 5.3.

A second order analysis by Stoyan [30, 14] yielded similar results. At close range, the plot

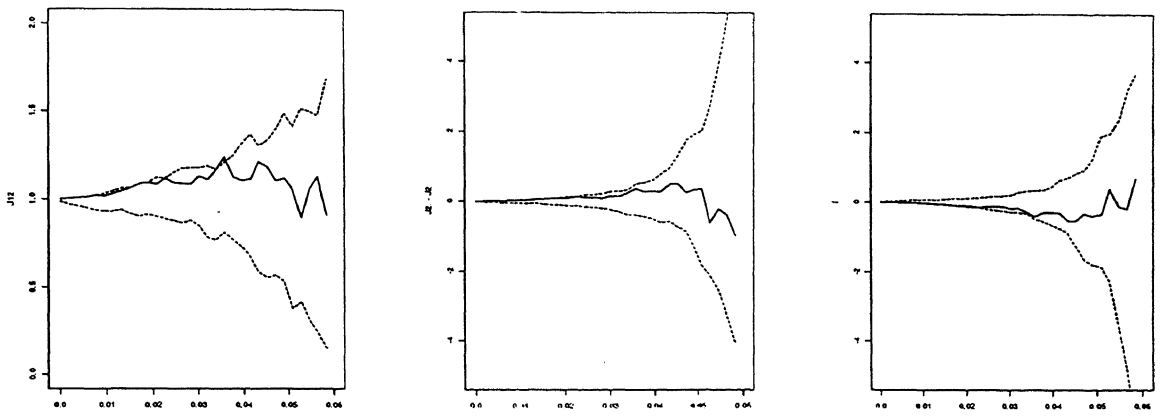

Figure 5.2: Envelopes based on 99 torus translations of X_1 (dashed) and empirical J_{12}, $J_{2\bullet} - J_{22}$ and I statistics for the cat retina data.

Figure 5.3: Envelopes based on 99 random labellings (dashed) and empirical J_{12} and I statistics for the cat retina data.

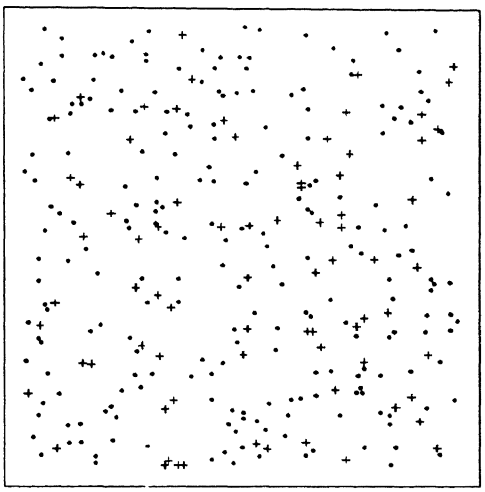

Figure 5.4: Dividing (•) and pyknotic (+) cells in a section of metastasising lymphoma in a hamster kidney.

of the mark correlation function p_{12} [29, p. 264–265] is high compared to the plots of p_{11} and p_{22}, before flattening down.

5.2 Hamster tumour

Figure 5.4, originally collected by Dr W A Aherne (Department of Pathology, University of Newcastle upon Tyne) and kindly provided by Professor P J Diggle, shows the positions of cell nuclei in an approximately .25mm square histological section of tissue from a laboratory–induced metastasising lymphoma in the kidney of a hamster. Two types of cells are distinguished: 77 pyknotic nuclei corresponding to dying cells and 226 nuclei arrested in metaphase. The background void is occupied by unrecorded, interphase cells in relatively large numbers. Both the marginal and joint J-functions are larger than 1, suggesting inhibition between the cells.

The spatial correlation between the dying and dividing cells is important for the study of tumour growth. Since the classification into pyknotic and metaphase cells is made after the spatial positions are determined, in this case random labelling seems a more natural hypothesis than independence of the components. We performed tests using each of the statistics J_{12}, $J_{1\bullet} - J_{\bullet\bullet}$, $J_{2\bullet} - J_{\bullet\bullet}$ and I with 99 random label allocations without replacement.

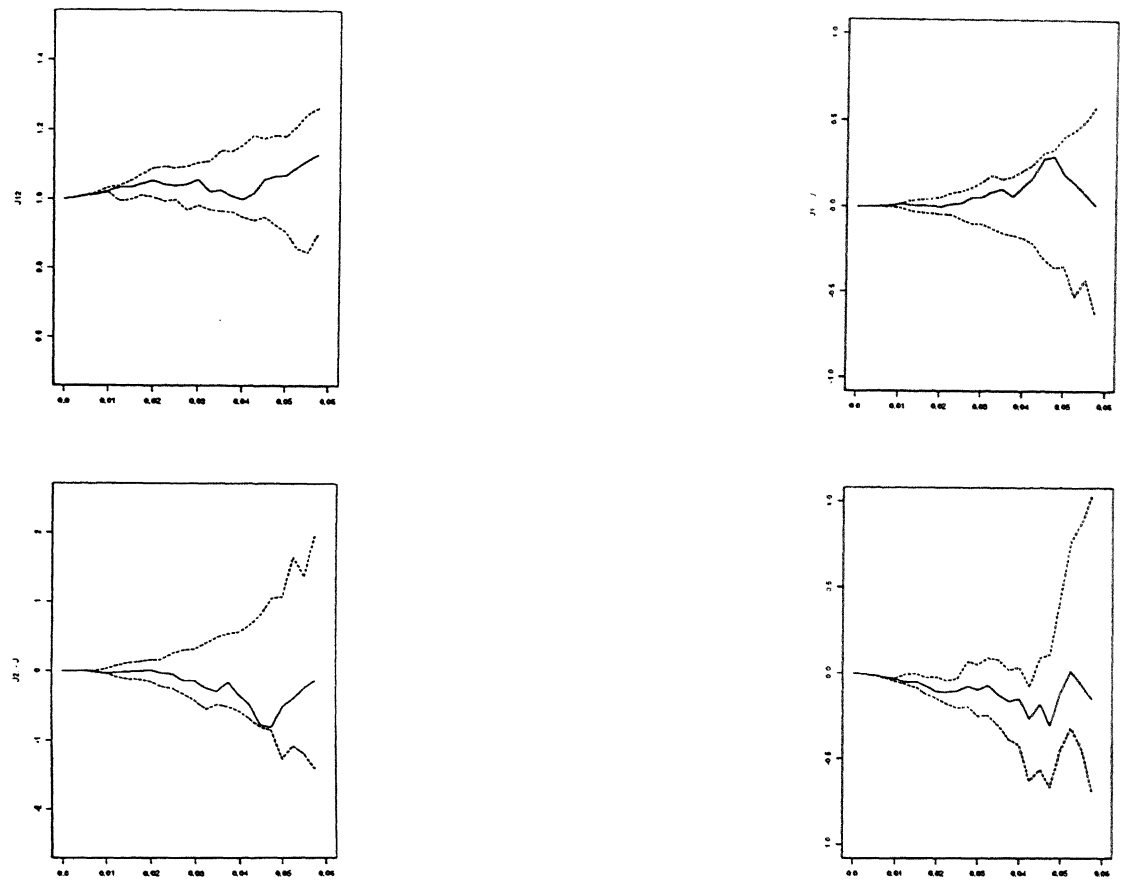

Figure 5.5: Envelopes based on 99 random labellings (dashed) and empirical J_{12}, $J_{1\bullet} - J_{\bullet\bullet}$, $J_{2\bullet} - J_{\bullet\bullet}$ and I statistics for the hamster tumour data.

The results are plotted in Figure 5.5. Although the sign of I suggests negative dependence between the two components, the null hypothesis cannot be rejected. Note that for most t, $J_{1\bullet}(t) - J_{\bullet\bullet}(t) > 0$, while $J_{2\bullet}(t) - J_{\bullet\bullet}(t)$ is predominantly negative. However, the deviations are not statistically significant at the 1% level.

Our conclusion is in keeping with other analyses reported in the literature. For instance, Diggle [11] accepted the random labelling hypothesis using the K-function; Stoyan [30, 14] plotted the mark correlation functions p_{ij}, $i, j \in \{1, 2\}$ and found nearly horizontal graphs, suggesting a random allocation of marks.

Although it is less meaningful from a biological point of view, one can perform a test for independence of the components. The results are plotted in Figure 5.6. The empirical J_{12}-function lies close to or above the upper envelope, and similarly for $J_{2\bullet} - J_{22}$, while the plot of the I-statistic is close to or below the lower envelope for small distances r. Diggle [11] also found deviations from independence with a Monte Carlo test of 99 simulations using the

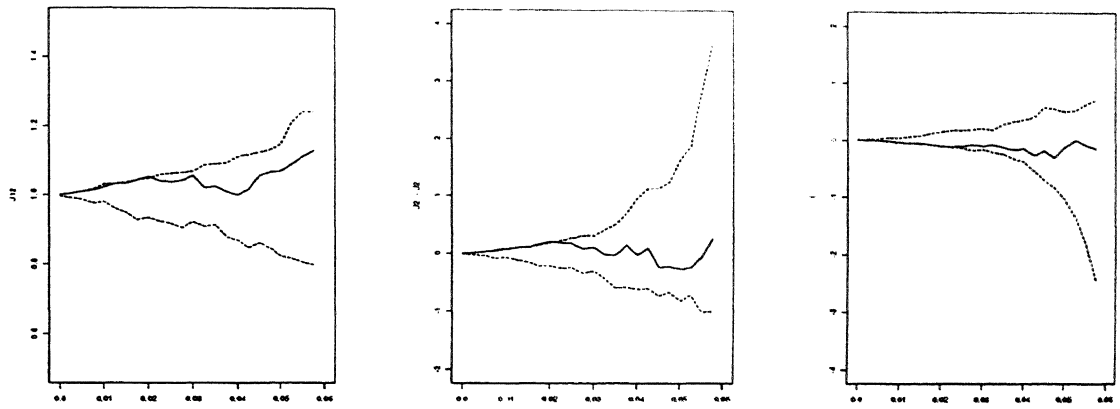

Figure 5.6: Envelopes based on 99 torus translations of X_1 (dashed) and empirical J_{12}, $J_{2\bullet} - J_{22}$ and I statistics for the hamster tumour data.

cross K–function.

5.3 Myrtle trees

Our third example is a pattern of 221 healthy and 106 diseased myrtles in a rectangle of 170.5 by 213.0 meters, depicted in Figure 5.7. The data set was collected by Dr G Kile and colleagues at CSIRO Tasmania and kindly supplied by Prof P J Diggle.

The empty spaces in three corners of the plot suggest that it would not be appropriate to treat these data as a realisation of a stationary point process viewed through the rectangular frame. Instead, we have arbitrarily marked out a smaller window with a polygonal boundary as shown in Figure 5.7, and computed all statistics with reference to this window.

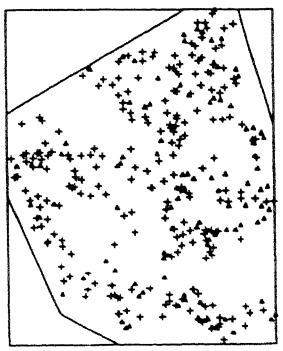

Figure 5.7: 221 healthy (+) and 106 diseased (Δ) myrtles.

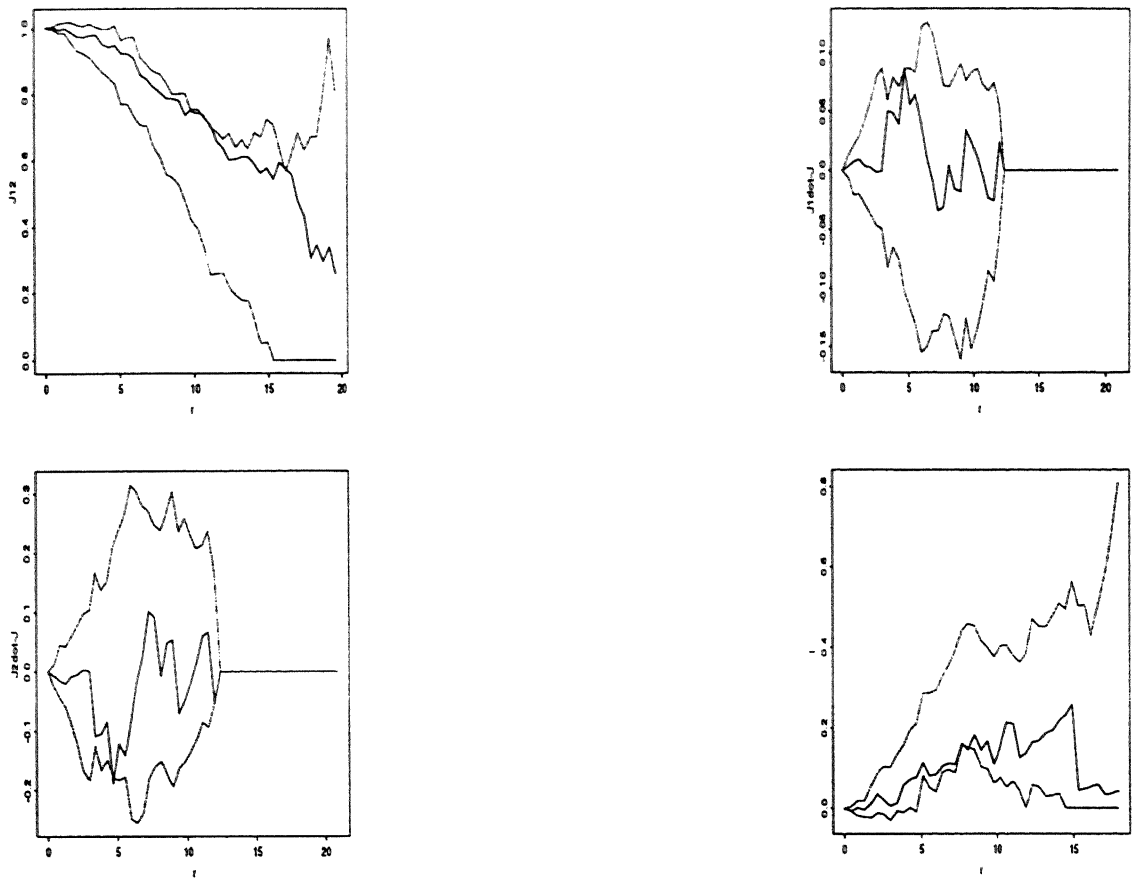

Figure 5.8: Envelopes based on 99 random labellings (dashed) and empirical J_{12}, $J_{1\bullet} - J_{\bullet\bullet}$, $J_{2\bullet} - J_{\bullet\bullet}$ and I statistics for the Tasmanian myrtle data.

Both components have J-functions that lie below 1, suggestive of clustered behaviour. The J-function of the superposition is also less than 1. The I-statistic is close to zero, mostly positive.

Since disease affects plants after their location has been fixed, we prefer to test for a random mark allocation conditional on the locations. Moreover, working on a non-rectangular window, the Lotwick-Silverman approach using independent toroidal shifts can no longer be applied. Secondly, if we would not have corrected for the corners, empty spaces would be overestimated, resulting in underestimates of F- and J-functions. However, the multiplicative bias (for fixed t) will be approximately the same, so should not effect a test for random labelling (cf. Definition 3) too much. This is confirmed by our analysis.

The Monte Carlo envelopes over 99 simulations are given in Figure 5.8. The empirical J_{12}, $J_{1\bullet} - J_{\bullet\bullet}$ and $J_{2\bullet} - J_{\bullet\bullet}$ and I functions lie between the simulation envelopes for almost all values of t, hence for most t tests based on these statistics accept the null hypothesis.

5.4 Ants' nests

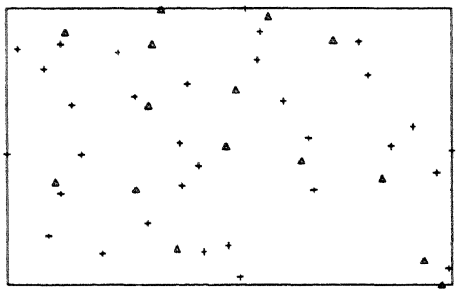

Figure 5.9: Nests of 32 Messor (+) and 15 Cataglyphis (Δ) ants

In our final example, we consider the distribution of the nests of two ant species, *Messor wasmanni* and *Cataglyphis bicolor* in a field in Northern Greece [15]. This data set was supplied by Professor V Isham with kind permission by Professor R.D. Harkness. These two species have different feeding patterns. The Messor ants collect seeds, while the Cataglyphis ants collect dead insects which are mostly dead Messor ants. For details see [15]. It is therefore of interest to see whether the functional dependence between the species is reflected in dependencies between the nest location patterns.

The original data set contains 68 Messor and 29 Cataglyphis nests in an area of about 1 hectare. This region is divided into two main parts, scrub land and field. As Cataglyphis ants tend not to build their nests in scrub, we only consider the field region (about 290 by 165 ft) with 32 Messor and 15 Cataglyphis nests. For convenience, we have rotated the data through -0.6 radians to align with a standard coordinate system (Figure 5.9). No coordinates of the boundary were given, so we took the smallest rectangle including all observed nests.

The data set was previously analysed by Harkness and Isham [15] using a K-function approach. They reported inhibition amongst Messor ants, but no evidence of interaction amongst the Cataglyphis and no evidence of dependence between the two species. See also [16, 26, 31]. The marginal J-functions of each species exceed 1, suggesting inhibitory patterns;

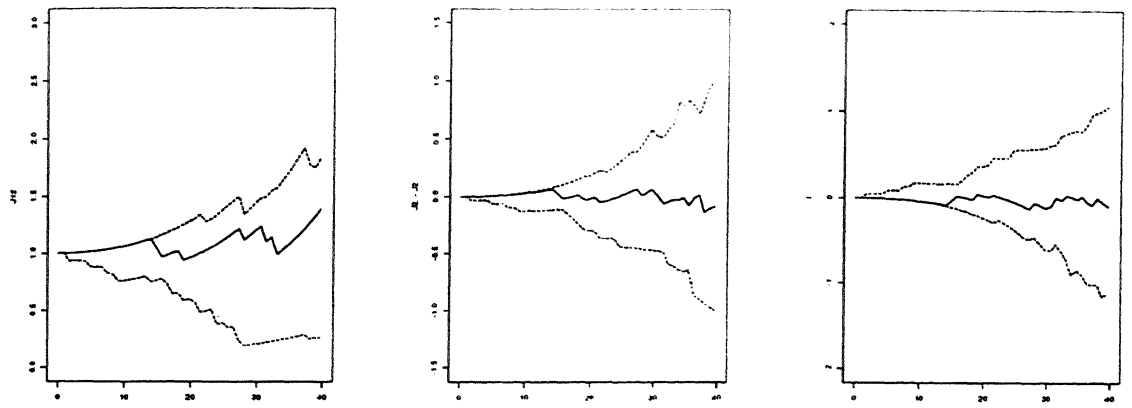

Figure 5.10: Envelopes based on 99 torus translations of X_1 (dashed) and empirical J_{12}, $J_{2\bullet} - J_{22}$ and I statistics for the ants' nests data.

the I-function is close to 0, mostly negative for small t. positive for larger t.

In 99 random torus translations, we found the empirical J_{12} function to lie at the upper envelope for small distances (due to the short-range repulsion between nests), and in between the envelopes at a larger range (see Figure 5.10). The I and $J_{2\bullet} - J_{22}$ graphs lie within the envelopes. Hence we too cannot find any significant positive correlation between the two components to reflect their functional dependence. As a possible explanation, Harkness and Isham suggest that Cataglyphis ants may prefer an overall strategic position to being close to one specific Messor nest.

Ants occupy the field simultaneously, hence a null hypothesis of random label allocation seems less natural than one of independence between the components. Nevertheless the Monte Carlo envelopes for the former hypothesis are given in Figure 5.11. For J_{12}, $J_{1\bullet} - J$ and $J_{2\bullet} - J_{\bullet\bullet}$ the random labelling null hypothesis is not rejected (except for the hard core distance between nests), but I provides evidence for positive association between the two ant species.

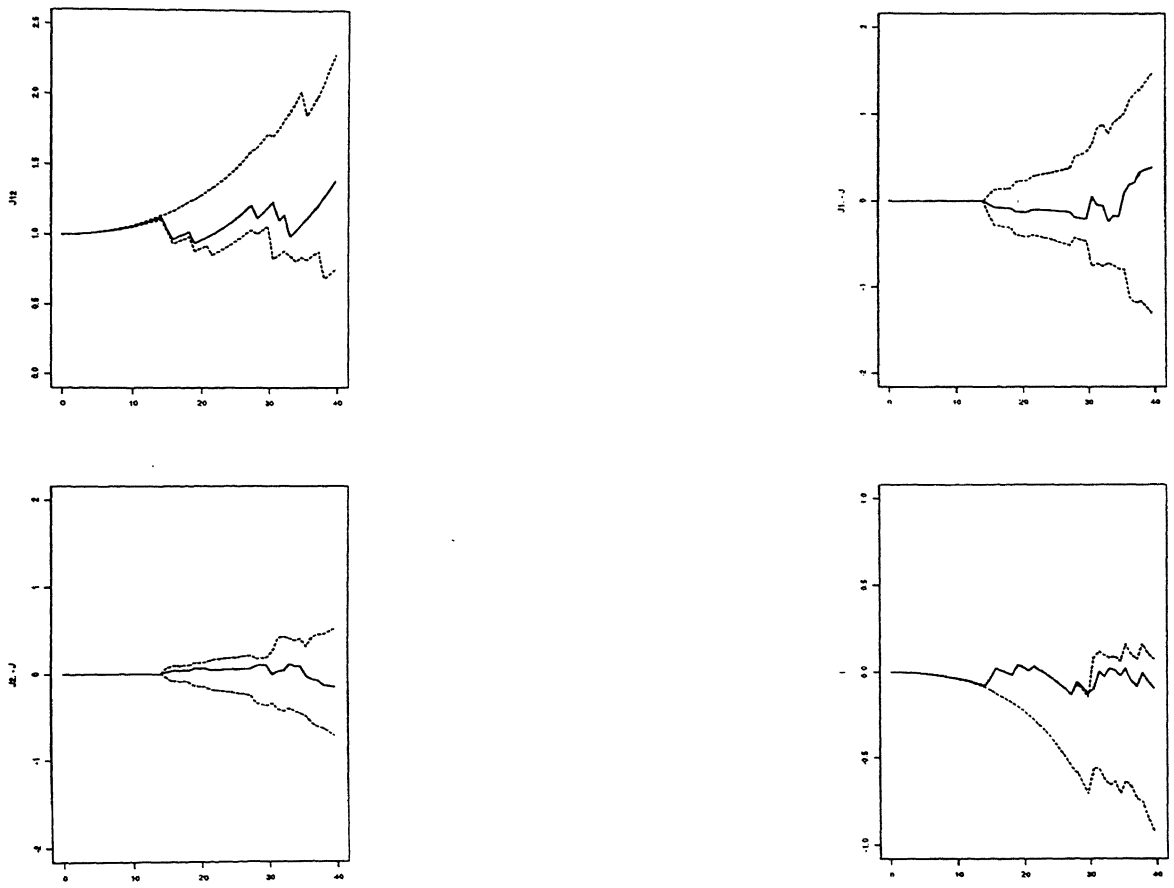

Figure 5.11: Envelopes based on 99 random labellings (dashed) and empirical J_{12}, $J_{1\bullet} - J_{\bullet\bullet}$, $J_{2\bullet} - J_{\bullet\bullet}$ and I statistics for the ants' nests data.

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