Quermass–interaction processes

by

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Abstract

This paper commences an investigation into a new class of random point and set processes, obtained using a rather natural weighting procedure. Given a Poisson point process, on each point one places a *grain*, a (possibly random) compact convex set. Let Ξ be the union of all grains. One can now weight the process using the exponential of a quermass functional of Ξ. If the functional is the area functional then we recover the area-interaction point process. New point processes arise if we take the perimeter length functional, or the Euler functional (number of components minus number of holes). The main question addressed by the paper is that of when the resulting point process is well-defined: geometric arguments are used to establish conditions for the point process to be stable in the sense of Ruelle.

Key words: area-interaction point process, Boolean model, germ-grain model, Markov point process, Minkowski functional, quermass integral, semi-Markov random closed set, spatial point process.

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There is still a shortage of good models for use in image analysis and spatial statistics. In this paper we commence the investigation of new point process and random set models which are constructed by weighting a Poisson point process with densities which are exponentials of (sums of) quermass integrals (Minkowski functionals) of a Boolean model based on the point process. These functionals are obtained from local geometric measurements including set volume and integrals of curvature over the boundary, and include the Euler-Poincaré characteristic.

In the point process case the model under investigation generalises the Widom-Rowlinson 'penetrable spheres model' from statistical physics and the area-interaction point process (from spatial statistics).

In this paper our main focus will be on the conditions under which planar quermass-interaction processes are stable (and hence exist) for all values of the weighting parameter. This has already been established for the special case of area-interaction [4]: we shall establish it in greater generality, and in particular under reasonable conditions for an interesting "topological" weighting based on the Euler-Poincaré characteristic. Our arguments are basically geometric covering arguments of a rather non-standard form, essentially elementary but of some intrinsic geometric interest. In further papers we plan to develop inferential and simulation theory as well as to explore the utility of this class of models in applications.

The paper is divided into 7 sections: §1 covers preliminaries on relevant concepts from stochastic geometry; §2 defines quermass-interaction point processes; §3 describes quermass-interaction random sets and germ-grain models; §4 begins the discussion of the important planar case, which introduces the main question to be dealt with in this initial study, namely the range of permissible parameter values under which one can weight using the Euler-Poincaré characteristic; §5 considers the case when grains are (planar) disks, in which case it turns out such weightings are always possible; §6 considers the case when grains are convex polygons, in which case it turns out such weightings are always possible if there is a lower bound on interior angles and side-lengths (otherwise weighting towards negative values of the characteristic produces divergence); finally §7 provides a conclusion indicating our plans for future investigation of these point processes, including simulation and inference issues.

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1 Preliminaries

In this section we briefly summarize relevant facts from the theories of Markov point processes, Boolean models, and quermass integrals.

1.1 Point processes

The basic benchmark process is a (stationary) Poisson point process in a bounded observation region A. This can be understood to express spatial independence in the

sense that points do not interact with each other. Formally, given that there are n points, these are independent and uniformly distributed over A. The total number of points in A is Poisson distributed with mean proportional to the area of A, giving the process its name. The constant of proportionality is called the intensity. The area measure can be replaced by any finite diffuse measure μ, yielding an inhomogeneous Poisson point process with intensity measure μ.

The benchmark Poisson point process can be used to define other processes by specifying their density $p(\cdot)$ with respect to the benchmark process. Such point processes may be viewed as *weighted Poisson point processes*. For a process defined in this way, the distribution $(q_0, q_1, q_2, ...)$ of the total number of points is given by

$$q_n \quad = \quad \frac{e^{-\mu(A)}}{n!} \int_A \cdots \int_A p(\{x_1, \ldots, x_n\})\, d\mu(x_1) \cdots d\mu(x_n)$$

and, given $N = n$, the joint conditional probability density of the point pattern is

$$p_n(x_1, \ldots, x_n) \quad = \quad e^{-\mu(A)} \mu(A)^n p(\{x_1, \ldots, x_n\})/(n! q_n)\,.$$

(Here the reference measure is provided by the distribution of n independent and identically distributed points distributed uniformly on A.)

It will be convenient to impose conditions on the density. Given a neighbourhood relation $\sim$, a *Markov point process* (following Ripley and Kelly [43]) satisfies

(M1) $p(\mathbf{x}) > 0$ implies $p(\mathbf{y}) > 0$ for all $\mathbf{y} \subset \mathbf{x}$;

(M2) if $p(\mathbf{x}) > 0$, then the *Papangelou conditional intensity*

$$\lambda(u; \mathbf{x}) \quad = \quad \frac{p(\mathbf{x} \cup \{u\})}{p(\mathbf{x})}$$

depends only on u and $\{x_i : u \sim x_i\}$. (The ratio $p(\mathbf{x} \cup \{u\})/p(\mathbf{x})$ is defined to be zero when $p(\mathbf{x})$ vanishes.)

Note that the density $p(\mathbf{x})$ can be reconstructed from the Papangelou conditional intensity up to a constant factor (and thus is completely defined, since it must integrate up to 1).

The famous Hammersley-Clifford theorem [5, 6, 8, 14, 41, 43, 49] then gives a simple interpretation in terms of interpoint interactions. A process with density p is a Markov point process if and only if

$$p(\mathbf{x}) \quad = \quad \prod_{\text{cliques } \mathbf{y} \subseteq \mathbf{x}} q(\mathbf{y}) \quad = \quad \alpha \prod_{\substack{\text{cliques } \mathbf{y} \subseteq \mathbf{x} \\ \mathbf{y} \neq \emptyset}} q(\mathbf{y})$$

for arbitrary non-negative interaction functions $q(\cdot)$, save that $\alpha = q(\emptyset)$ is determined by the requirement that the total integral of p equals 1.

Because of this property, Markov point processes are natural models for problems involving derivation of conditional probabilities and also are easy to simulate using Markov chain Monte Carlo methods, and hence are amenable to iterative statistical techniques (ICM etc) [7, 9, 13, 27, 31].

A generalisation can be obtained by allowing the neighbourhood relation to depend on the configuration (Baddeley and Møller [5]). See also section 3.5.

1.2 Boolean models

When considering random sets we can use the *Boolean model* as a benchmark process instead of the Poisson point process. Indeed the Boolean model can be viewed as a Poisson point process on a suitable space (of non-empty compact (often convex) sets). Here we will describe the Boolean model from the related germ-grain perspective. A set, called a *grain*, is placed at each point of a (possibly inhomogeneous) Poisson point process of *germs* in Euclidean space. Different random grains are random convex compact sets which are independent of each other (conditional on the realization of the point process of germs) with a distribution which depends continuously on the location of the respective germ (here of course we use the topology of weak convergence for probability measures on $C(\mathcal{K})$ the family of compact convex sets, metrizing $C(\mathcal{K})$ with the *Hausdorff metric* distance between two compact sets K and L given by $\mathrm{dist}(K,L) = \inf\{r : K \subset L \oplus B(o,r), L \subset K \oplus B(o,r)\}$, for $B(o,r)$ the (closed) ball of centre o and radius r). (Here the *Minkowski sum* $A \oplus B$ of two Euclidean sets A and B is given by $A \oplus B = \{a+b : a \in A, b \in B\}$.) The two most important examples are

(a) Different grains are independent both of each other and of locations, and follow the same distribution ν,

(b) A grain depends continuously (Hausdorff metric) on the location of the respective germ.

Either way, this produces a marked Poisson point process, by marking the germ process with the grains. Finally, the Boolean model is the random set obtained by the union of all the grains. In this paper, unless specifically stated otherwise, we will assume that the grains are ν-almost surely *ovoids* (that is to say, nonempty convex compact sets).

By virtue of the Choquet theorem [29, theorem 2-2-1], a random closed set Ξ is determined by its *avoidance function* on $\mathcal{K}$,

$$Q(K) \quad = \quad \mathbb{P}(\Xi \cap K = \emptyset). \tag{1}$$

For a Boolean model based on a homogeneous Poisson point process with random grains (case (a)),

$$Q(K) \quad = \quad \exp\left[-\beta\, \mathbb{E}\nu_d(K \oplus \check{Z})\right] \tag{2}$$

where $\mathbb{E}$ denotes the expectation with respect to ν of the typical grain Z, and β is the intensity of the underlying Poisson point process.

We should distinguish between the case where we can observe both the germs and the grains (for example in the area-interaction point process model described below where the configuration of grains can be deduced from the configuration of germ points) and the random set case where only the union of the grains is observed and not the underlying germ process. This distinction has important consequences for statistical inference, which has to be based only on observable quantities, though it largely does not affect the arguments of this paper which focus on stability and existence considerations.

1.3 Quermass integrals

The quermass integrals or Minkowski functionals are fundamental concepts of geometry [15, 48] generalizing (in the planar case) the notions of area and perimeter. In d dimensions and for $r \leq d-1$, they are defined for ovoids (convex compact sets) $K \in C(\mathcal{K})$ by

$$W_r^d(K) \quad = \quad \frac{b_d}{b_{d-r}} \int_{L_r} \nu_{d-r}(\mathrm{proj}_{S^\perp}(K)) d\mu_r(S) \tag{3}$$

where L_r is the class of all r-dimensional subspaces S, μ_r is the unique probability measure on L_r that is invariant under rigid motions, $\mathrm{proj}_{S^\perp}$ is the map projecting onto $S^\perp$ the subspace orthogonal to S, and ν_j is Lebesgue measure on j-dimensional space. Finally, $b_d = \pi^{d/2}/\Gamma(1+d/2)$ denotes the $d-$volume of the d-dimensional unit ball. (If $r = d$ then we set $W_d^d(\cdot) \equiv b_d$.) Simple Haar measure considerations show that it is equivalent to define $W_r^d(K)$ for $r \leq d-1$ as the unique invariant measure on the family of all affine r-dimensional subspaces intersecting K, normalised so that the unit ball has quermass integral b_d.

A different but still equivalent definition is via the *Steiner formula*

$$\nu_d(K \oplus B(0,t)) \quad = \quad \sum_{r=0}^{d} \binom{d}{r} W_r^d(K) t^r.$$

Interesting special cases include the following:

$W_0^d(K)$ is the Lebesgue measure $\nu_d(K)$ of K;

$d \times W_1^d(K)$ is the surface (hyper-) area of K;

$d \times W_2^d(K)$ is the mean integrated curvature over the boundary of K;

and $W_{d-1}^d(K) = (b_d/2) \times \bar{b}(K)$ is proportional to the mean breadth $\bar{b}(K)$ of the ovoid K.

If the boundary ∂K is sufficiently regular (for example if it is possible to define at each point $t \in \partial K$ the $d-1$ main curvatures) then the Minkowski functionals admit simple integral representations using symmetric functions of these curvatures (see Matheron [29]). Thus for example

$$d \times W_2^d(K) \quad = \quad \int_{\partial K} m(t)dt$$

where $m(t)$ is the mean curvature at t.

Let ψ be a functional defined for all ovoids. It is called *C-additive* if

$$\psi(K_1 \cup K_2) \quad = \quad \psi(K_1) + \psi(K_2) - \psi(K_1 \cap K_2)$$

for any ovoids $K_1, K_2 \in C(\mathcal{K})$ for which the union is again an ovoid ($K_1 \cup K_2 \in C(\mathcal{K})$). The Minkowski functionals are C-additive, and also increasing, continuous with respect to Hausdorff distance and invariant under rigid motions. Hadwiger's characterisation theorem delivers a converse to this observation: any sufficiently well-behaved ovoid-functional can be written as a linear combination of Minkowski functionals. More specifically

Theorem 1.1 (Hadwiger's characterization theorem [15]) *Suppose that ψ is a C-additive ovoid functional (hence $\psi(K)$ is defined for $K \in C(\mathcal{K})$) which is continuous with respect to the Hausdorff metric on $C(\mathcal{K})$ and is invariant under rigid motions. Then it can be written as a linear combination of quermass integrals*

$$\psi \quad = \quad \sum_{r=0}^{d} a_r W_r^d$$

where the coefficients a_r are uniquely defined. (If "continuous" is replaced by "increasing" (with respect to set-inclusion) or "non-negative" then the same statement holds under the further condition that the a_r are non-negative.)

We intend to use quermass integrals to weight Boolean models. Hence we will be interested in evaluation of quermass integrals on locally finite unions of convex compact sets, which form the *convex ring* $\mathcal{R}$. The quermass integrals can be extended onto the convex ring in several ways. The most direct is the *additive extension*

$$W_r^d(K) \quad = \quad \frac{b_d}{b_{d-r}} \int_{L_r} \left(\int_{S^\perp} \chi(K \cap S_x) dx \right) d\mu_r(S)$$

where χ denotes the Euler-Poincaré characteristic, and S_x the translation of the subspace S using the vector x. This equals 1 for any ovoid; while for any $K = \cup_{i=1}^p K_i$ (for $K_i \in C(\mathcal{K})$) we have an inclusion-exclusion formula:

$$\chi(K) \quad = \quad \sum_i \chi(K_i) - \sum_{i_1 < i_2} \chi(K_{i_1} \cap K_{i_2}) + \cdots + (-1)^{p+1} \chi(K_1 \cap \cdots \cap K_p) \, .$$

In particular the right-hand side does not depend on the particular representation $K = \cup_{i=1}^{p} K_i$: thus the inclusion-exclusion formula can be used to define the additive extension of quermass integrals. An equivalent definition uses the (generalised) Steiner formula

$$\int \chi(K \cap B(x,t))dx \quad = \quad \sum_{r=0}^{d} \binom{d}{r} W_r^d(K) t^r.$$

This extension is by no means unique; another possibility is to require the original Formula (3) to hold for all K in the convex ring, resulting in a different extension. But the Euler-Poincaré extension has a useful relationship to 'number' which is exploited in various applications of stochastic geometry; in the planar case $W_r^d(K)/\pi$ is equal to the number of components of K minus the number of holes of K. (For an example of this in the theory of high-level excursions of random fields, see Adler [1, chapter 4])

Yet another possibility (albeit computationally more involved) is the positive extension, defined via a different generalised Steiner formula

$$\int n(K;r;x)dx \quad = \quad \sum_{r=0}^{d} \binom{d}{r} \overline{W}_r^d(K) t^r$$

where $n(K;r;x)$ is the number of *projections* $y \in K$ (points in K locally closest to x) lying within distance r of x (see Matheron [29, (4-7-8)] also Schneider [48]). A significant subtlety in this definition is that it is possible to have $n(K;r;x) > 1$ for $x \in \mathrm{int}(K)$. The $\overline{W}_r^d(K)$ defined here satisfy

$$\overline{W}_r^d(\bigcup_i K_i) \quad \leq \quad \sum_i \overline{W}_r^d(K_i)$$

for any compact convex sets K_1, ..., K_n. However in the following we shall focus our attention mainly on the simpler and more intuitive additive extension.

It is important that $\overline{W}_r^d(\cdot) = W_r^d(\cdot)$ for $r = 0, 1$.

2 Quermass-interaction point processes

We now introduce the idea of point processes weighted by exponentials of (linear combinations of) quermass integrals applied to a Boolean model based on the point process in question.

2.1 Notation and framework

Let $\mathcal{X}$ be a locally compact complete separable metric space, for example $\mathbb{R}^d$ or a compact subset.

Definition 2.1 *A quermass-interaction point process on $\mathcal{X}$ has density $p(\mathbf{x})$ with respect to a (possibly inhomogeneous) benchmark Poisson process on $\mathcal{X}$ (of finite intensity measure μ), where $p(\mathbf{x})$ is given by*

$$p(\mathbf{x}) = \alpha\beta^{n(\mathbf{x})}\gamma^{-W_r^d(\mathcal{U}_\mathbf{x})}. \tag{4}$$

Here β and γ are strictly positive model parameters, $\mathbf{x} = \{x_1, \ldots, x_n\}$ is the configuration in question, and $\mathcal{U}_\mathbf{x}$ is the set union $\bigcup_{i=1}^n Z(x_i)$ where the $Z(x_i)$ are independent random compact convex sets (conditioned on the realization $\mathbf{x}$ of the Poisson point process) such that the distribution of $Z(x)$ depends continuously on x.

Here of course we use the topology of weak convergence on $C(\mathcal{K})$ using the Hausdorff metric. When the $Z(x)$ have degenerate distributions then we may suppose $Z : \mathcal{X} \to C(\mathcal{K})$ is a continuous set-valued function using the Hausdorff metric.

Note that we can absorb β into the definition of the intensity measure μ.

Special cases of this model have been discussed in the literature. Widom and Rowlinson [54] (see also [18, 44, 45]) introduced "penetrable sphere" models for liquid-vapour transitions in statistical physics. This is the special case of (4) in which $\gamma > 1$, $r = 0$ and $Z(u) = B(u,t)$ the closed ball of radius t. Hence the model describes attraction by spherical molecules.

The area-interaction process [4] is a generalisation of the Widom-Rowlinson model which includes inhibition models, and which has $r = 0$ and general (not necessarily convex) Z, but also allows the replacement of area by any finite Borel regular measure. It has an intriguing relationship to "selfish herd" arguments in theoretical biology [17]. Note that even though the perimeter (case $r = 1$) is a positive functional it is not a measure, and so the corresponding quermass-interaction point process is not included in the area-interaction case.

Area-interaction models are Markov in the sense of Ripley and Kelly with respect to the *overlapping object relationship* [3]:

$$u \sim v \quad \text{if and only if} \quad Z(u) \cap Z(v) \neq \emptyset$$

and satisfy Ruelle's stability condition for all values of γ. This condition requires that the energy $E(\cdot) = -\log(p(\cdot)/p(\emptyset))$ is bounded below by a linear bound in the number of points:

$$E(\mathbf{x}) \quad \geq \quad -Bn(\mathbf{x}) \tag{5}$$

for some $B > 0$. Indeed, the area-interaction density with respect to a Poisson(β) process (restricted to a bounded window, as is always the case here) is uniformly bounded.

Care is needed in the interpretation of this if $Z(x)$ is random, since then the neighbourhood relationship $\sim$ will also be random, thus linking to the setting of [5].

An non-stable point process using the attractive version of the Strauss weighting [50] is obtained by considering a process conditioned on having exactly n points and then randomizing n using a Poisson distribution. This model resembles a quermass-interaction density but uses a weight based on the number of pairs of points less than a distance r apart. However if the attractive version of the Strauss weighting is applied directly to a Poisson process then the weighting diverges (as shown by Kelly and Ripley [22]). Indeed Ruelle [46, §3.2] shows that, for (upper semi-continuous) pairwise interaction processes, unless the energy is stable the weight functional will diverge for a motion-invariant point process extending over the whole of space. Gates and Westcott [12] show that, even in a bounded window and conditional on the number of points, unstable point processes may yield problems in simulations (published examples are typically not yet in equilibrium, and results will be very sensitive to boundary conditions) and approximations of the partition function can be wrong by many orders of magnitude. See also Møller [30].

Before we start to investigate quermass-interaction point processes we must carry out the tedious chore of establishing the (obvious) measurability of the density.

Lemma 2.2 *In the case when $Z(x)$ is non-random, the quermass-interaction density is measurable with respect to $\mathcal{N}^f$, the Borel σ-algebra corresponding to the weak topology on the space of all integer-valued simple finite measures $\mathfrak{N}^f$.*

Remark: A similar measurability result (more tedious to state) follows for *random* $Z(x)$ by a routine variation of the argument given below.

Proof : We follow the general approach of [4, Lemma 2.1], but care has to be taken because the general quermass integral is no longer monotonic. To show the measurability of the density

$$p(\mathbf{x}) \quad = \quad \alpha\beta^{n(\mathbf{x})}\gamma^{-W_r^d(\mathcal{U}_\mathbf{x})}$$

it suffices to consider $W_r^d(\mathcal{U}_\mathbf{x})$, since $n(\mathbf{x})$ will be $\mathcal{N}^f$-measurable. Now following Theorem 1.1, from the integral expression for the additive extension $W_r^d(\cdot)$ and an integration argument (using 1.3), it suffices to show measurability of $\chi(\mathcal{U}_\mathbf{x} \cap S)$ where χ is the Euler-Poincaré characteristic on a subspace S. But this measurability follows by the inclusion-exclusion formula immediately following the additive expression, and by the fact that $(A, B) \mapsto A \cap B$ is upper-semicontinuous. (For the upper-semicontinuity of intersection, see Corollary 1 to Proposition 1-2-4 in Matheron [29].) □

There is a similar and routine argument which yields measurability when the quermass integral is replaced by the positive extension.

We turn to the issue of when the quermass-interaction density is integrable, which is the main question addressed by this paper.

Note that the energy is

$$-n(\mathbf{x})\log\beta + W_r^d(\mathcal{U}_\mathbf{x})\log\gamma.$$

The stability condition for $\gamma > 1$ amounts to

$$W_r^d(\mathcal{U}_\mathbf{x}) \quad > \quad -Bn(\mathbf{x})$$

whilst for $\gamma < 1$ we need

$$W_r^d(\mathcal{U}_\mathbf{x}) \quad < \quad Bn(\mathbf{x})$$

(for some $B > 0$). Since the β term is linear in $n(\mathbf{x})$, it will not affect questions of stability.

We note in passing that the positive extension of Minkowski functionals always produces stability:

Lemma 2.3 *Assume the positive extensions $\overline{W}_r^d(\cdot)$ of the Minkowski functionals are used. Then a quermass-interaction process with $\gamma \geq 1$ is stable. If $\overline{W}_r^d(Z(a))$, $a \in \mathcal{X}$ is bounded above then the quermass-interaction model with $\gamma < 1$ is also stable.*

Proof : For $\gamma \geq 1$, any B satisfies the stability equation. For $\gamma < 1$, use subadditivity (1.3): if $\mathbf{x} = \{x_1, ..., x_n\}$ then

$$0 \leq \overline{W}_r^d(\mathcal{U}_\mathbf{x}) \quad \leq \quad \sum_{i=1}^{n} \overline{W}_r^d((Z_{x_i})).$$

□

In fact even more can be said for the perimeter interaction when the grains are disks of constant radius. Stability follows as a consequence of work by Baddeley and Gill [2], which actually proves the stronger result of a uniform bound on the density with respect to a Poisson process over a compact region, by geometric reasoning involving the correspondence of the exposed boundaries to the sectors which they define. Complementary geometric arguments yield a uniform bound on the Papangelou conditional intensity [25].

Corollary 2.4 *Under the conditions in the previous Lemma, when the positive extension is used the quermass-interaction density is always integrable.*

The situation is much more interesting for the additive extensions, (except for $r = 0, 1$ in which case the positive and additive extensions are identical). We shall focus on the planar Euler-Poincaré characteristic, so $d = r = 2$. Further interest is added to this case by the considerations

This provide a *topological* weighting, in the sense that the weighting depends only on the topology of the union of grains;

There is a weak link to Markov-like properties described in the next sub-section.

In this case $W_r^d(K)/\pi$ equals the number of components of K minus the number of holes of K. Clearly $n(\mathbf{x})$ provides an upper bound, hence the associated 'repulsive' quermass-interaction process is stable. For the 'attractive' counterpart $\gamma > 1$, we need an upper bound on the number of holes. This problem is dependent on the geometry of the grains and is treated in Sections 4, 5, 6.

2.2 Markov properties

As with the area-interaction model [4], the quermass-interaction generalisations are Markovian (this is also true for the positive extension).

Theorem 2.5 *Whenever $p(\cdot)$ is integrable then it is Markovian in the Ripley-Kelly sense with respect to the overlapping objects relationship.*

Proof : Property (**M1**) of the definition of a Markov point process is straightforward, so it suffices to establish property (**M2**).

The case $r = 0$ has been established in [4], so we can assume $r > 0$. First observe that since both β and γ are positive, the hereditary property is trivial. The result follows by considering the log probability ratio (which is to say, the log Papangelou conditional intensity): we apply the inclusion-exclusion law to $W_r^d(\mathcal{U}_{\mathbf{x}} \cup Z(a))$.

$$\begin{aligned} -\log \frac{p(\mathbf{x} \cup \{a\})}{p(\mathbf{x})} &= \left[W_r^d(\mathcal{U}_{\mathbf{x}} \cup Z(a)) - W_r^d(\mathcal{U}_{\mathbf{x}})\right] \log\gamma - \log\frac{1}{\beta} \\ &= \left[W_r^d(Z(a)) - W_r^d(Z(a) \cap \mathcal{U}_{\mathbf{x}})\right] \log\gamma - \log\frac{1}{\beta} \\ &= \left[W_r^d(Z(a)) - W_r^d(Z(a) \cap \bigcup_{y \sim a} Z(y))\right] \log\gamma - \log\frac{1}{\beta}. \end{aligned}$$

Thus the Papangelou conditional intensity for adding a to $\mathbf{x}$ depends only on the sub-configuration of points of $\mathbf{x}$ neighbouring a. Hence (**M2**) follows. (There is a corresponding and straightforward argument for the positive extension, depending on the fact that for x_i for which $Z(x_i) \cap Z(a) = \emptyset$ the exposed boundary in $\mathbf{x} \cup \{a\}$ is the same as in $\mathbf{x}$, so that a similar cancellation occurs.) □

This makes strict sense only when the grain $Z(x)$ is non-random. More generally, one can argue as above for the density of the marked point process or germ-grain process as appropriate.

If it can be shown that the grain is always contained in a disk centred on $\mathbf{o}$ of fixed radius r then the above argument establishes a local Markov property with respect to the conventional neighbourhood relationship $x \sim y$ when x, y are closer than $2r$.

A fortiori the process is nearest-neighbour Markov with respect to the connected component relation of Baddeley and Møller [5]. Note also that Møller proposed a model in which the weighting is carried out by counting connected components instead of the Euler characteristic. In Møller's model the Markov property is replaced by a nearest-neighbour Markov property.

By the Hammersley-Clifford theorem (see §1.1 above), the density $p(\cdot)$ can be written as a product of clique interaction terms

$$p(\mathbf{x}) \quad = \quad \prod_{\mathbf{y} \subseteq \mathbf{x}} q(\mathbf{y})$$

where $q(\mathbf{y}) = 1$ unless $y_i \sim y_j$ for all elements of $\mathbf{y}$. The interaction functions resemble those of the area-interaction model. For the additive case, or for $r = d$,

$$\begin{aligned} q(\emptyset) &= \alpha \\ q(\{a\}) &= \beta\gamma^{-W_r^d(Z(a))} \\ q(\{y_1, \ldots, y_k\}) &= \gamma^{(-1)^k W_r^d(\bigcap_{i=1}^k Z(y_i))}; \end{aligned} \tag{6}$$

(for positive extensions one can replace $W_r^d(Z(a))$ by a boundary integral.) In particular, the model has (*numerical*) interaction of all orders, but note that the model interaction can be considered to be pairwise where the interactions combine according to Boolean logic rather than arithmetic.

In the case where the grain depends continuously on location the process moreover satisfies a spatial Markov property (see [23, 43]). Define the *dilation* of a set $E \subseteq \mathcal{X}$ by

$$D_Z(E) \quad = \quad \{u \in \mathcal{X} : \text{ there is } e \in E \text{ such that } Z(u) \cap Z(e) \neq \emptyset\}. \tag{7}$$

If $Z(a) = a \oplus Z_0$ this becomes the classical dilation of mathematical morphology. Then the spatial Markov property states that the restriction of the process to E is conditionally independent of the restriction to $D_Z(E)^c$ given the information in $D_Z(E) \setminus E$.

Reverting to the case when the grain $Z(u)$ is a random ovoid, we may view the process as a germ-grain process in the sense of [51, Section 6.4]. Write $X = \{[x; Z(x)]\}$. Suppose that the grains $Z(x)$ are *observed*. The distribution of X is obtained from that of a benchmark marked Poisson process (of finite total intensity, each point x marked by its grain $Z(x)$) by using a weighting factor $\exp(-\psi(\mathcal{U}))$ where

$$\mathcal{U} \quad = \quad \bigcup\{Z(x) : [x; Z(x)] \in X\}.$$

It is interesting to note that we can rephrase C-additivity of a functional $\psi : C(K) \to \mathbb{R}$ in terms of *multiple conditional grain intensities*, and thus motivate the study of quermass-interactions.

Definition 2.6 *Consider the procedure of augmenting X by adding k new points*

$$\{[u_1; Z(u_1)], ..., [u_k; Z(u_k)]\}$$

to X. Set $\mathcal{U} = \bigcup\{Z(x) : [x; Z(x)] \in X\}$ and $\mathcal{V} = Z(u_1) \cup ... \cup Z(u_k)$. Then the multiple conditional grain intensity *is*

$$\lambda(\mathcal{V}; \mathcal{U}) \quad = \quad \exp(-(\psi(\mathcal{U} \cup \mathcal{V}) - \psi(\mathcal{U}))),$$

the ratio of the density of the augmented process to that of the unaugmented process.

Of course the notion of multiple conditional grain intensity is closely related to that of n-fold Palm distribution (see [21]).

Consider the effect of a *localization condition*, encapsulating the idea that the addition of $\mathcal{V}$ depends only on $\mathcal{U} \cap \mathcal{V}$ and $\mathcal{V}$. This forces ψ to be a *valuation* ([28, page 171]) on the class $\mathcal{L}$ of finite unions of possible grains. If *either* we can assume this class to be the convex ring or the family of polyhedra, *or* we can use it to approximate the convex ring appropriately, then we can limit the form of possible ψ by applying characterization theorems for valuations.

Lemma 2.7 *Suppose that $X = \{[x; Z(x)]\}$ is a germ-grain process, produced from an (inhomogeneous) marked Poisson process by a weighting $e^{-\psi(\mathcal{U}_\mathbf{x})}$ depending on $\mathcal{U}_\mathbf{x} = \bigcup_{i=1}^n Z(x_i)$. If the multiple conditional grain intensity (2.6) localizes:*

$$\lambda(\mathcal{V}; \mathcal{U}) \quad = \quad \exp(-(\alpha(\mathcal{V}) - \beta(\mathcal{U} \cap \mathcal{V}))) \tag{8}$$

for functionals α defined on $\mathcal{L}$, β defined on $\mathcal{L}_2 = \{\mathcal{U} \cap \mathcal{V} : \mathcal{U}, \mathcal{V} \in \mathcal{L}\}$ then the functional ψ may be taken to be a valuation on $\mathcal{L}$ and indeed

$$\psi(\mathcal{U} \cup \mathcal{V}) \quad = \quad \psi(\mathcal{U}) + \psi(\mathcal{V}) - \psi(\mathcal{U} \cap \mathcal{V})$$

whenever $\mathcal{U}$, $\mathcal{V}$, $\mathcal{U} \cap \mathcal{V}$ belong to $\mathcal{L}$.

Proof : We may suppose (without loss of generality) that $\psi(\emptyset) = 0$, since this can be absorbed in the normalizing constant. Also without loss of generality we may assume that $\beta(\emptyset) = 0$.

Set $\mathcal{V} = Z_1 \cup ... \cup Z_k$ and observe that from the localization of multiple conditional grain intensity (8) we have

$$\psi(\mathcal{U} \cup \mathcal{V}) - \psi(\mathcal{U}) \quad = \quad \alpha(\mathcal{V}) - \beta(\mathcal{U} \cap \mathcal{V})$$

for any finite union $\mathcal{U}$ of grains.

If $\mathcal{U} = \mathcal{V} = Z_1 \cup ... \cup Z_k$ then

$$0 = \psi(\mathcal{V}) - \psi(\mathcal{V}) \quad = \quad \alpha(\mathcal{V}) - \beta(\mathcal{V})$$

and therefore $\alpha(\mathcal{V}) = \beta(\mathcal{V})$.

If $\mathcal{U} = \emptyset$ then

$$\psi(\mathcal{V}) = \psi(\mathcal{V}) - \psi(\emptyset) \quad = \quad \alpha(\mathcal{V}) - \beta(\emptyset)$$

and therefore $\alpha(\mathcal{V}) = \psi(\mathcal{V})$.

Consequently we discover that ψ is a *valuation* on the family $\mathcal{L}$ [28, page 181]:

$$\psi(\mathcal{U} \cup \mathcal{V}) \quad = \quad \psi(\mathcal{U}) + \psi(\mathcal{V}) - \psi(\mathcal{U} \cap \mathcal{V})$$

whenever $\mathcal{U}$, $\mathcal{V}$, $\mathcal{U} \cup \mathcal{V}$, $\mathcal{U} \cap \mathcal{V}$ all belong to $\mathcal{L}$. □

Suppose the additional conditions hold that ψ is motion invariant and extends *as a valuation* to the whole convex ring. If also ψ is a continuous function when restricted to convex compact sets then Hadwiger's characterization result (Theorem 1.1) immediately shows us that ψ is a linear combination of quermass integrals:

$$\exp(-\psi(\mathcal{U}_{\mathbf{x}})) \quad = \quad \exp(-\sum \alpha_r W_r^d((\mathcal{U}_{\mathbf{x}}))) .$$

This provides some motivation for the study of quermass-interactions as related to the question of whether the multiple conditional grain intensity localizes as in Definition 2.6.

It is reasonable to ask for an appropriate continuity requirement on ψ, such that if the family $\mathcal{L}$ of finite unions of grains is sufficiently rich then ψ extends as a valuation to the whole convex ring. Work is in progress to elicit such a condition, which should certainly apply when $\mathcal{L}$ is the family of finite unions of disks of arbitrary radii, as studied in Section 5. Note that if $\mathcal{L}$ is the family of finite unions of convex polygons of positive area (as might be appropriate in the case of Section 6) then it is easy to formulate a suitable condition.

3 Quermass-interaction random sets

In the previous sections we were concerned with models for point processes on $\mathbb{R}^d$. These can be used to build random set models analogous to the way a Boolean model is constructed from a Poisson germ process.

Definition 3.1 *Let X be an integrable quermass-interaction process on a bounded subset $A \subseteq \mathbb{R}^d$, with density*

$$p(\mathbf{x}) \propto \gamma^{-W_r^d(\mathcal{U}_{\mathbf{x}})}$$

with respect to an (inhomogeneous) Poisson process on A of finite intensity measure μ. (Integrability here means the density is integrable) Then

$$\Xi = \mathcal{U}_{\mathbf{x}} = \bigcup_{i=1}^{n} Z(x_i)$$

is a random set, a quermass-interaction random set.

As in the Boolean model, the component grains $Z(x_i)$ are not observable since their boundaries may be occluded by other grains. Indeed, there is no way of determining even the number of germs giving rise to a union set Ξ, complicating estimation of model parameters (see later).

We need to check that the random set is well-defined, which is to say that "hitting events" such as $\{\Xi \Uparrow K\}$, the event that the set Ξ has non-void intersection with the set K, are weakly measurable.

Lemma 3.2 *The quermass-interaction random set (Definition 3.1) is well-defined.*

Proof : We have to prove that $\{\Xi \Uparrow K\}$ is measurable for all $K \in \mathcal{K}$. As $Z(\cdot)$ is continuous,

$$L = \{a \in A : Z(a) \cap K \neq \emptyset\}$$

is closed in A. Hence

$$\{\Xi \Uparrow K\} = \left\{\mathbf{x} \in \mathfrak{N}^f : n(\mathbf{x}_L) = 0\right\}^c$$

is closed, hence measurable. □

The quermass-interaction *point processes* are Markov with respect to the overlapping objects relation and satisfy a spatial Markov property

$$X \cap E \quad \perp \quad X \cap D_Z(E)^c \quad | \quad X \cap D_Z(E) \setminus E.$$

In words, the random point pattern $X \cap E$ is independent of the random point pattern $X \cap D_Z(E)^c$ when conditioned on the realization of the "frontier" point pattern $X \cap D_Z(E) \setminus E$.

This property has an implication for the induced random set,

$$\Xi \cap E \quad \perp \quad \Xi \cap D_Z(E)^c \quad | \quad X \cap D_Z(E) \setminus E. \tag{9}$$

Unfortunately we cannot in general choose the conditioning event to depend on $\Xi \cap D_Z(E) \setminus E$ rather than the (typically unobservable) $X \cap D_Z(E) \setminus E$.

It is possible to derive a random set Markov property in the 1-dimensional case. For example in $\mathbb{R}^d$, Matheron defines two compact sets K and K' as separated by another compact set $C \in \mathcal{K}$ if any line segment joining $x \in K$ with $x' \in K'$ hits C. Furthermore, the random set Ξ is said to be *semi-Markovian* if (9) holds for any $E, F \in \mathcal{K}$ separated by $G \in \mathcal{K}$ and the conditioning is on $\Xi \cap G = \emptyset$. It is then easy to show that the 1 dimensional quermass-interaction random sets (3.1) are semi-Markov.

No such result can be expected in higher dimensions, as separation no longer implies topological separation.

Finally we turn to non-deterministic grains.

Definition 3.3 *Let*

$$p(\mathbf{z}) \quad = \quad \alpha\gamma^{-W_r^d(\mathcal{U}_\mathbf{Z})}$$

be a marked point process on $A \times C(\mathcal{K}')$ defined by its density with respect to a Poisson $(\mu \times \nu)$ process, where μ is the (finite) intensity measure for the unmarked Poisson process, and ν is any probability measure on $C(\mathcal{K}')$. Then

$$\Xi \quad = \quad \bigcup_{i=1}^{n} x_i \oplus K_i$$

is called a quermass-interaction random closed set model with random grains.

Note that the random marks will *not* be independent under the weighted probability distribution. The marginal germ process of Z yields a further generalisation of quermass-interaction point processes, but we will not discuss this here, as the random set approach seems more natural.

Lemma 3.4 *The random set Ξ (Definition 3.3) is well-defined.*

Proof : Observe

$$\{\Xi \Uparrow K\} \quad = \quad \{(x_i, Z_i)_i : n(\mathbf{z}_L) = 0\}^c$$

where

$$L \quad = \quad \{(x, Z) : x \oplus Z \cap K \neq \emptyset\}.$$

As the mapping $f : \mathcal{X} \times C(\mathcal{K}') \to C(\mathcal{K}')$ $(x, Z) \mapsto x \oplus Z$ is continuous, L is closed. Thus $\{\Xi \Uparrow K\}$ is weakly measurable by definition. □

4 Planar case: counterexamples

In this section we consider in detail the two-dimensional case $d = 2$ of additive quermass-interaction point processes in the plane. In this case, for $K \in C(\mathcal{K})$, we have

$$\begin{aligned} W_0^2(K) &= \nu_2(K) = \text{ area} \\ W_1^2(K) &= \frac{1}{2}U(K) = \frac{1}{2} \text{ perimeter } = \frac{\pi}{2} \text{ mean breadth} \\ W_2^2(K) &= \pi \end{aligned}$$

and we study the point process whose density with respect to a Poisson point process or Boolean model / marked Poisson point process, if the grains are random) is given by

$$p(\mathbf{x}) \quad = \quad \alpha\beta^{n(\mathbf{x})}\gamma^{-\sum_{r=0}^{2} W_r^d(\mathcal{U}_\mathbf{X})}.$$

In the instances $r = 0$ and $r = 1$, the functional W_r^2 is positive, both extension methods coincide, and Lemma 2.3 leads to the simple conclusion that integrability holds for all values of γ (at least for bounded convex grains as above). So it remains to consider the case $r = 2$.

Here the extension methods do *not* coincide and we have to argue in detail. The additive extension of W_2^2 has a simple interpretation: it is proportional to the Euler-Poincaré characteristic χ ("the number of components minus the number of holes" for this planar case). In fact $W_2^2(K) = \pi\chi(K)$. For Ruelle stability (5) to hold for all parameter values, we require

$$-B_1\, n(\mathbf{x}) \quad \leq \quad \chi(\Xi) \quad \leq \quad B_2 n(\mathbf{x}) \tag{10}$$

where B_1, B_2 are positive constants and $\Xi = \mathcal{U}_{\mathbf{x}}$. The right-hand inequality is immediate from C-additivity, but the left-hand inequality is actually false in general (see the counterexamples below). Let us examine what can go amiss. First note that if $\gamma = 1$ then the weighting has no effect and everything is trivial. The case $\gamma < 1$ (inhibition) is also clear:

$$\int \cdots \int \gamma^{-\pi\chi(\mathcal{U}_{\mathbf{x}})} d\mu(x_1) \cdots d\mu(x_n) \quad \leq \quad \gamma^{-\pi n(\mathbf{x})} \mu(\mathcal{X})^{n(\mathbf{x})}.$$

So the process is then bounded above by a Poisson process with intensity measure $\gamma^{-\pi}\mu(\mathcal{X})$ and therefore is integrable and indeed stable. However stability does not hold in general for the clustered case of $\gamma > 1$, as we now show by exhibiting counterexamples.

The first counterexample is suggested by the observation that n lines in general position in the plane produce $(n-1)(n-2)/2$ (bounded) holes. (Proof by induction: adding a line in general position to an assembly of n lines in general position produces $n-1$ new bounded holes.)

Counterexample 1: Poisson line process in the plane. *Let the germ process be an inhomogeneous Poisson point process Φ of finite total intensity. Let the typical grain be a line randomly oriented with some fixed directional distribution. Suppose that the intensity measure of the resulting line process is diffuse and has topological support containing two lines which intersect (of course this second requirement will be fulfilled unless all lines in the process are almost surely parallel!). Then the expectation*

$$\mathbb{E}\left\{\gamma^{-W_2^2(\Xi)}\right\} \quad = \quad \mathbb{E}\left\{\exp\left(-(\ln\gamma) \times W_2^2(\Xi)\right)\right\} \tag{11}$$

is infinite if $\gamma > 1$.

Since $W_2^2(\Xi)/\pi = \chi(\Xi)$ is "the number of components minus the number of holes" this means that we cannot weight the model towards having more holes than would be expected in the unweighted (Poisson) case.

Proof : Note that $-\chi(\Xi)$ is bounded above by the number of holes in Ξ. The topological support condition means that we can choose two compact sets K_1, K_2 in line space such that (a) the intensity measure charges both K_1 and K_2, and (b) all lines in K_1 intersect all lines in K_2. We condition on the event that *all* lines of the process belong to $K_1 \cup K_2$. Under this conditioning event (which is of positive probability) the number of holes is given by $(N_1 + 1)(N_2 + 1)$ where N_i is the (random) number of lines in K_i and has a nondegenerate Poisson distribution. But this means that the expectation in Equation (11) is infinite, because the moment generating function of the product of two non-degenerate Poisson distributions is infinite for positive argument. □

It might be objected that the above counterexample uses non-compact grains. Basic arguments using Boolean models readily yield the following localization and conditioning argument which replaces unbounded lines by bounded line segments.

Counterexample 2: Poisson segment process in the plane. *Let the germ process be an inhomogeneous Poisson point process Φ of finite total intensity. Let the typical grain be a line segment randomly oriented with some fixed directional distribution. Suppose that the intensity measure of the resulting line segment process is diffuse and has topological support containing two line segments which intersect. Then the expectation*

$$\mathbb{E}\left\{\gamma^{-W_2^2(\Xi)}\right\} \quad = \quad \mathbb{E}\left\{\exp\left(-(\ln\gamma) \times W_2^2(\Xi)\right)\right\} \tag{12}$$

is infinite if $\gamma > 1$.

Proof : We can argue exactly as in Counterexample 1, except that this time the topological support condition allows us to choose compact sets in segment space K_1 and K_2, such that (a) the intensity measure charges both K_1 and K_2, and (b) each segment in K_1 intersects all segments in K_2. □

The problems in the above two counterexamples appear to be related to the pathological "sharpness" of the grains, and in particular to the fact that they have negligible area. A natural condition to exclude this pathology is to require a lower bound on the internal angles of convex polygonal grains.

Definition 4.1 *A convex grain G is said to satisfy a "local wedge condition of angle $\phi > 0$" if for any point $\omega \in \partial G$ there is a disk $B(\omega, r)$ (centred at ω, of positive radius $r = r(\omega)$) such that $B(\omega, r) \cap G$ is a sector of the disk of angle at least ϕ. (No lower bound is placed on the radius of the disk, other than the requirement that it be positive.)*

Note that a convex grain satisfying this condition is automatically polygonal.

Here is a counterexample to show that care is required even when the grains satisfy a local wedge condition.

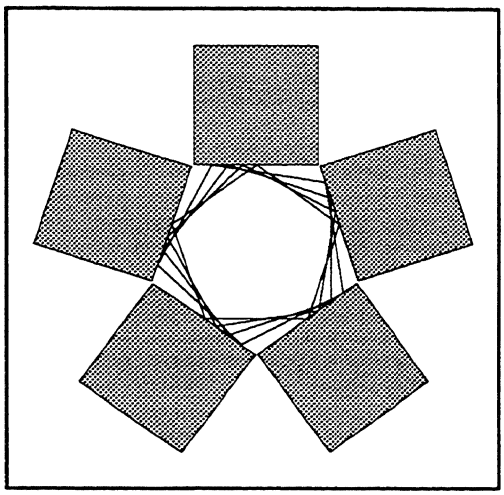

Figure 1: How to build configurations of n polygonal grains which create $O(n^2)$ holes (pentagonal case). We can arrange $k = n/2$ pentagons one on top of the other so that by adding $r = n/2$ other polygons we can create $n/4$ holes per polygon.

Counterexample 3: Boolean model with convex grains satisfying a local wedge condition. *In general the weighting need not satisfy the stability condition when* $\gamma > 1$.

Proof : The grains are regular k-gons, of varying side-number k ($k > 3$) and size. To establish failure of stability, we have to show how to construct configurations of n grains which possess $O(n^2)$ holes.

Fix $k > 3$, $r > 1$ and set $\epsilon = (r-1)\pi/(k(k-1)^2)$. Notice that the local wedge condition is satisfied for $\phi = \pi/2$, since $k > 3$.

Consider r similar k-gons, of which the first is inscribed in a circle of unit radius centered on the origin $\mathbf{o}$, and such that the i^{th} k-gon is obtained from the first by rotation about $\mathbf{o}$ through an angle of $(i-1)\epsilon$ and scaling (again about $\mathbf{o}$) by a factor of $\sec((i-1)\epsilon)$. At each vertex of the first k-gon place a square with sides of unit length, tangent to the inscribing circle at the midpoint of a side. (See Figure 4 for the case $k = 5$, $r = 5$.)

For all sufficiently large k, each square intersects each of the k-gons at a vertex, and none of the intersections of squares with k-gons are covered by other squares or k-gons. (This follows from the observation that the k-gons intersect in singleton sets with lines through vertices of the first polygon which are perpendicular to radii of the circle which it inscribes.) Consequently this configuration of $r-1$ k-gons and k squares creates at least $k(r-1)$ holes. Setting $k = r = n/2$ for even n delivers the required violation of stability. □

It is important to note that the above counterexample works only if we allow polygonal grains of arbitrarily small sidelength. Later on we shall see that an additional lower bound on sidelength (obtained by requiring a *uniform* local wedge condition) is sufficient to ensure stability for polygonal grains.

In this paper we confine ourselves to the planar case, which is the case of principal importance for image analysis (though not for physics!). However it is interesting to note that things can go even more badly wrong for the Euler-Poincaré characteristic in the spatial case.

Counterexample 4: Process of flats in space. *Divergence can occur for all parameter values except for the trivial (unweighted) case of $\gamma = 1$. Take the Poisson point process Φ of germs to be inhomogeneous and of finite total intensity. Fix an orthonormal basis. Let the typical grain be a "flat" or 2-plane, normal to a vector chosen randomly from the orthonormal basis. Suppose that the intensity measure of the underlying flat process is diffuse. Then the expectation*

$$\mathbb{E}\left\{\gamma^{-W_3^3(\Xi)}\right\} \quad = \quad \mathbb{E}\left\{\exp\left(-(\ln\gamma) \times W_3^3(\Xi)\right)\right\} \tag{13}$$

is finite if and only if $\gamma = 1$ (the trivial unweighted case!).

Proof : First note that $W_3^3(\Xi)$ is no longer proportional to the number of holes minus the number of components, but is proportional to the three-dimensional Euler-Poincaré characteristic. However (in the simple case which we have chosen to consider) it is easily computed from first principles using the inclusion-exclusion formula of C-additivity. Let N_1, N_2, N_3 be the numbers of flats normal to each of the three basis vectors. Then

$$\begin{aligned} W_3^3(\Xi) &= (N_1 + N_2 + N_3) - (N_1N_2 + N_2N_3 + N_3N_1) + N_1N_2N_3 \\ &= (N_1 - 1)(N_2 - 1)(N_3 - 1) + 1 \end{aligned}$$

(since intersections of more than three flats will be almost surely void, because the underlying intensity measure of the flat process is diffuse). It suffices to show divergence of $\mathbb{E}\left\{\gamma^{-(N_1-1)(N_2-1)(N_3-1)-1}\right\}$.

Suppose that $\gamma < 1$. Then

$$\begin{aligned} \mathbb{E}\left\{\gamma^{-(N_1-1)(N_2-1)(N_3-1)-1}\right\} &= \sum_{n_1}\sum_{n_2}\sum_{n_3} e^{-1-\lambda_1-\lambda_2-\lambda_3}\frac{\lambda_1^{n_1}\lambda_2^{n_2}\lambda_3^{n_3}}{n_1!n_2!n_3!}\gamma^{-(n_1-1)(n_2-1)(n_3-1)} \\ &\geq \sum_{n} e^{-1-\lambda_1-\lambda_2-\lambda_3}\frac{\lambda_1^{n}\lambda_2^{n}\lambda_3^{n}}{n!^3}\gamma^{-(n-1)^3} \quad = \quad \infty \end{aligned}$$

where the divergence follows from Stirling's formula.

Suppose $\gamma > 1$. Then consider the bound obtained by restricting the above expectation to the event, which is of positive probability, that $N_3 = 0$. We have

$$\begin{aligned}
\mathbb{E}\left\{\gamma^{-(N_1-1)(N_2-1)(N_3-1)-1}\right\} &\geq \mathbb{E}\left\{\gamma^{-(N_1-1)(N_2-1)(N_3-1)-1}|N_3=0\right\} \times \mathbb{P}\left\{N_3=0\right\} \\
&= \sum_{n_1}\sum_{n_2} e^{-1-\lambda_1-\lambda_2}\frac{\lambda_1^{n_1}\lambda_2^{n_2}}{n_1!n_2!}\gamma^{(n_1-1)(n_2-1)} \times \mathbb{P}\left\{N_3=0\right\} \\
&\geq \sum_{n} e^{-1-\lambda_1-\lambda_2}\frac{\lambda_1^{n}\lambda_2^{n}}{n!^2}\gamma^{(n-1)^2} \times \mathbb{P}\left\{N_3=0\right\} \quad = \quad \infty
\end{aligned}$$

where once again the divergence follows from Stirling's formula. □

Naiman and Wynn have generously contributed the following counterexample, which shows that in 4-space one cannot expect convergence for all parameter values even in the well-behaved case of balls of unit radius.

Counterexample 5: Unit balls in 4-space. *Consider the Boolean model Ξ based on an inhomogeneous Poisson process of finite total intensity in 4-space with grains which are unit balls. Suppose that the intensity measure has a density which is constant over the ball centred on the origin and of radius $\sqrt{2}$. Then for $\gamma < 1$ the distribution produced by weighting using*

$$\gamma^{-W_4^4(\Xi)} \tag{14}$$

is not stable in Ruelle's sense.

Proof : First note that $W_4^4(\Xi)$ is proportional to the Euler-Poincaré characteristic of Ξ.

Consider the ensemble of $2n$ balls of unit radius, of which the first n are centred respectively at $(\sqrt{2}\cos(2k\pi/n), \sqrt{2}\sin(2k\pi/n), 0, 0)$ for $k = 1, ..., n$, and the second n are centred respectively at $(0, 0, \sqrt{2}\cos(2k\pi/n), \sqrt{2}\sin(2k\pi/n))$ for $k = 1, ..., n$. (The condition on the density of the intensity measure is imposed in order to ensure that such a configuration is feasible for Ξ.) The first n balls form a sub-ensemble whose union is homotopic to a circle (for large enough n) and therefore has Euler-Poincaré characteristic 0, and similarly for the sub-ensemble of the other n balls.

However intersections between balls from the first and second sub-ensembles are pairwise only, and are singleton sets. It follows from the inclusion-exclusion identity that the Euler-Poincaré characteristic of the union of all $2n$ balls is $-n^2$. Hence Ruelle stability fails. □

It is an open question whether stability fails for the weighting $\gamma^{-W_3^3(\Xi)}$ (case $\gamma < 1$) when Ξ is the Boolean model produced by using unit balls in 3-space. However Naiman and WSK have independently produced a counterexample for the case of balls of random radius in 3-space, based on Diagram 4.7.1 from [39] (see also [26]). See Figure 4 for an indication of the construction.

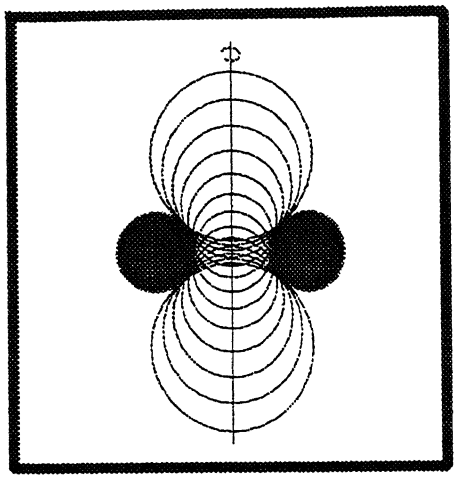

Figure 2: Vertical section view of $2n$ balls of varying radius in 3-space, whose union has Euler-Poincaré characteristic $1-n^2$. First arrange n unit-radius balls in an overlapping horizontal ring (two of these seen in section as dark circles). Then build a fan of n balls with centres located along axis of symmetry, so that the balls in the fan form a connected union and each ball in the fan touches each of the first n balls in one point only.

These counterexamples show that even in the planar case some conditions are needed if the range of γ is to be unconstrained. On the other hand they appear somewhat pathological. Note that the problems are local (the treatment of the line process case makes this clear) and appear in Counterexamples 1, 2, 4 to be related to the "sharpness" of the constituent grains, while Counterexample 3 shows problems arise when grains of "small" sidelength are allowed.

There are two positive results which cover an important range of practical examples, and which serve to clarify the sense in which the above counterexamples are pathological. These cover the complementary cases of (a) random disk grains and (b) random polygon grains which are neither too sharp nor too small. We deal with these results in the two following sections.

As a final remark on the topic of counterexamples, note that it is natural to enquire whether the divergence (in $d = 2$ at least) can ever occur if the grains are non-random. Divergence can occur for simple non-convex non-random grains: consider the case of a grain composed of intersecting horizontal and vertical line segments, and apply the ideas underlying Counterexample 2. In the case of convex non-random grains which are polygons, one can argue that *either* the grains are parallel line segments or parallel lines (in which trivial case stability is immediate, as there will be no holes!) *or* they must satisfy a uniform local wedge condition (given below as Definition 6.1), in which

case stability follows from the arguments in Section 6. The case of non-polygonal convex non-random grains is currently open, with the exception of grains which are disks, which case is covered by the results in the following section.

5 Planar case: when grains are disks

In this section we show that if the Boolean grains are random disks then the Euler quermass-interaction is stable, and hence convergent for all parameter values. Remarkably, no size constraint is required: the disk radii can be random and need only be strictly positive. This is particularly striking in the light of Counterexample 3 above, which suggests that stability problems arise when side length is small. Here we see such problems need not occur at the limit! The argument is strictly geometrical, and is to be found in the theorem below: an ensemble of N disks has a union with at most $2N - 5$ holes.

If the disks are of constant size then there is an easy argument using the Dirichlet tessellation based on the disk centres. Let $B(x_1, r_1)$, $B(x_2, r_2)$, ..., $B(x_N, r_N)$ be the (closed) disks. In each component of the complement of $\bigcup_i B(x_i, r_i)$ there must be at least one *node* of the tessellation (a node is a vertex of the planar linear graph formed by the tessellation, including the "vertex at infinity"), for otherwise the boundary of this component would have to be made out of the boundaries of at most two disks (which would force the "vertex at infinity" to belong to the complement). Hence the number of holes in the union $\bigcup_i B(x_i, r_i)$ is dominated by the number of nodes of the Dirichlet tessellation, which by planar graph theory (using the Euler formula; see for example [55, Theorem 13A]) is itself dominated by the bound $2N - 5$, since N is the number of faces of the tessellation (note the bound is not $2N - 6$, as we exclude the hole at infinity).

Unfortunately this simple argument appears not to generalize, being tied to the Euclidean metric structure underlying the definitions of a disk and of a Dirichlet tessellation. For disks of arbitrary radius we have to argue carefully about how to reduce the union of disks to a planar network without decreasing the number of holes. The reduction uses line segments connecting certain of the disk centres (together with some polygons): the main technical issue is to choose a set of such line segments which leave connectivity unchanged and which do not cross each other. Naiman and Wynn have recently discovered a delightful argument deriving Theorem 5.3 from their work on abstract tube theory [33, 35], based on an algebraic topology argument related to the Morse inequalities. However the argument given below is more self-contained, and in particular avoids algebraic topology.

We commence by introducing some notation and proving a couple of preliminary lemmas.

Consider an ensemble $B(x_1, r_1)$, $B(x_2, r_2)$, ..., $B(x_N, r_N)$ of N overlapping closed disks of varying sizes all lying in the plane. Set $\mathcal{D}$ to be the union of the disks, and $\mathcal{D}_0$

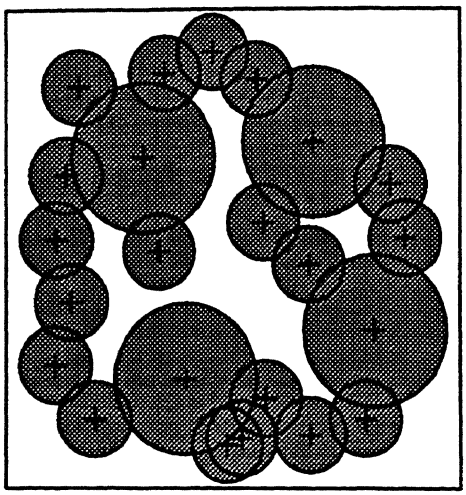

Figure 3: A typical field of overlapping disks $B(x_1, r_1)$, $B(x_2, r_2)$, ..., $B(x_N, r_N)$ of varying sizes.

to be the union of the interiors of the disks, so that

$$\begin{aligned} \mathcal{D} &= \bigcup_{i=1}^{N} B(x_i, r_i) \\ \mathcal{D}_0 &= \bigcup_{i=1}^{N} int\,(B(x_i, r_i)). \end{aligned}$$

We suppose that they are placed in general position, so that no more than two disk boundaries intersect at any given point, and so that if two disk boundaries do intersect then they intersect at two distinct points. Figure 3 illustrates a possible arrangement: *close* inspection will reveal that the disks here are in fact in general position!

Each pair of overlapping disks $B(x_i, r_i)$, $B(x_j, r_j)$ has boundaries intersecting in two points x_{ij}^-, x_{ij}^+, where the sign is chosen by an arbitrary convention so that if $i < j$ then x_{ij}^+ is on the clockwise side of $B(x_i, r_i) \cap B(x_j, r_j)$ when viewed from the centre of $B(x_i, r_i)$.

For each point of intersection $x_{ij}^\pm$ of the boundaries of two disks $B(x_i, r_i)$, $B(x_j, r_j)$, if $x_{ij}^\pm$ is not covered by $\mathcal{D}_0$ then define $T_{ij}^\pm$ to be the closed triangular region with vertices at $x_{ij}^\pm$ and the centres of $B(x_i, r_i)$, $B(x_j, r_j)$. Define S_{ij} to be the line segment running between the centres of $B(x_i, r_i)$, $B(x_j, r_j)$.

We say that $T_{ij}^\pm$ is *not defined* if the corresponding $x_{ij}^\pm$ is covered by $\mathcal{D}_0$. We say that S_{ij} is *not defined* if *both* the corresponding x_{ij}^+ and x_{ij}^- are covered by $\mathcal{D}_0$.

Figure 4 illustrates the definition of x_{ij}^+, T_{ij}^+ and S_{ij}. Note that if T_{ij}^+ (respectively T_{ij}^-) is defined then x_{ij}^+ (respectively x_{ij}^-) is a "corner" of the union $\mathcal{D}$. We now make

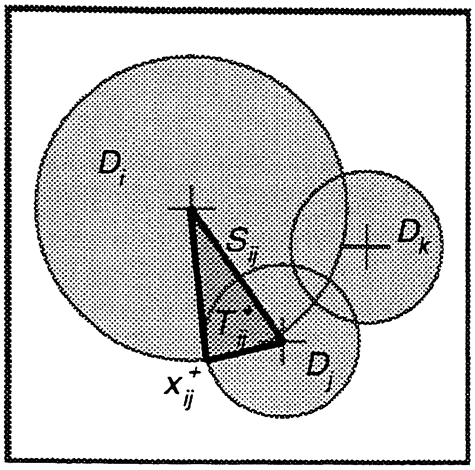

Figure 4: The closed triangular region T_{ij}^+ with vertices at x_{ij}^+ and the centres of $B(x_i, r_i)$, $B(x_j, r_j)$. In this example T_{ij}^- is not defined, since x_{ij}^- is covered by the interior of a third disk $B(x_k, r_k)$.

some observations about these triangular regions. Firstly we note that they serve as "dead areas" for disks, in the sense that if a $B(x_k, r_k)$ overlaps a $T_{ij}^{\pm}$ (for i, j, k distinct) then it cannot contribute any exposed $x_{k\ell}$. This follows readily from geometric intuition, but here we give a rigorous proof based on homogeneous coordinates.

Lemma 5.1 *If T_{ij}^+ (respectively T_{ij}^-) is defined then any further disk $B(x_k, r_k)$ with centre in T_{ij}^+ (respectively T_{ij}^-) must be wholly contained in int $(B(x_i, r_i)) \cup$ int $(B(x_j, r_j))$.*

Proof : Without loss of generality consider T_{ij}^+. Because T_{ij}^+ is defined, x_{ij}^+ must lie outside $B(x_k, r_k)$ (recall that the disks are placed in general position, so we can replace *int* $(B(x_k, r_k))$ by $B(x_k, r_k)$ here). (Figure 5 illustrates the situation.)

Choose coordinates such that $x_{ij}^+ = 0$ and the centres of $B(x_i, r_i)$, $B(x_j, r_j)$ are at a, b respectively. If the centre of $B(x_k, r_k)$ lies in T_{ij}^+ then it is at $\lambda a + \mu b$, for $\lambda + \mu \leq 1$, $\lambda \geq 0$, $\mu \geq 0$. (In fact λ, μ, $1 - \lambda - \mu$ provide a system of homogeneous coordinates for the centre of $B(x_k, r_k)$.)

Consider a point y lying outside the interiors of both $B(x_i, r_i)$ and $B(x_j, r_j)$. This means

$$\begin{aligned} \|y - a\| &\geq \|a\|, \\ \|y - b\| &\geq \|b\|, \end{aligned}$$

and on squaring and simplifying we find

$$\|y\|^2 - 2\langle y, a \rangle \geq 0,$$

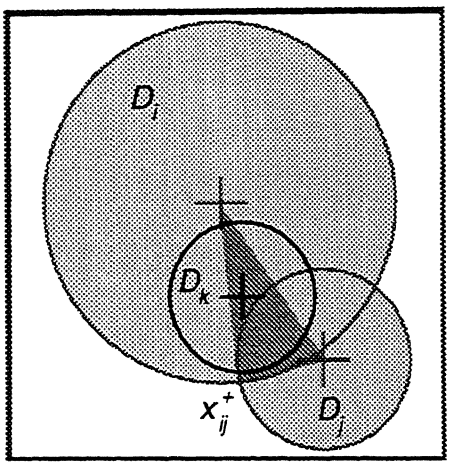

Figure 5: An argument using homogeneous coordinates, based on $0 = x_{ij}^+$ and the centres of $B(x_i, r_i)$ and $B(x_j, r_j)$, shows that if T_{ij}^+ is defined, and if $B(x_k, r_k)$ is centred in T_{ij}^+, then $B(x_k, r_k)$ is contained in the union of the interiors of $B(x_i, r_i)$ and $B(x_j, r_j)$.

$$\|y\|^2 - 2\langle y, b\rangle \quad \geq \quad 0\,.$$

Hence we deduce

$$(\lambda + \mu)\|y\|^2 - 2\langle y, (\lambda a + \mu b)\rangle \quad \geq \quad 0 \tag{15}$$

(note that λ and μ are both nonnegative!) and therefore, because $\lambda + \mu \leq 1$,

$$\|y - (\lambda a + \mu b)\| \quad \geq \quad \|\lambda a + \mu b\|\,. \tag{16}$$

But $B(x_k, r_k)$ must not contain x_{ij}^+, and this means that its radius must be strictly less than $\|\lambda a + \mu b\|$. So y must lie outside $B(x_k, r_k)$, and so $B(x_k, r_k)$ must be contained in $int\,(B(x_i, r_i)) \cup int\,(B(x_j, r_j))$. □

Therefore *no* $T_{k\ell}^+$ or $S_{k\ell}$ can be defined for such a $B(x_k, r_k)$; T_{ij}^+ is a "dead area" for disks. A similar argument holds for T_{ij}^-.

Secondly we note that no two of these "dead-area" triangles can have overlapping interiors.

Lemma 5.2 *No two triangles $T_{ij}^{\pm}$, $T_{rs}^{\pm}$ can have overlapping interiors. (Here the $\pm$ superscript refers systematically to one of $+$ or $-$ in each of the two cases of $T_{ij}^{\pm}$, $T_{rs}^{\pm}$).*

Proof : Let c_i, c_j, c_r, c_s be the centres of disks $B(x_i, r_i)$, $B(x_j, r_j)$, $B(x_r, r_r)$, $B(x_s, r_s)$ respectively. Let $x_{ij}^{\pm}$, $x_{rs}^{\pm}$ be exposed intersections of the respective disk boundaries.

Suppose that a point u is in the interiors of both the triangle $c_i c_j x_{ij}^{\pm}$ and the triangle $c_r c_s x_{rs}^{\pm}$. We derive a contradiction from this and the requirement of the disks being in general position, as follows.

First observe that by the previous lemma we can deduce that the open disk $\tilde{D}$ of centre u and radius $|u - x_{ij}^{\pm}|$ is contained in $int\ (B(x_i, r_i)) \cup int\ (B(x_j, r_j))$. Thus we can add a further closed disk $B(x_{N+1}, r_{N+1})$ to the original assembly of disks $B(x_1, r_1)$, $B(x_2, r_2)$, ..., $B(x_N, r_N)$ without altering the union of all the disks, where $B(x_{N+1}, r_{N+1})$ is a closed disk of centre u and radius less than but arbitrarily close to $|u - x_{ij}^{\pm}|$. Consequently $B(x_{N+1}, r_{N+1})$ cannot cover $x_{rs}^{\pm}$, since otherwise $x_{rs}^{\pm}$ would be covered by $\mathcal{D}_0 = \bigcup_{i=1}^{N} int\ (B(x_i, r_i))$, contradicting our assertion that $T_{rs}^{\pm}$ is defined.

Working with the new assembly $B(x_1, r_1)$, $B(x_2, r_2)$, ..., $B(x_N, r_N)$, $B(x_{N+1}, r_{N+1})$, we can also apply the previous lemma to $T_{rs}^{\pm}$ and $B(x_{N+1}, r_{N+1})$, to deduce that $B(x_{N+1}, r_{N+1}) \subseteq int\ (B(x_r, r_r)) \cup int\ (B(x_s, r_s))$. Since the radius of $B(x_{N+1}, r_{N+1})$ is arbitrarily close to $|u - x_{ij}^{\pm}|$, we deduce that $\tilde{D} \subset int\ (B(x_r, r_r)) \cup int\ (B(x_s, r_s))$.

But now we have shown that the open disk $\tilde{D}$ of center u and radius $|u - x_{ij}^{\pm}|$ is contained in $int\ (B(x_r, r_r)) \cup int\ (B(x_s, r_s))$, while $x_{ij}^{\pm}$ is not (being exposed). So $x_{ij}^{\pm}$ lies on the boundaries of $B(x_r, r_r)$, $B(x_s, r_s)$, as well as on the boundaries of $B(x_i, r_i)$, $B(x_j, r_j)$. At least three of these disks are distinct, so this violates the requirement for the disks to be in general position. We deduce that the interiors of the triangles $c_i c_j x_{ij}^{\pm}$ and $c_r c_s x_{rs}^{\pm}$ are disjoint, as required. □

We now turn to the main result of this section.

Theorem 5.3 *For $\mathcal{D}$ a union of N closed disks in the plane, the number of holes in $\mathcal{D}$ is bounded above by $2N - 5$.*

Proof: We may suppose the disks are in general position as described at the beginning of this section. We use the notation established above.

For every (exposed) "corner" $x_{ij}^{\pm}$ of $\mathcal{D}$ we have defined a "dead-area" triangle $T_{ij}^{\pm}$ with vertices at $x_{ij}^{\pm}$ and the centres of the two disks $B(x_i, r_i)$, $B(x_j, r_j)$ whose overlapping forms the "corner". Moreover we have shown that the interiors of distinct defined "dead-area" triangles do not overlap. The resulting configuration of defined triangles $T_{ij}^{\pm}$ is shown in Figure 6(a).

The "corners" of $\mathcal{D}$ divide the boundary $\partial\mathcal{D}$ into "edges" (circular arcs). To each "edge" we can associate two bounding "corners", p_1 and p_2, except when the "edge" is a complete circle, corresponding to a disk separated from all the others (note that the configuration of general position removes ambiguous cases). We need not consider the exceptional case, as this makes no contribution to the number of holes of $\mathcal{D}$. For the non-exceptional edges the corresponding triangles share a vertex which is a disk centre c. The non-overlapping property given in Lemma 5.2 means we can retract each "edge" back to the joined segments $p_1 \to c \to p_2$, without altering the number of holes of $\mathcal{D}$.

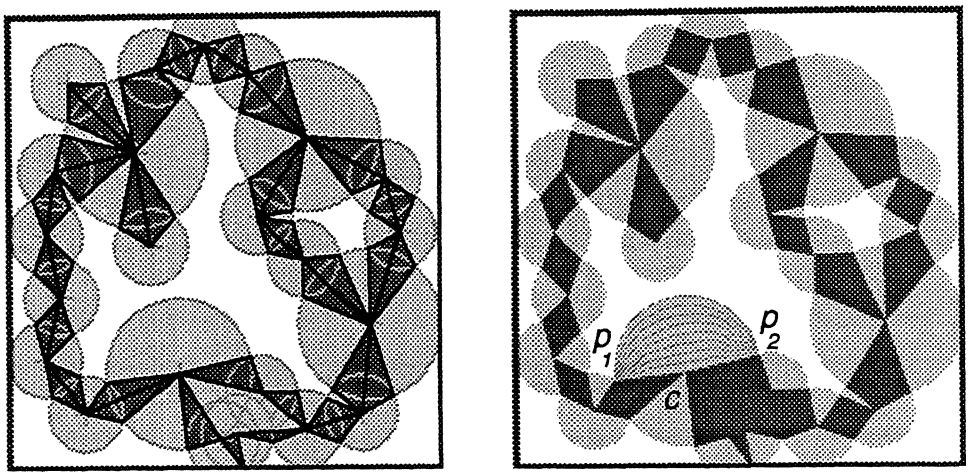

Figure 6: (a): $\mathcal{D}$ together with the configuration of defined triangles $T_{ij}^{\pm}$.
(b): Construction of $\tilde{\mathcal{D}}$ from $\mathcal{D}$ by retracting the "edges" back to joined pairs of triangle segments.

We can do this by the mapping $F : [0,1] \times H \to H$, defined for a circular sector $H = p_1 c p_2$ by

$$F(t,(r,\theta)) \quad = \quad ((1 - t^{\frac{\theta_0 - \theta}{\theta_0}})r, \theta)$$

where we coordinatize the sector H by polar coordinates such that

$$H \quad = \quad \{(r,\theta) : r \in [0, r_0], |\theta| \leq \theta_0\} \ .$$

Call the resulting region $\tilde{\mathcal{D}}$. Figure 6(b) illustrates the construction.

Now notice that each triangle $T_{ij}^{\pm}$ can be retracted back to the line segment S_{ij} running between the centres of the two defining disks without altering the number of holes in $\tilde{\mathcal{D}}$. (This follows from general position and Lemma 5.2. Call the resulting region $\mathcal{E}$. Figure 7 illustrates the construction.

Finally consider the holes in $\mathcal{E}$. If we replace $\mathcal{E}$ by the network of line segments S_{ij} then we can only increase the number of holes (points disconnected by $\mathcal{E}$ will remain disconnected by the network).

But we can now use planar graph theory as in the constant-radius case (Euler's formula as in formula; see for example [55, Theorem 13A]) to bound above the number of holes in the network by $2N - 5$ as required. □

We owe the application of planar graph theory here to Mike Alder: a previous version of the argument used a simple angle-counting argument. Note that the major part of the effort in the proof of an apparently simple result goes towards establishing

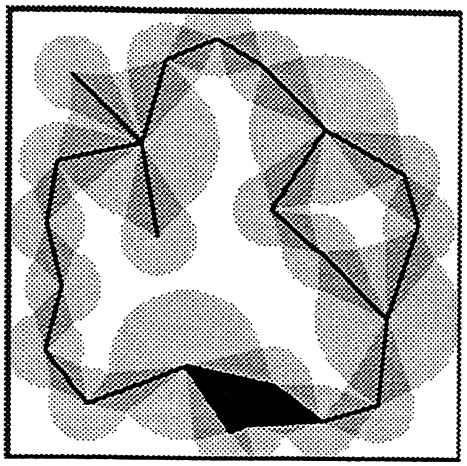

Figure 7: Construction of $\mathcal{E}$ from $\tilde{\mathcal{D}}$ by retracting the triangles $T_{ij}^{\pm}$ back to line segments S_{ij}.

that we can shrink the union of disks to a planar graph of which nodes are disk centres, without decreasing the number of holes.

Arguments about the Poisson distribution lead to the following corollary, which applies the result to the context of our paper.

Corollary 5.4 *Let X be a point process with quermass-interaction based on a Boolean model Ξ based on X and using random disks, with arbitrary positive radius distribution, and with finite intensity for the underlying Poisson germ process. The Euler-Poincaré quermass-interaction weighting $\gamma^{-W_2^2(\Xi)/\pi}$ is stable for all values of γ.*

The main result of this section, Theorem 5.3, is of independent geometric interest. Simple periodic examples show asymptotic sharpness of the bound of at most $2N - 5$ holes for the union of N disks.

Extreme Euler-Poincaré quermass-interactions which weight "against holiness" are also of interest: if the intensity is high enough to force overlaps then it is an interesting question as to what are the most probable configurations, and indeed whether phase-transitions appear.

We plan to investigate both ranges of extremes using simulation.

6 Planar case: when grains are polygons

In this section we establish stability for the Euler-Poincaré quermass-interaction when the typical grain is a randomly rotated polygon (or more generally a random polygon

which is neither too small nor too sharp). More precisely, we consider the case when the grains satisfy a uniform version of Definition 4.1:

Definition 6.1 *A convex grain G is said to satisfy a "uniform wedge condition of angle $\phi > 0$ and radius $r > 0$" if for any point $\omega \in \partial G$ the disk $B(\omega, r)$ (of radius r and centred at ω) when intersected with G produces a circular sector $B(\omega, r) \cap G$ of angle at least ϕ.*

This holds for example (for some r, ϕ) if G is a convex polygon of positive area. It corresponds to the wedge condition of Definition 4.1 together with a lower bound on side-length.

Theorem 6.2 *Let Ξ be a planar Boolean model based on an inhomogeneous Poisson point process Φ of finite total intensity, with typical grain G almost surely satisfying a uniform wedge condition of angle ϕ and radius r for some fixed r, ϕ. Then $W_2^2(\Xi)$ is bounded above and below by a constant times the total number $\Phi(\mathbb{R}^2)$ of germs.*

Proof : The proof begins with a series of reductions directed at resolving the question down to an unusual but deterministic geometric packing problem.

A: It suffices to bound the number of holes.

Arguing as before, the Euler-Poincaré characteristic $\chi(\Xi) = W_2^2(\Xi)/\pi$ is equal to the number of components of Ξ minus the number of holes of Ξ, and the number of components is bounded above by the number of germs $\Phi(\mathbb{R}^2)$. It therefore suffices to obtain a suitable upper bound for the number of holes.

B: It suffices to consider the case of grains which are random wedges.

Localizing to a disk of radius r, it suffices to consider the case when G is an infinite convex planar wedge of angle exceeding $\phi > 0$. To see this, note that the observation window can be covered by discs of radius r, and that there is a many-to-one correspondence between holes produced by the various intersections of Ξ with covering disks and holes produced by the original Ξ. Let $N = \Phi(\mathbb{R}^2)$ be the total number of wedges, equivalently the total number of germs.

C: Discretization of wedge angle and orientation.

It suffices to consider the case of grains which are randomly oriented wedges of fixed positive angle $\theta/2$, with clockwise-edge orientations distributed over a finite set of orientations 0, θ, 2θ, ..., $k\theta$, where $5\theta/2 < \phi$ depends only on the original ϕ and k is given by $(k + 1/2)\theta < 2\pi \leq (k + 1)\theta$. (Here "clockwise" edge refers to the view from the wedge vertex. This is illustrated in Figure 8(a).)

To analyze the discretization, note that each original wedge can be replaced by a shrunken wedge, sharing the same vertex and contained in the original wedge, but of angle θ and of clockwise-edge orientation belonging to the finite set described in the above sentence. It is possible for this replacement to decrease the number of holes,

but only by at most N. In fact suppose the original wedges are W_1, ..., W_N, and the shrunken wedges are U_1, ..., U_N. Let $U_i(t)$ be a continuously shrinking wedge, changing monotonically from $U_i(0) = W_i$ to $U_i(1) = U_i$ by reducing wedge angle while keeping the vertex fixed. Consider the procedure which shrinks the wedges one after the other in order, and consider the stage at which W_i is shrunk to U_i.

As $t \in [0,1]$ increases so the number of connected components of the complement of $\left(\bigcup_{j<i} U_j\right) \cup U_i(t) \cup \left(\bigcup_{j>i} W_j\right)$ (the number of holes of the union of wedges) decreases only when there is an exposure of the vertex of one of the wedges U_j or W_j. But this can happen only once for each index j in the entire sequence of shrinkages $W_i \to U_i$, $i = 1, ..., N$.

Consequently the total reduction of the number of holes cannot exceed N, which therefore does not alter the required conclusion.

D: It suffices to bound the number of exposed intersections of edges of wedges.

Except in the trivial case of $\Xi = \emptyset$, every hole of Ξ has a boundary possessing at least one *exposed* intersection of edges of wedges (meaning an edge intersection not itself covered by Ξ). It therefore suffices to obtain an upper bound on the number of exposed edge intersections which is linear in N the number of germs.

E: We need only consider the case when there are two distinct orientations of wedges.

Let us call the collection of wedges of a given orientation a *wedge packet.*

The number of wedge packets being finite and depending only on the wedge-angle bound ϕ, it suffices to bound intersections between just two wedge packets. If these are the same packet then all wedges are parallel. But then there can be only at most two exposed edge intersections per wedge and the required bound follows.

F: For the purposes of exposition we consider only the number of exposed intersections of clockwise edges of wedges.

It will be observed that the argument below applies equally to the other forms of intersection (counter-clockwise to clockwise, clockwise to counter-clockwise, counter-clockwise to counter-clockwise).

Orient the configuration so that clockwise edges of wedges from one wedge packet are all vertical. We call the wedges from this packet *vertical.* We call the wedges from the other packet *slanted.* Let V be the number of vertical wedges and S be the number of slanted wedges. Say that one vertical wedge is *downwind* of another if it is further from the vertex of a slanted clockwise edge intersecting both (and of course the other wedge is said to be *upwind* of the first!). This is illustrated in Figure 8(b).

Now we proceed to assign each exposed intersection to a unique wedge, though *not* necessarily one of the two wedges directly involved in the intersection in question. To do this we must distinguish two kinds of exposed intersection:

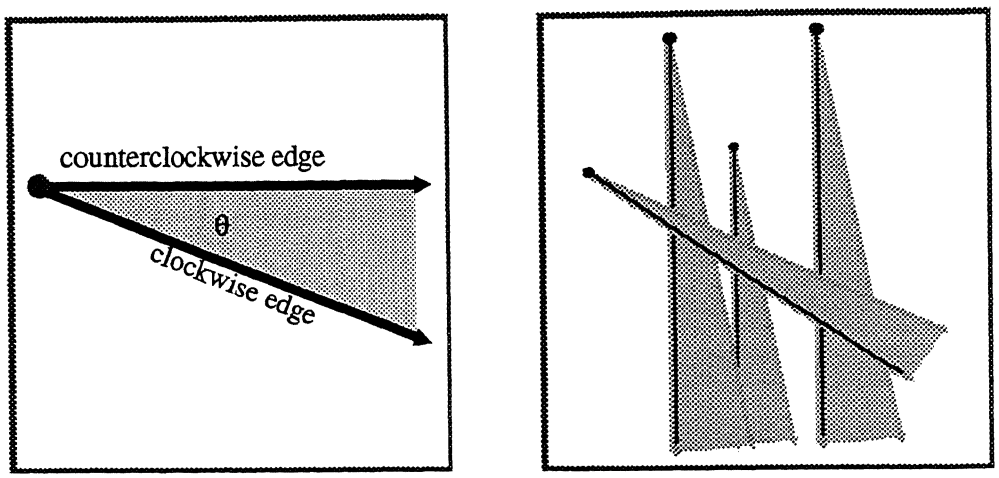

Figure 8: (a) Illustration of clockwise and counterclockwise edges of a wedge of vertex angle θ.
(b) The two *vertical* wedges to the right are *downwind* of the *vertical* wedge to the left: there is one *slanted* wedge.

(a) exposed intersections such that the slanted wedge has no (exposed or unexposed) intersections upwind on its clockwise edge;

(b) exposed intersections such that the slanted wedge does have (exposed or unexposed) intersections upwind on its clockwise edge.

We shall assign an exposed intersection of type (a) to its slanted wedge. There can be only one such wedge per slanted intersection, therefore the total number of exposed intersections of type (a) is bounded by S the number of slanted wedges.

The total number of type (b) intersections is bounded linearly in V the number of vertical wedges, as follows.

To each type (b) intersection we assign a *predecessor* vertical wedge which provides the first upwind intersection (exposed *or* unexposed!) with the slanted wedge. Now each vertical wedge can be predecessor to at most $M(\alpha,\theta)$ type (b) intersections, where

$$M(\alpha,\theta) \quad = \quad \left[1 + \frac{\cot(\theta) - \cot(\alpha)}{\cot(\alpha) - \cot(\alpha+\theta)}\right] \tag{17}$$

and α is the angle of intersection between the slanted and vertical clockwise edges (see Figure 9). This follows because exposed type (b) intersections owning the same predecessor wedge P must involve slanted wedges which do not overlap on L, where L is the vertical line determined by the most upwind of the vertical wedges providing

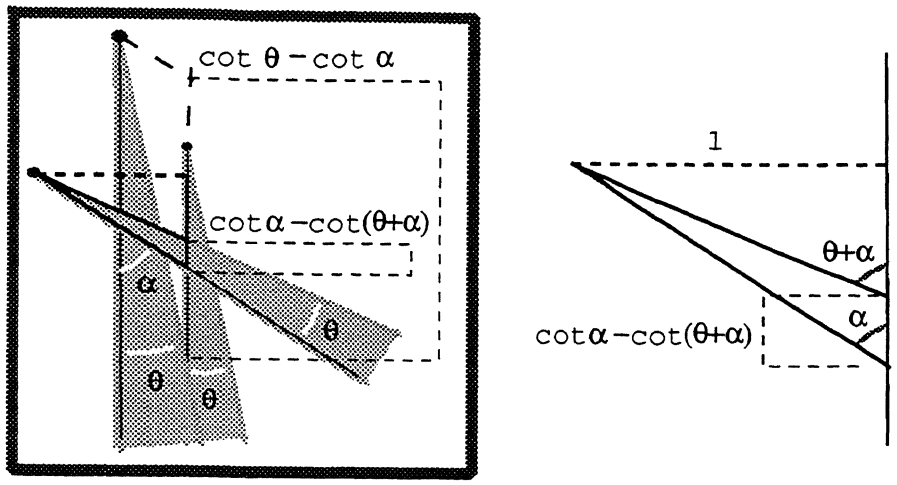

Figure 9: Space taken up by an exposed clockwise-clockwise edge intersection.

type (b) intersections which own P. Figure 10(a, b) illustrates these considerations, especially the predecessor relationship.

Note that we must have $\theta \leq \alpha \leq \pi - \theta$, since we are dealing with distinct wedge packets and orientations are multiples of θ which itself is of the form π/m.

Calculus shows that the number $M(\alpha, \theta)$ in Equation (17) is bounded above for this range of α by $[1 + \cot^2(\theta)]$.

This achieves a bound which is linear in the number of wedges, as required.

Thus the number of exposed clockwise-clockwise edge intersections between two distinct wedge packets is bounded above by

$$S + \left[1 + \cot^2 \theta\right] \times V \tag{18}$$

(recall S is the number of wedges in the slanted wedge packet and V is the number of wedges in the vertical wedge packet).

Together with the reduction steps listed above, this establishes the result. □

7 Conclusion

7.1 Simulation

There is much further work to be done on these models. For example how can they best be simulated? After the recent work of Propp and Wilson [42] the ambitions of stochastic geometers have been raised. It is no longer satisfactory to produce approximate simulations *via* long-run equilibria of spatial birth-and-death processes or

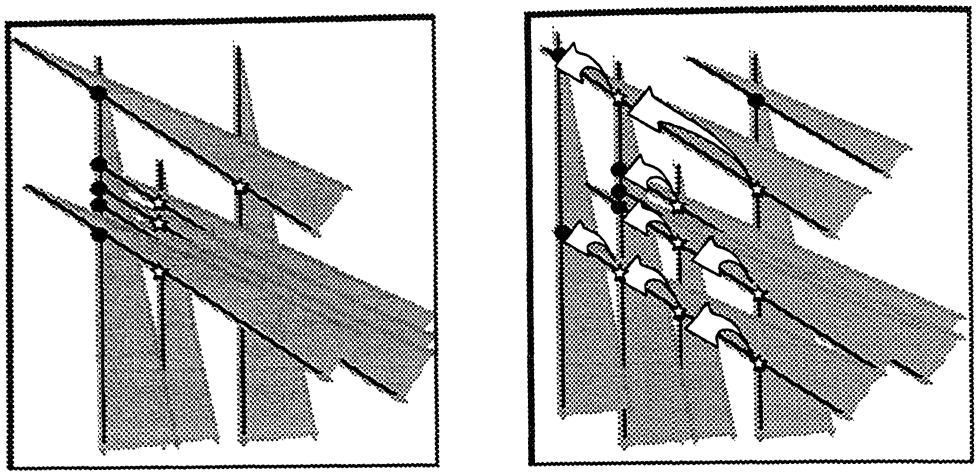

Figure 10: (*a*) The exposed clockwise-clockwise edge intersections owning the most upwind vertical wedge as predecessor are marked by stars; those on the most upwind wedge itself are marked by disks.
(*b*) The most upwind exposed intersections of slanted wedges are marked by disks, others by stars. The predecessor relationship is indicated by arrows.

Markov chains. Instead one should aim to construct simulation algorithms which sample from equilibrium exactly using reverse-time coupled Markov chains as in [42]. This has already been done for the area-interaction point process in [16, 24]; indeed the algorithms presented there generalize easily to cover a variety of other point process models. However the Euler weighting is less amenable, since the local energy is not bounded. One of us (WSK) is working on this and will report progress at a later date.

7.2 Inference

For point processes the methods described in [4] can be adapted quite easily. In particular, in the planar case the proposed quermass-interaction provides an exponential family of $1+3$ parameters (intensity β and coefficients of quermass integrals) and the sufficient statistic is the pair composed of the total number of objects and the vector of values of the quermass integral. We plan to investigate inference and maximum likelihood via Markov chain Monte Carlo techniques, as in [13], and by approximation methods as in [32, 36, 37, 38, 40].

The pseudo-likelihood is

$$PL(\beta,\gamma;\mathbf{x}) \quad = \quad \exp\left\{-\int_W \lambda(u;\mathbf{x})du\right\} \prod_{i=1}^{n} \lambda(x_i;\mathbf{x}). \tag{19}$$

where $\lambda(u;\mathbf{x})$ denotes the Papangelou conditional intensity [6, 10, 11, 19, 20, 47, 52, 53]. This takes on a convenient form for Markovian densities,

$$\lambda(u;\mathbf{x}) \quad = \quad \frac{p(\mathbf{x}\cup\{u\})}{p(\mathbf{x}\setminus\{u\})} = \beta\gamma^{-[W_r^d(U(\mathbf{x})\cup Z(u))-W_r^d(U(\mathbf{x}\setminus\{u\})]}.$$

Dropping the dependence on x, we write $t(u) = W_r^d(U(\mathbf{x})\cup Z(u)) - W_r^d(U(\mathbf{x}\setminus\{u\})$. The task is to optimise

$$\log PL(\mathbf{x}) \quad = \quad -\beta\int_A \gamma^{-t(u)}du + n\log\beta - \sum_{i=1}^{n} t(x_i)\log\gamma.$$

The partial derivatives are

$$\frac{\partial}{\partial\beta} \quad = \quad -\int_A \gamma^{-t(u)}du + \frac{n}{\beta} \quad = \quad 0$$

and

$$\frac{\partial}{\partial\gamma} \quad = \quad \frac{\beta}{\gamma}\int_A t(u)\gamma^{-t(u)}du - \frac{1}{\gamma}\sum_{i=1}^{n} t(x_i) \quad = \quad 0$$

and hence the pseudo-likelihood equations are similar to those of the area-interaction or indeed the Strauss process. Again, they are a special case of the Takacs-Fiksel method (for derivative type functionals). To estimate the quermass integrals it is possible to apply standard stereological techniques to reduce dimensions.

It should be noted that for the random set case the unobservability issue is likely to make estimation *very* difficult.

7.3 Preston extensions

One may ask whether these processes can be extended to the whole of Euclidean space. Following the arguments in Preston's book, as in [4], it can be shown that we can always extend the notion of a quermass-interaction to the whole of Euclidean space so long as (a) the interaction is stable, and (b) the diameters of the grains are bounded above. Thus the work described above does indeed set the scene for quermass-interaction point processes.

7.4 Relationship to abstract tube theory

We have already noted (in §5) an intriguing overlap with the work of Naiman and Wynn on abstract tubes and inclusion-exclusion identities [33, 35], which can be used to provide an alternative proof of Theorem 5.3. We hope to pursue this relationship in joint work with Naiman and Wynn. The intriguing question is, to what extent the relationship can be developed in order to exploit the results of §6 in a more general context, since these results currently appear to go beyond what may be obtained from abstract tube theory. (But see the work on Vapnis-Chervonenkis dimension in [34].)

References

[1] R.J. Adler. *The geometry of random fields*, John Wiley and Sons, New York and Chichester, 1981.

[2] A.J. Baddeley and R.D. Gill. Kaplan-Meier estimators for interpoint distance distributions of spatial point processes. Research Report 718, Mathematical Institute, University of Utrecht, March 1992.

[3] A.J. Baddeley and M.N.M. van Lieshout. ICM for object recognition. In *Computational statistics*, Volume 2, pages 271-286, edited by Y. Dodge and J. Whittaker, Physica/Springer, Heidelberg-New York, 1992.

[4] A.J. Baddeley and M.N.M. van Lieshout. Area-interaction point processes. *Annals of the Institute of Statistical Mathematics* 47:601-619, 1995.

[5] A.J. Baddeley and J. Møller. Nearest-neighbour Markov point processes and random sets. *International Statistical Review* 57:89-121, 1989.

[6] J. Besag. Spatial interaction and the statistical analysis of lattice systems (with discussion). *Journal of the Royal Statistical Society, Series B* 36:192-236, 1974.

[7] J. Besag. On the statistical analysis of dirty pictures (with discussion). *Journal of the Royal Statistical Society, Series B*, 48:259-302, 1986.

[8] P. Clifford. Markov random fields in statistics. In *Disorder in physical systems*, pages 19-32, edited by G.R. Grimmett and D.J.A. Welsh, Clarendon Press, Oxford, 1990.

[9] P. Clifford and G. Nicholls. Comparison of birth-and-death and Metropolis-Hastings Markov chain Monte Carlo for the Strauss process. Manuscript, Department of Statistics, Oxford University.

[10] T. Fiksel. Estimation of parametrized pair potentials of marked and non-marked Gibbsian point processes. *Elektronische Informationsverarbeitung und Kybernetika*, 20:270-278, 1984.

[11] T. Fiksel. Estimation of interaction potentials of Gibbsian point processes. *Statistics*, 19:77-86, 1988.

[12] D.J. Gates and M. Westcott. Clustering estimates for spatial point distributions with unstable potentials. *Annals of the Institute of Statistical Mathematics* 38:123-135, 1986.

[13] C.J. Geyer and J. Møller. Simulation procedures and likelihood inference for spatial point processes. *Scandinavian Journal of Statistics* 21:359-373, 1994.

[14] G.R. Grimmett. A theorem about random fields. *Bulletin of the London Mathematical Society* 5:81-84, 1973.

[15] H. Hadwiger. *Vorlesungen über Inhalt, Oberfläche und Isoperimetrie*, Springer Verlag, Berlin, 1957.

[16] O. Häggström, M.N.M. van Lieshout and J. Møller. Characterisation results and Markov chain Monte Carlo algorithms including exact simulation for some spatial point processes. In preparation.

[17] W.D. Hamilton. Geometry for the selfish herd. *Journal of Theoretical Biology* 31:295-311, 1971.

[18] J.M. Hammersley, J.W.E. Lewis and J.S. Rowlinson. Relationships between the multinomial and Poisson models of stochastic processes, and between the canonical and grand canonical ensembles in statistical mechanics, with illustrations and Monte Carlo methods for the penetrable sphere model of liquid-vapour equilibrium. *Sankhya: The Indian Journal of Statistics, series A* 37:457-491, 1975.

[19] J.L. Jensen. Asymptotic normality of estimates in spatial point processes. *Scandinavian Journal of Statistics*, 20:97-109, 1993.

[20] J.L. Jensen and J. Møller. Pseudolikelihood for exponential family models of spatial point processes. *Annals of Applied Probability*, 1:445-461, 1991.

[21] O. Kallenberg. *Random Measures* (revised and enlarged). Akademie-Verlag Berlin and Academic Press London, 1983.

[22] F.P. Kelly and B.D. Ripley. A note on Strauss's model for clustering. *Biometrika* 63:357-360, 1976.

[23] W.S. Kendall. A spatial Markov property for nearest-neighbour Markov point processes. *Journal of Applied Probability* 28:767-778, 1990.

[24] W.S. Kendall. Perfect simulation for the Area-Interaction Point Process. In *Probability towards the Year 2000* edited by L. Accardi and C.C. Heyde. To appear, 1996.

[25] W.S. Kendall (1996): On some weighted Boolean models. In: *Advances in Theory and Applications of Random Sets* edited by D. Jeulin and L. Decker. Ecole des Mines, Fontainebleau

[26] V. Klee. On the complexity of d-dimensional Voronoi diagrams. *Archiv de Mathematik* 34:75-80, 1980.

[27] M.N.M. van Lieshout. Stochastic annealing for nearest-neighbour point processes with application to object recognition. *Advances in Applied Probability* 26:281-300, 1994.

[28] P. McMullen and R. Schneider. Valuations on convex bodies. In *Convexity and its Applications*, pages 170-247, edited by P. Gruber and J.M. Wills, Birkhäuser, Basel, 1983.

[29] G. Matheron. *Random sets and integral geometry*, John Wiley and Sons, New York and Chichester, 1975.

[30] J. Møller. Markov chain Monte Carlo and spatial point processes. *Probability Theory and Related Fields* 33:269-272, 1995.

[31] J. Møller. Markov chain Monte Carlo and spatial point processes. In: *Proceedings Seminaire Européen de Statistique, "Current trends in stochastic geometry with applications"*. O. Barndorff-Nielsen, W.S. Kendall and M.N.M. van Lieshout (Eds), Chapman and Hall, 1996. To appear.

[32] R.A. Moyeed and A.J. Baddeley. Stochastic approximation for the MLE of a spatial point process. *Scandinavian Journal of Statistics* 18:39-50, 1995.

[33] D.Q. Naimann and H.P. Wynn. Inclusion-exclusion-Bonferroni identities and inequalities for discrete tube-like problems via Euler characteristics. *Annals of Statistics* 20:43-76.

[34] D. Naimann and H.P. Wynn. Independence number, Vapnis-Chervonenkis dimension, and the complexity of families of sets. Research preprint 1993, to appear in *Discrete Mathematics.*

[35] D. Naimann and H.P. Wynn. Abstract tubes, improved inclusion-exclusion identities and inequalities, and importance sampling. Research Report, University of Warwick, 1996.

[36] Y. Ogata and M. Tanemura. Estimation for interaction potentials of spatial point patterns through the maximum likelihood procedure. *Annals of the Institute of Statistical Mathematics* 33:315-338, 1981.

[37] Y. Ogata and M. Tanemura. Likelihood analysis of spatial point patterns. *Journal of the Royal Statistical Society, Series B* 46: 496-518, 1984.

[38] Y. Ogata and M. Tanemura. Likelihood estimation of soft-core interaction potentials for Gibbsian point patterns. *Annals of the Institute of Statistical Mathematics* 41:583-600, 1989.

[39] A. Okabe, B. Boots and K. Sugihara. *Spatial Tessellations: Concepts and applications of Voronoi diagrams.* John Wiley and Sons, New York and Chichester, 1992.

[40] A. Penttinen. *Modelling Interaction in Spatial Point Patterns: Parameter Estimation by the Maximum Likelihood Method.* Number 7 in Jyväskylä Studies in Computer Science, Economics and Statistics, University of Jyväskylä. 1984.

[41] C.J. Preston. Generalised Gibbs states and Markov random fields. *Advances in Applied Probability* 5:242-261, 1973.

[42] J.G. Propp and D.B. Wilson. Exact sampling with coupled Markov chains and application to statistical mechanics. *Random Structure and Algorithms.* To appear, 1996.

[43] B.D. Ripley and F.P. Kelly. Markov point processes. *Journal of the London Mathematical Society* 15:188-192, 1977.

[44] J.S. Rowlinson. Penetrable sphere models of liquid-vapor equilibrium. *Advances in Chemical Physics* 41:1-57, 1980.

[45] J.S. Rowlinson. Probability densities for some one-dimensional problems in statistical mechanics. In *Disorder in physical systems*, pages 261-276, edited by G.R. Grimmett and D.J.A. Welsh, Clarendon Press, Oxford, 1990.

[46] D. Ruelle. *Statistical mechanics*, John Wiley and Sons, New York and Chichester, 1969.

[47] A. Särkkä. *Pseudo-likelihood approach for pair potential estimation of Gibbs processes.* Jyväskylä Studies in Computer Science, Economics and Statistics 22, 1993.

[48] R. Schneider. *Convex bodies: the Brunn-Minkowski theory*, Encyclopedia of mathematics and its applications, Volume 44, Cambridge University Press, Cambridge, 1993.

[49] S. Sherman. Markov random fields and Gibbs random fields. *Israel Journal of Mathematics* 14:92-103, 1973.

[50] D.J. Strauss. A model for clustering. *Biometrika* 63:467-475, 1975.

[51] D. Stoyan, W.S. Kendall and J. Mecke. *Stochastic Geometry and its Applications*, John Wiley and Sons, New York and Chichester, 1995.

[52] R. Takacs. Estimator for the pair-potential of a Gibbsian point process. Institutsbericht 238, Institut für Mathematik, Johannes Kepler Universität Linz, Austria, 1983.

[53] R. Takacs. Estimator for the pair potential of a Gibbsian point process. *Statistics*, 17:429-433, 1986.

[54] B. Widom and J.S. Rowlinson. New model for the study of liquid-vapor phase transitions. *The Journal of Chemical Physics* 52:1670-1684, 1970.

[55] R.J. Wilson. *Introduction to Graph Theory*, Oliver and Boyd, Edinburgh. 1972.