An index of spatial interaction in multivariate point patterns

by

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1 Introduction

In [15] we introduced a new statistic $J(t)$ for the analysis of spatial interaction in (single-type) stationary point patterns N on a complete seperable metric space $(\mathcal{X}, \rho)$ (usually $\mathbb{R}^d$ with the Euclidean metric). It is defined by

$$J(t) = \frac{1 - G(t)}{1 - F(t)} \tag{1}$$

for all $t \geq 0$ for which $F(t) \neq 1$. Here F denotes the empty space function

$$F(t) = \mathbb{P}(\rho(y, N) \leq t),$$

the distribution function of the distance of a fixed point y to the nearest point of N [8]. Further

$$G(t) = \mathbb{P}^y(\rho(y, N \setminus \{y\}) \leq t)$$

where $\mathbb{P}^y$ is the Palm distribution [7, 14, 22] of N at y, which can intuitively be regarded as the distribution of the entire process conditional on there being a point at y. This is called the *nearest-neigbhour distance distribution function*. For reasons of symmetry we might prefer to condition on the remainder of the process, i.e. $N \setminus \{y\}$, yielding the reduced Palm distribution $\mathbb{P}^{!y}$ and

$$G(t) = \mathbb{P}^{!y}(\rho(y, N) \leq t).$$

Note that by stationarity, these definitions do not depend on the choice of y.

Useful properties of J are (a) $J(t)$ is constant for t greater than the effective range of interaction, (b) $J(t) > 1$ indicates the process tends to give ordered patterns; $J(t) < 1$ suggests clustering and (c) the J-function of the superposition of two independent point processes is an appropriately weighted convex combination of the J-functions of the component processes.

The aim of the present paper is to adapt the J-function to multivariate point processes

$$Y = (X_1, \ldots, X_m), \qquad m \in \mathbb{N}.$$

Here each of the X_i is a configuration of points in (say) $\mathbb{R}^d$, and we can distinguish between points coming from different components X_i. Multivariate patterns arise naturally in a wide variety of applications. In forestry, several species of trees coexist in a given field; in ecology one may be interested in the interaction between animal species; in microscopy the spatial arrangement of different cell types gives information about tissue strength or disease etcetera.

Statistical inference for multivariate patterns is usually based on cross versions of the familiar single-type summary statistics F, G or K [19]. For instance, $G_{ij}(t)$ [8] denotes, roughly speaking, the probability of finding a type j-event within distance t of a type i-event. In order to give a more rigorous definition, let $\mathbb{P}^{!(y,i)}$ denote the Palm distribution of Y with respect to an event with mark i at y. Then

$$G_{ij}(t) = \mathbb{P}^{!(y,i)}(\rho(y, X_j) \leq t).$$

By stationarity, this definition is independent of the choice of y.

We will need the following multivariate version of the fundamental *Nguyen-Zessin formula* [24]. Assuming the conditional intensity $\lambda_Y((y,i);Y)$ of Y exists, it satisfies

$$\lambda_i \mathbb{E}^{!(0,i)} f(Y) = \mathbb{E}[\lambda_Y((0,i);Y) f(Y)] \tag{2}$$

for any nonnegative measurable function f on the space of realsations of Y. In particular, choosing $f \equiv 1$ yields

$$\lambda_i = \mathbb{E}\lambda_Y((0,i);Y) = \mathbb{E}\lambda_i(0;X_i).$$

Indeed,

$$\mathbb{E}[\lambda_Y((0,i);Y)|X_i] = \lambda_i(0;X_i) \quad \text{a.e.}$$

by equation (2). If we define

$$\lambda_+(0;Y) = \lambda_Y((0,1);Y) + \lambda_Y((0,2);Y) \tag{3}$$

to be the Y–conditional intensity of a point of either type at 0, it follows that

$$\mathbb{E}[\lambda_+(0;Y)|X] = \lambda(0;X) \quad \text{a.e.}$$

where X denote the unmarked points $X_1 \cup X_2$.

Dependence between the components may be investigated by comparing a test statistic estimated from the superposition data X to the value that would have been obtained if X_1, X_2 satisfied an appropriate independence assumption. For instance, one could subtract the empty space function of the superposition from

$$1 - (1 - F_1)(1 - F_2)$$

where F_j, $j = 1, 2$, denotes the marginal empty space function of X_j or study the sign of [16]

$$T = \log(1 - F) - \log(1 - F_1) - \log(1 - F_2).$$

Our proposal is to exploit property (c) above to base inference on the J–statistic which has the advantage of simultaneously providing information on the type and range of spatial interaction.

The plan of this paper is as follows. In Section 2 we fix notation and give the main definitions. In Section 3 we show that the proposed statistics are computable for a wide range of models, and in Section 4 we give some applications to bivariate data sets.

2 Definitions and notation

Throughout this paper we are concerned with the dependence structure in jointly stationary point processes $Y = (X_1, X_2)$. For details of point processes, see [6, 7, 8, 20, 22].

Recall that for independent stationary point processes X_1 and X_2 with intensities λ_1, λ_2 and J-statistics J_1, J_2 the J-function of the superposition $X = X_1 \cup X_2$ is

$$\frac{\lambda_1}{\lambda_1 + \lambda_2} J_1 + \frac{\lambda_2}{\lambda_1 + \lambda_2} J_2, \tag{4}$$

a convex combination of the J-functions of the components [15, Theorem 2]. This motivates the following definition.

Definition 1 *Let X_1 and X_2 be jointly stationary point processes with intensities λ_1, λ_2 and J-statistics J_1, J_2. Write $J(t)$ for the J-function of $X = X_1 \cup X_2$ and let $J_c(\cdot)$ be defined as in (4). Then we define*

$$I(t) = J_c(t) - J(t)$$

for every t for which J, J_1 and J_2 are well-defined.

The type of spatial correlation is indicated by the sign of $J_c(t) - J(t)$. Note that if X_1 and X_2 are independent, $J_c - J \equiv 0$. A positive value indicates positive correlation between the components, a negative sign suggests negative correlation. A heuristic explanation is that in case of negative dependence the superposition has relatively large inter-event distances due to the larger distances between type 1 and type 2 points. Hence $G_1(t) > G(t)$, $G_2(t) > G(t)$, where G_i denotes the G-function of X_i. On the other hand, the point-event distances in the total configuration are relatively small, as type 1 points tend to fill the space left by type 2 points and vice versa. Hence $F_1(t) < F(t)$, $F_2(t) < F(t)$, again writing F_i for the empty space function of component process X_i. Thus

$$\frac{1 - G_i(t)}{1 - F_i(t)} < \frac{1 - G(t)}{1 - F(t)} = J(t)$$

and hence $J_c(t) < J(t)$. The case of positive correlation can be dealt with similarly.

Definition 2 *For jointly stationary point processes X_1, X_2 define*

$$J_{ij}(t) = \frac{1 - G_{ij}(t)}{1 - F_j(t)} \tag{5}$$

($i, j \in \{1, 2\}$ different) for all $t \geq 0$ for which $F_j(t) < 1$.

In words, J_{ij} compares distances to the nearest type j-event from a type i-event to those from an arbitrary point in space. Note that the definition is not symmetric in i and j. While this may be undesirable in inference, it may be easier to interpret, expecially when considering qualitatively different point patterns.

Analogous to the univariate case, values of $J_{ij} > 1$ indicate inhibition between type i and type j, while values less than 1 suggest positive association. Heuristically, if there are relatively few type j-events within a radius r of a type i-event, we have $G_{ij}(t) \leq F_j(t)$ and hence $J_{ij} > 1$. A similar argument holds for the clustered case.

Restricting to i and j different is no loss of generality as $J_{ii}(\cdot) = J_i(\cdot)$.

Definition 3 *For jointly stationary point processes X_1, X_2 and unmarked points $X = X_1 \cup X_2$ define*

$$J^{!i}(t) = \frac{\mathbb{P}^{!(0,i)}(X \cap B(0,t) = \emptyset)}{\mathbb{P}(X \cap B(0,t) = \emptyset)} \tag{6}$$

for all t for which $F(t) = \mathbb{P}(X \cap B(0,t) \neq \emptyset) < 1$.

For an interpretation of this definition, see Lemma 3.

In the remainder of this Section we collect primary properties of the statistics introduced in Definitions 1–3.

Lemma 1 *Let X_1 and X_2 be two independent stationary point processes on $\mathbb{R}^d$. Then*

1. *for every t for which $I(t)$ is well-defined, $I(t) = 0$;*
2. *for every t for which $F_j(t) < 1$ ($j = 1,2$), $J_{12}(t) = J_{21}(t) = 1$;*
3. *for every t for which $F(t) < 1$, $J^{!j}(t) = J_j(t)$, $j = 1,2$.*

Proof : We already saw property 1 (cf. [15]). For 2 use the fact that for independent point processes, $1 - G_{ij}(t) = 1 - F_j(t)$ [8, p. 92] or [6, p. 700]. Finally for 3, eg.

$$J^{!1}(t) = \frac{\mathbb{P}(X_2 \cap B(0,t) = \emptyset)\mathbb{P}^{!0}(X_1 \cap B(0,t) = \emptyset)}{\mathbb{P}(X_1 \cap B(0,t) = \emptyset)\mathbb{P}(X_2 \cap B(0,t) = \emptyset)} = J_1(t).$$

□

Lemma 2 *Let X_1 and X_2 be stationary point processes. Then*

$$J_{ij}(t) = \mathbb{E}\left[\frac{\lambda_Y((0,i);Y)}{\lambda_i} \mid X_j \cap B(0,t) = \emptyset\right]$$

and

$$J_{ij}(t) = \left(\mathbb{E}^{!(0,1)}\left[\frac{\lambda_i}{\lambda_Y((0,i);Y)} \mid X_j \cap B(0,t) = \emptyset\right]\right)^{-1}$$

This result should be compared to similar expressions in the univariate case, see [15].

Proof : Use the Nguyen-Zessin formula (2) taking $f(Y) = 1\{X_j \cap B(0,t) = \emptyset\}$ or $f(Y) = 1\{X_j \cap B(0,t) = \emptyset\}/\lambda_Y((0,i);Y)$. □

In contrast, the following lemma shows that using the I-statistic involves computing

$$J^{!i}(t) = \mathbb{E}\left[\frac{\lambda_Y((0,i);Y)}{\lambda_i} | X \cap B(0,t) = \emptyset\right]. \tag{7}$$

Lemma 3 *For any t for which $I(t)$ is well-defined,*

$$I(t) = \frac{\lambda_1}{\lambda_1 + \lambda_2}\left[J_1(t) - J^{!1}(t)\right] + \frac{\lambda_2}{\lambda_1 + \lambda_2}\left[J_2(t) - J^{!2}(t)\right].$$

Proof : Write V for the event $\{X \cap B(0,t) = \emptyset\}$. Note that

$$\mathbb{E}[\lambda_+(0;Y)\mathbb{1}_V] = \mathbb{E}[\lambda(0;X)\mathbb{1}_V]$$

as $\mathbb{1}_V$ is X-measurable (cf. equation (3)). Hence

$$\begin{aligned} J(t) &= \frac{1}{\lambda_1 + \lambda_2}\frac{\mathbb{E}[\lambda(0;X)\mathbb{1}_V]}{\mathbb{P}(V)} = \frac{1}{\lambda_1 + \lambda_2}\frac{\mathbb{E}[\lambda_+(0;Y)\mathbb{1}_V]}{\mathbb{P}(V)} \\ &= \frac{\lambda_1}{\lambda_1 + \lambda_2}\frac{1 - G_{X|X_1}(t)}{1 - F_X(t)} + \frac{\lambda_2}{\lambda_1 + \lambda_2}\frac{1 - G_{X|X_2}(t)}{1 - F_X(t)} \end{aligned}$$

where $1 - G_{X|X_i}(t) = \mathbb{P}^{!(0,i)}(V)$ denotes the conditional probability of no point of either type in $B(0,t)$ given an event of type i at 0. By arguments similar to those in Lemma 2, $\frac{1-G_{X|X_i}(t)}{1-F_X(t)} = J^{!i}(t)$ and the result follows. □

An expression for $I(\cdot)$ in terms of correlations is obtained by noting that (say) $J_1(t) - J^{!1}(t)$ can be written as

$$\frac{\mathrm{Cov}\,(\lambda_1(0;X_1), 1\{X_1 \cap B(0,t) = \emptyset\})}{\lambda_1 \mathbb{P}(X_1 \cap B(0,t) = \emptyset)} - \frac{\mathrm{Cov}\,(\lambda_Y((0,1);Y), 1\{X \cap B(0,t) = \emptyset\})}{\lambda_1 \mathbb{P}(X \cap B(0,t) = \emptyset)}.$$

Lemma 4 *If*

$$\mathbb{E}[\lambda_1(0;X_1)|X_1 \cap B(0,t) = \emptyset] \geq \mathbb{E}[\lambda_Y((0,1);Y)|X \cap B(0,t) = \emptyset]$$

and

$$\mathbb{E}[\lambda_2(0;X_2)|X_2 \cap B(0,t) = \emptyset] \geq \mathbb{E}[\lambda_Y((0,2);Y)|X \cap B(0,t) = \emptyset]$$

then $I(t) \geq 0$. Reversal of signs gives a similar condition for $I(t) \leq 0$.

In words, if additional conditioning on no type 2-events in the vicinity makes the likelihood of a type 1-event smaller, there is positive association; conditioning increasing the likelihood suggests negative association between the component processes. In principle, mixtures of different signs are possible. In that case the largest in absolute value dominates.

Lemma 5

$$J_{12}(t) \geq 1 \Leftrightarrow \mathrm{Cov}(\lambda_Y((0,1);Y), 1\{X_2 \cap B(0,t) = \emptyset\}) \geq 0.$$

and

$$J^{!1}(t) \geq 1 \Leftrightarrow \mathrm{Cov}(\lambda_Y((0,1);Y), 1\{X \cap B(0,t) = \emptyset\}) \geq 0.$$

Proof : Note that

$$J_{12}(t) \geq 1 \Leftrightarrow \mathbb{E}[\lambda_Y((0,1);Y)\,\mathbf{1}\{X_2 \cap B(0,t) = \emptyset\}] \geq \lambda_1\,\mathbb{P}(X_2 \cap B(0,t) = \emptyset).$$

Similarly

$$J^{!1}(t) \geq 1 \Leftrightarrow \mathbb{E}[\lambda_Y((0,1);Y)\,\mathbf{1}\{X \cap B(0,t) = \emptyset\}] \geq \lambda_1\,\mathbb{P}(X \cap B(0,t) = \emptyset).$$

□

Regarding the interaction range, we have the following results.

Lemma 6 *Let X be a stationary point process that is the superposition of two stationary point processes X_1 and X_2. Write $\lambda(\cdot;\cdot)$ for the conditional intensity [14] of the unmarked superposition process, and λ_i for the conditional intensity of X_i, $i = 1,2$. If $\lambda(0;X) = \lambda(0;\emptyset)$ whenever $1\{X \cap B(0,r) = \emptyset\}$ for all $r \geq R_s$, and moreover $\lambda_1(0;X) = \lambda_1(0;\emptyset)$ when $1\{X \cap B(0,r) = \emptyset\}$ (all $r \geq R_1$) and $\lambda_2(0;X) = \lambda_2(0;\emptyset)$ when $1\{X \cap B(0,r) = \emptyset\}$ (all $r \geq R_2$) then*

$$I(t) = \frac{\lambda_1(0;\emptyset) + \lambda_2(0;\emptyset) - \lambda(0;\emptyset)}{\lambda_1 + \lambda_2}$$

is constant for all $t \geq \max\{R_s, R_1, R_2\}$.

Proof : By Theorem 1 in [15]

$$I(t) = \frac{\lambda_1}{\lambda_1+\lambda_2}\frac{\lambda_1(0;\emptyset)}{\lambda_1} + \frac{\lambda_2}{\lambda_1+\lambda_2}\frac{\lambda_2(0;\emptyset)}{\lambda_2} - \frac{\lambda(0;\emptyset)}{\lambda_1+\lambda_2}$$

and the result follows. □

Lemma 7 *Let $Y = (X_1, X_2)$ be a jointly stationary point process whose conditional intensity $\lambda((0,i);Y)$ ($i \in \{1,2\}$) exists and depends only on $Y \cap B(0,R)$. Then for $t \geq R$, conditional on $X_j \cap B(0,t) = \emptyset$, $J_{ij}(t)$ depends on $X_i \cap B(0,R)$ only.*

Proof : By Lemma 2,

$$\begin{aligned}
J_{ij}(t) &= \mathbb{E}\left[\frac{\lambda_Y((0,i);Y)}{\lambda_i} \mid X_j \cap B(0,t) = \emptyset\right] \\
&= \mathbb{E}\left[\frac{\lambda_Y((0,i);Y \cap B(0,R))}{\lambda_i} \mid X_j \cap B(0,t) = \emptyset\right] \\
&= \mathbb{E}\left[\frac{\lambda_Y((0,i);X_i \cap B(0,R))}{\lambda_i} \mid X_j \cap B(0,t) = \emptyset\right]
\end{aligned}$$

□

Generalisations to three or more components are straightforward.

3 Examples

In this Section we discuss in detail a variety of multivariate point pattern models. As before, let $Y = (X_1, X_2)$ be a marked point process that is stationary and has stationary components. Write X for the superposition $X_1 \cup X_2$.

3.1 Two–type Gauss–Poisson process

A Gauss–Poisson process [3, 17, 18] is a Poisson cluster processs in which each point of a stationary Poisson process gives birth to another point with probability p_2 and with probability $p_1 = 1 - p_2$ does not have any offspring. The positions of offspring relative to the 'mother' are iid, with probability density $h(\cdot)$. The observed configuration consists of both the 'parents' and the offspring, if any. We modify the process, by assuming that for each observation we can see whether it is offspring (type 2) or not (type 1).

Lemma 8 *For a two-type Gauss-Poisson process,*

$$I(t) = \frac{2p_2}{p_1 + 2p_2} \int\int_{B(0,t)} h(x)dx.$$

Proof : Both X_1 and X_2 are stationary Poisson processes, hence $J_c \equiv 1$. Furthermore, the superposition is a cluster process and we can invoke a formula by Bartlett [2, p. 8–9] to see that $J(t)$ is the probability of there being no other point of the same cluster in a ball with radius t of a given point. This probability is

$$\frac{p_1}{p_1 + 2p_2} + \frac{2p_2}{p_1 + 2p_2} \left(1 - \int\int_{B(0,t)} h(x)dx \right).$$

Hence

$$I(t) = \frac{2p_2}{p_1 + 2p_2} \int\int_{B(0,t)} h(x)dx.$$

□

Note that I is non-negative, increasing and bounded above by $\frac{2p_2}{p_1+2p_2}$. If $h(\cdot)$ is concentrated on $B(0, R)$, then $I(t) = I(R)$ for all $t \geq R$.

Lemma 9 *For a two-type Gauss-Poisson process,*

$$J_{12}(t) = J_{21}(t) = 1 - \frac{2p_2}{p_1 + 2p_2} \int\int_{B(0,t)} h(x)dx.$$

Proof : By independence of clusters in a Poisson cluster process,

$$\begin{aligned}
1 - G_{12}(t) &= \mathbb{P}^{!0,1}(X_2 \cap B(0,t) = \emptyset) \\
&= \mathbb{P}^{!0,1}(\text{ no type 2 associated point ; no type 2 point from another cluster }) \\
&= \left(1 - \frac{2p_2}{p_1 + 2p_2} \int\int_{B(0,t)} h(x)dx \right) (1 - F_2(t)).
\end{aligned}$$

For $1 - G_{21}(t)$ we can use the same argument by noting that the process can be seen as a Poisson cluster process with parents X_2 and displacements $-V$, where V is a random variable with probability density h. $\square$

The functions J_{12} and J_{21} are decreasing to $\frac{p_1}{p_1+2p_2}$ and do not exceed 1, suggesting attraction between type 1 and type 2 points as expected. If type 2 points are scattered within a ball of radius R of type 1 points, $J_{ij}(t) = \frac{p_1}{p_1+2p_2}$ for $t \geq R$.

3.2 Bivariate Poisson processes

A *bivariate Poisson process* is a two-type process in which the marginal distribution of each of the components is that of a stationary Poisson process. The dependence structure is not specified.

3.2.1 Linked Poisson

A *linked Poisson process* [9] is constructed by associating with each type 1 event in a stationary Poisson process a type 2 event; the displacements are iid, with density h. A useful rephrasing is as a Poisson cluster process, where each cluster contains exactly two points: the parent x (say) and one daughter at $x + v$, where v has probability density $h(\cdot)$. Note that this is a special case of the two–type Gauss-Poisson process described above ($p_2 = 1$).

Lemma 10 *For a linked Poisson process, $I(t) = \int\int_{B(0,t)} h(x)dx$.*

Proof : Since the marginal processes are Poisson, $J_1 \equiv J_2 \equiv 1$ and hence $J_c(t) = 1$ for all $t \geq 0$. To find the J-function of the superposition we use the interpretation above as a Poisson cluster process. Then,

$$J(t) = C_0(Z \cap B(0,t) = \{0\}) = 1 - \int\int_{B(0,t)} h(x)dx$$

where C_0 is the Palm distribution at 0 of the typical cluster Z. $\square$

In particular, the I-function is non-negative, suggesting positive association between type 1 and type 2 points (as expected). The function is increasing and bounded between 0 and 1, since h is a probability density. Furthermore, if the secondary points are scattered within a ball of radius R, that is $h(\cdot)$ is concentrated on $B(0, R)$, $I(t) = 1$ for all $t \geq R$.

Lemma 11 *For a linked Poisson process, $J_{12}(t) = J_{21}(t) = 1 - \int\int_{B(0,t)} h(x)dx$.*

Proof : We need to compute $1 - G_{12}(t)$. As follows,

$$\begin{aligned}
1 - G_{12}(t) &= \mathbb{P}^{!0,1}(X_2 \cap B(0,t) = \emptyset) \\
&= \mathbb{P}^{!0,1}(\text{ no type 2 associated point ; no type 2 point from another cluster }) \\
&= \left(1 - \int\int_{B(0,t)} h(x)dx\right)(1 - F_2(t)).
\end{aligned}$$

For the last equation we need the fact that in a Poisson cluster process, different clusters are independent (Slivnyak theorem).

For $1 - G_{21}(t)$ we can use the same argument by noting that the process can be seen as a Poisson cluster process with parents X_2 and displacements $-V$, where V is a random variable with probability density h. □

In particular, both J_{12} and J_{21} are decreasing and less than or equal to 1 (since h is a probability density). If we assume that h is concentrated on a ball $B(0, R)$, again $J_{ij}(t) = 0$ for all $t \geq R$, mirroring the results for a univariate Poisson cluster process [15].

3.3 Bivariate Cox processes

A bivariate Cox process [5, 10] is defined by a joint distribution of two random measures (Λ_1, Λ_2). Conditional on $(\Lambda_1, \Lambda_2) = (\lambda_1, \lambda_2)$, type 1 and type 2 events form a pair of independent inhomogeneous Poisson processes with intensity measures λ_1 and λ_2. We assume that Λ_i is absolutely continuous, and will interpret λ_i as an intensity function.

3.3.1 Linked Cox

An example of positive dependence between the random measures is a linked Cox process [5] where $\Lambda_1 = \nu\Lambda_2$ for some fixed positive constant ν.

Lemma 12 *Assume (X_1, X_2) is a linked Cox process on $\mathbb{R}^2$ with a 'mixed Poisson' law $\Lambda_2 = \alpha m$, for some non-negative random variable α with finite positive expectation and where m is Lebesgue measure. Then*

$$\begin{aligned} I(t) &= \frac{\nu}{\mathbb{E}\alpha\,(1+\nu)}\left[\frac{\mathbb{E}\alpha e^{-\nu\alpha\pi t^2}}{\mathbb{E}e^{-\nu\alpha\pi t^2}} - \frac{\mathbb{E}\alpha e^{-(1+\nu)\alpha\pi t^2}}{\mathbb{E}e^{-(1+\nu)\alpha\pi t^2}}\right] \\ &+ \frac{1}{\mathbb{E}\alpha\,(1+\nu)}\left[\frac{\mathbb{E}\alpha e^{-\alpha\pi t^2}}{\mathbb{E}e^{-\alpha\pi t^2}} - \frac{\mathbb{E}\alpha e^{-(1+\nu)\alpha\pi t^2}}{\mathbb{E}e^{-(1+\nu)\alpha\pi t^2}}\right]. \end{aligned}$$

Note that $\mathbb{E}e^{-\alpha\pi t^2} = L(\pi t^2)$, the Laplace transform of α evaluated in πt^2.

Generalisations to higher dimensions are straightforward; in the general case (Λ_1, Λ_2) are stationary random measures, α-weighted means have to be replaced by expectation under the Palm distribution at 0 of the random measure Λ_2.

Proof : By the superposition property of independent Poisson processes, $X_1 \cup X_2$ is a mixed Poisson process with intensity measure $(1+\nu)\alpha m$. By Theorem 6 in [15] and the discussion following,

$$J(t) = \frac{\mathbb{E}\left[\alpha e^{-(1+\nu)\alpha\pi t^2}\right]}{\mathbb{E}\alpha\;\mathbb{E}e^{-(1+\nu)\alpha\pi t^2}}.$$

Similar expressions hold for J_1 and J_2 with $(1+\nu)$ replaced by ν and 1 respectively. Furthermore, $\lambda_1 = \mathbb{E}[\nu\alpha], \lambda_2 = \mathbb{E}[\alpha]$. Thus

$$J_c(t) = \frac{\nu\,\mathbb{E}\alpha}{(1+\nu)\,\mathbb{E}\alpha} J_1(t) + \frac{\mathbb{E}\alpha}{(1+\nu)\,\mathbb{E}\alpha} J_2(t)$$

$$= \frac{\nu}{1+\nu} \frac{\mathbb{E}\left[\alpha e^{-\nu\alpha\pi t^2}\right]}{\mathbb{E}[\alpha]\,\mathbb{E}e^{-\nu\alpha\pi t^2}} + \frac{1}{1+\nu} \frac{\mathbb{E}\left[\alpha e^{-\alpha\pi t^2}\right]}{\mathbb{E}\alpha\,\mathbb{E}e^{-\alpha\pi t^2}}$$

□

It can be shown that I is non-negative and converges to 0 as $t \to \infty$.

Lemma 13 *Assume (X_1, X_2) is a linked Cox process ($\Lambda_1 = \nu\Lambda_2$) with $\Lambda_2 = \alpha m$, for some non-negative random variable α with finite positive expectation. Then*

$$J_{12}(t) = \frac{\mathbb{E}\left[\alpha e^{-\alpha\pi t^2}\right]}{\mathbb{E}\alpha\,\mathbb{E}e^{-\alpha\pi t^2}} \qquad \textit{and} \qquad J_{21}(t) = \frac{\mathbb{E}\left[\alpha e^{-\nu\alpha\pi t^2}\right]}{\mathbb{E}\alpha\,\mathbb{E}e^{-\nu\alpha\pi t^2}}.$$

Proof: The marginal distribution of X_2 is a mixed Poisson process, thus $1-F_2(t) = \mathbb{E}e^{-\alpha\pi t^2}$. The Palm distribution of the superposition given a point of type 1 at the origin is $\nu\alpha$ weighted, hence

$$1 - G_{12}(t) = \frac{\mathbb{E}\left[\nu\alpha e^{-\alpha\pi t^2}\right]}{\mathbb{E}[\nu\alpha]} = \frac{\mathbb{E}\left[\alpha e^{-\alpha\pi t^2}\right]}{\mathbb{E}\alpha}.$$

The result for J_{21} follows similarly. □

It can be verified that $J_{ij} \leq 1$, suggesting positive correlation between the component processes. Moreover, J_{ij} is decreasing with $\lim_{t\to\infty} J_{12}(t) = \lim_{t\to\infty} J_{21}(t) = \frac{\text{essinf}\,\alpha}{\mathbb{E}\alpha}$.

3.3.2 Balanced Cox

An example of negative dependence is formed by the class of balanced Cox processes [10] where

$$\Lambda_1 + \Lambda_2 = \nu m,$$

m again denoting Lebesgue measure. Note that the superposition is always distributed as a Poisson process with intensity ν.

Lemma 14 *Let (X_1, X_2) be a balanced Cox process on $\mathbb{R}^2$ with $\Lambda_2 = \alpha m$, for a random variable α concentrated on $(0, \nu)$ with $0 < \mathbb{E}\alpha < \nu$. Then*

$$I(t) = \frac{1}{\nu}\left\{ \frac{\mathbb{E}\left[\alpha e^{-\alpha\pi t^2}\right]}{\mathbb{E}e^{-\alpha\pi t^2}} - \frac{\mathbb{E}\left[\alpha e^{-(\nu-\alpha)\pi t^2}\right]}{\mathbb{E}e^{-(\nu-\alpha)\pi t^2}} \right\}$$

Proof : As the superposition is a Poisson process, $J(t) = 1$. To compute the convex combination

$$\begin{aligned} J_c(t) &= \frac{\nu - \mathbb{E}\alpha}{\nu} \frac{\mathbb{E}\left[(\nu-\alpha)e^{-(\nu-\alpha)\pi t^2}\right]}{(\nu - \mathbb{E}\alpha)\mathbb{E}e^{-(\nu-\alpha)\pi t^2}} + \frac{\mathbb{E}\alpha}{\nu} \frac{\mathbb{E}\left[\alpha e^{-\alpha\pi t^2}\right]}{\mathbb{E}\alpha\,\mathbb{E}e^{-\alpha\pi t^2}} \\ &= \frac{1}{\nu} \frac{\mathbb{E}\left[(\nu-\alpha)e^{-(\nu-\alpha)\pi t^2}\right]}{\mathbb{E}e^{-(\nu-\alpha)\pi t^2}} + \frac{1}{\nu} \frac{\mathbb{E}\left[\alpha e^{-\alpha\pi t^2}\right]}{\mathbb{E}e^{-\alpha\pi t^2}} \end{aligned}$$

$$= \frac{1}{\nu}\left\{\nu - \frac{\mathbb{E}\left[\alpha e^{-(\nu-\alpha)\pi t^2}\right]}{\mathbb{E}e^{-(\nu-\alpha)\pi t^2}} + \frac{\mathbb{E}\left[\alpha e^{-\alpha\pi t^2}\right]}{\mathbb{E}e^{-\alpha\pi t^2}}\right\}$$

$$= 1 + \frac{1}{\nu}\left\{\frac{\mathbb{E}\left[\alpha e^{-\alpha\pi t^2}\right]}{\mathbb{E}e^{-\alpha\pi t^2}} - \frac{\mathbb{E}\left[\alpha e^{-(\nu-\alpha)\pi t^2}\right]}{\mathbb{E}e^{-(\nu-\alpha)\pi t^2}}\right\}$$

□

Moreover, $I(t) \leq 0$ (indicating negative dependence) and decreases to (essinf α - esssup $\alpha)/\nu$ as $t \to \infty$.

Lemma 15 *Let (X_1, X_2) be a balanced Cox process with $\Lambda_2 = \alpha m$, for a random variable α concentrated on $(0, \nu)$ with $0 < \mathbb{E}\alpha < \nu$. Then*

$$J_{12}(t) = \frac{\mathbb{E}\left[(\nu-\alpha)e^{-\alpha\pi t^2}\right]}{\mathbb{E}(\nu-\alpha)\,\mathbb{E}e^{-\alpha\pi t^2}} \qquad \textit{and} \qquad J_{21}(t) = \frac{\mathbb{E}\left[\alpha e^{-(\nu-\alpha)\pi t^2}\right]}{\mathbb{E}\alpha\,\mathbb{E}e^{-(\nu-\alpha)\pi t^2}}$$

Proof: The marginal distribution of X_2 is a mixed Poisson process; thus $1-F_2(t) = \mathbb{E}e^{-\alpha\pi t^2}$. The Palm distribution of the superposition given a point of type 1 at the origin is $\nu - \alpha$ weighted, hence

$$1 - G_{12}(t) = \frac{\mathbb{E}\left[(\nu-\alpha)e^{-\alpha\pi t^2}\right]}{\mathbb{E}(\nu-\alpha)}$$

The result for J_{21} follows similarly. □

We can verify that $J_{ij} \geq 1$, indicating negative correlation between the component processes. For example consider J_{21}. Then

$$\begin{aligned}\frac{\mathbb{E}\left[\alpha e^{-(\nu-\alpha)\pi t^2}\right]}{\mathbb{E}\alpha\,\mathbb{E}e^{-(\nu-\alpha)\pi t^2}} &= -\frac{\mathbb{E}\left[(\nu-\alpha)e^{-(\nu-\alpha)\pi t^2}\right]}{\mathbb{E}\alpha\,\mathbb{E}e^{-(\nu-\alpha)\pi t^2}} + \frac{\nu}{\mathbb{E}\alpha} \\ &\geq -\frac{(\nu-\mathbb{E}\alpha)\,\mathbb{E}e^{-(\nu-\alpha)\pi t^2}}{\mathbb{E}\alpha\,\mathbb{E}e^{-(\nu-\alpha)\pi t^2}} + \frac{\nu}{\mathbb{E}\alpha} = \frac{\mathbb{E}\alpha}{\mathbb{E}\alpha} = 1.\end{aligned}$$

Moreover, both $J_{12}(\cdot)$ and $J_{21}(\cdot)$ are monotonically increasing with limits $\frac{\nu - \text{essinf }\alpha}{\mathbb{E}(\nu-\alpha)}$ and $\frac{\text{esssup }\alpha}{\mathbb{E}\alpha}$ respectively.

3.4 Markov processes

Let $Y = \{(x_i, m_i)\}$ be the marked Markov point process on a bounded subset $B \subset \mathbb{R}^d \times \{1, \ldots, m\}$ with density

$$p(\mathbf{y}) \propto \prod_i \beta_{m_i} \prod_{i<j} \gamma_{m_i m_j}(||x_i - x_j||) \tag{8}$$

with respect to the law of a Poisson process [1]. Strictly speaking, the process is not stationary, but we will conveniently ignore this. Here without loss of generality $\gamma_{ij} = \gamma_{ji}$. Note that in general, terms γ_{ii} appear.

Lemma 16 *For the two-type Markov process (8) and all t for which the appropriate statistics are well-defined*

1. $J(t) = \sum_i \frac{\beta_i}{\lambda}\mathbf{E}\left[\prod_{(x,j)\in Y}\gamma_{ij}(||x||) \mid X \cap B(0,t) = \emptyset\right]$

2. $J_{ij}(t) = \frac{\beta_i}{\lambda_i}\mathbf{E}\left[\prod_{(x,k)\in Y}\gamma_{ik}(||x||) \mid X_j \cap B(0,t) = \emptyset\right]$

3. $J^{i\cdot}(t) = \frac{\beta_i}{\lambda_i}\mathbf{E}\left[\prod_{(x,j)\in Y}\gamma_{ij}(||x||) \mid X \cap B(0,t) = \emptyset\right]$

If $\gamma_{ij}(||x||) = 1$ for $||x|| > r_{ij}$, the formulae above reduce to $J(t) = \sum \frac{\beta_i}{\lambda}$ for $t \geq r = \max r_{ij}$ and $J^{i\cdot}(t) = \frac{\beta_i}{\lambda_i}$ for $t \geq r_i = \max_j r_{ij}$. For J_{ij}, note that for $t \geq r_{ij}$, J_{ij} depends on X_j only through the conditioning. If the γ_{ii} terms are absent, it further reduces to $\frac{\beta_i}{\lambda_i}$.

Proof : It is easily seen that

$$\lambda_Y((0,i);Y) = \frac{p(\mathbf{y} \cup \{(0,i)\})}{p(\mathbf{y})} = \beta_i \prod_{(\mathbf{x},j)\in Y}\gamma_{ij}(||x||).$$

Hence, formula 1. follows. Using Lemma 2, one obtains formula 2. and finally by (7), 3. follows. □

In order to compute $I(t)$ we need the marginal distributions of X_j, $j = 1, 2$. By integration over $\mathbf{x}_2$, the density of X_1 with respect to the reference Poisson process on B is

$$p(\mathbf{x}_1) = \alpha\beta_1^{n(\mathbf{x}_1)}\prod_{i<j}\gamma_{11}(||x_{1i} - x_{1j}||)I_2(\mathbf{x}_1)$$

where

$$I_2(\mathbf{x}_1) = \int \beta_2^{n(\mathbf{x}_2)}\prod_{i<j}\gamma_{22}(||x_{2i} - x_{2j}||)\prod_{k,l}\gamma_{12}(||x_{1k} - x_{2l}||)d\pi(\mathbf{x}_2)$$

and π denotes the law of a Poisson process on B.

If $\gamma_{22} \equiv 1$, X_1 is nearest-neighbour Markov with respect.to Baddeley and Møller's connected component relation [1] at range $2R = 2r_{12}$. Hence for $t \geq 2R$, Nguyen-Zessin for $g(X_1) = \mathbf{1}\{X_1 \cap B(0,t) = \emptyset\}/\lambda_1(0;\emptyset)$ and the fact that $\lambda_1(0;X_1) = \lambda_1(0;\emptyset)$ on $\{X_1 \cap B(0,t) = \emptyset\}$ yield that $J_1(t)$ is constant with value $\frac{\lambda_1(0;\emptyset)}{\lambda_1}$.

For $t < 2R$, an explicit formula can be given, but is not illuminating.

4 Applications

In this Section we analyse four data sets with a range of correlation structures between the component processes, taking a Monte Carlo approach with test statistic $I(t)$.

We took a conditional random labelling null-hypothesis, that is given a data set consisting of n_i type i events ($i = 1, 2$), the labels are assigned randomly without replacement. An alternative is to condition only on the location of the events and to sample the labels with replacement. A disadvantage of the latter is that the label probabilities are unknown (usually estimated by $p_i = n_i/n$ where n_i is the number of observed i-events and n the total number of events) and that the relative frequency of the labels is variable.

In general, non-parametric sampling from the unconditional null-hypothesis of independent components is hard. For rectangular windows, Lotwick and Silverman proposed to identify opposite sides to obtain a torus, and then translate one of the patterns randomly over the torus [16]. Note that since the locations of events vary, in contrast to the random labelling hypotheses above, the J-function of $X = X_1 \cup X_2$ needs to be recomputed for every simulation.

4.1 Ants

Our first example considers the distribution of the nests of two ant species, *Messor wasmanni* and *Cataglyphis bicolor* in a field in Northern Greece [12]. The question is of interest as Cataglyphis ants feed on dead insects, mostly Messor ants killed by a hunting spider. For details see [12].

The original data contains 68 Messor and 29 Cataglyphis nests in an area of about 1 hectare. This region is divided into two main parts, scrub land and field. As Cataglyphis ants tend not to build their nests in scrub, we only consider the field region with 32 Messor and 15 Cataglyphis nests. For convenience, we have rotated the data to align with a standard coordinate system (Figure 1).

In Figure 2 we plotted J_c and J curves. The graphs are similar and rather close to 1. Figure 3 shows Monte Carlo envelopes based on 99 simulations. The dotted data curve lies at the top end of the curve, (with some points actually above the upper envelope, giving slight evidence of positive association).

Harkness and Isham performed a K–function analysis [16] and found no evidence for dependence between the two species. See also [13].

4.2 Myrtles

The second example is of a pattern of 221 healty and 106 diseased myrtles in a rectangle of 170.5 by 213.0 meter. The data set was obtained and studied by G Kile and colleagues at CSIRO Tasmania. All patterns are clustered ($J < 1$) and the graph of J_c lies on (small distances) or above the plot of J, indicating positive correlation between the two patterns.

In a 99 simulations Monte Carlo example, the positive correlation is significant in the middle distance range. It is not at the extremes.

4.3 Retina

The beta–type of ganglion cell in a cat retina can be subdivided in 'on' and 'off' depending on the branching level of the dendritic tree in the inner plexiform layer. Analysis of the spatial pattern provides information on the cat's visual discrimination. For details see [25].

Our analysis showed that J_c is of parabola shape, above the graph of J. The top is at approximately twice the breaking point in the J–graph. A possible explanation is the trend of alternating "on" and "off" cells (expressing itself as positive correlation, significant at 1 percent level).

A second order analysis [21] yielded similar results. At close range, the plot of the mark correlation function p_{12} [23, p. 264–265] is high compared to the plots of p_{11} and p_{22}, before flatting down (at approximately 0.11).

4.4 Hamster tumours

Figure 4 provided by Dr W A Aherne (Department of Pathology, University of Newcastle upon Tyne) shows the positions of the centers of nuclei of certain cells in an approximately .25mm square histological section of tissue from a laboratory–induced metastasing lymphoma, in the kidney of a hamster. The two types of events are (i) 77 pyknotic nuclei, corresponding to dying cells; (ii) 226 nuclei arrested in metaphase, cooresponding to cells which have been "frozen" in the act of division. The background void is occupied by unrecorded, interphase cells in relatively large numbers.

Both nuclei patterns are inhibitory ($J > 1$). The graph of J_c lies below that of J, suggesting negative correlation. This proved to be not significant based on 99 simulations. This is in keeping with other analyses reported in the literature. Diggle's analysis [8] based on the K-function yielded the same conclusion. However, his test for independent components proved to be significant (based on 99 simulations). Stoyan [21] also used second order techniques by plotting the mark correlation functions p_{ij}, $i, j \in \{1, 2\}$. The graphs are nearly horizontal suggesting random allocation of marks.

F I G U R E S   1 - 3   A B O U T   H E R E

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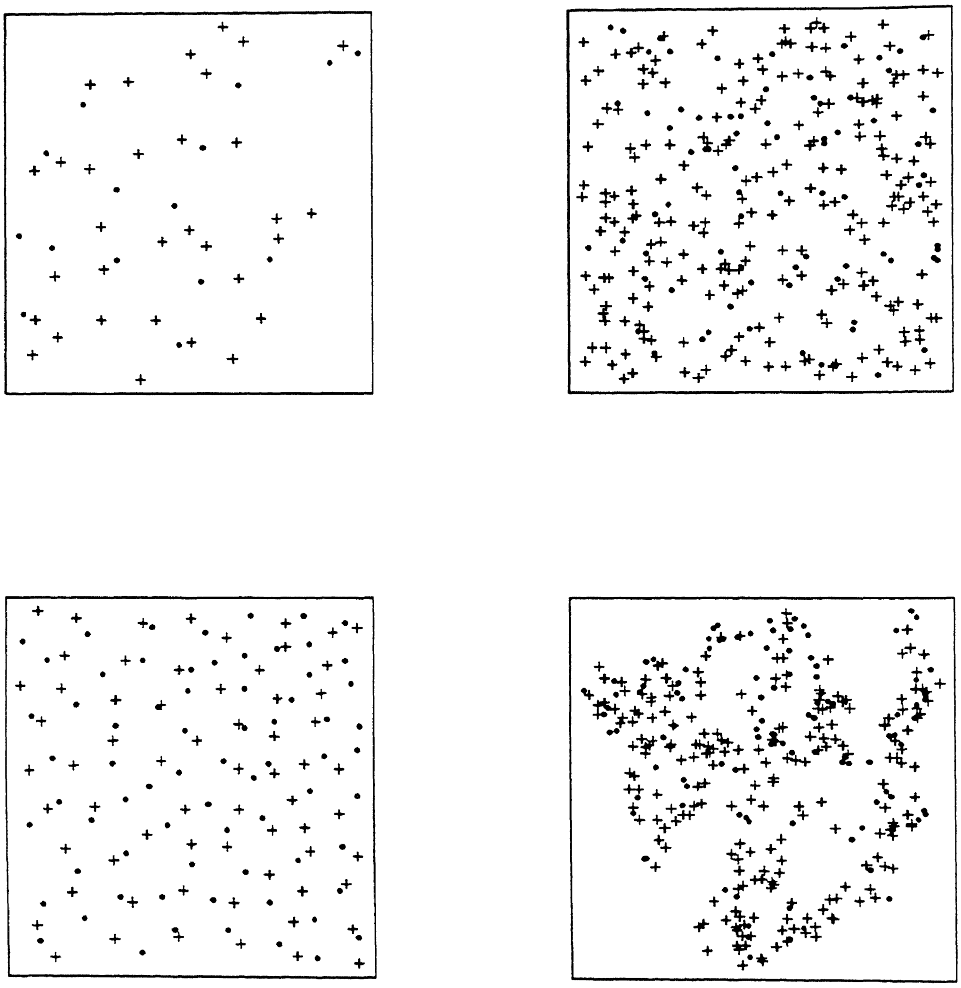
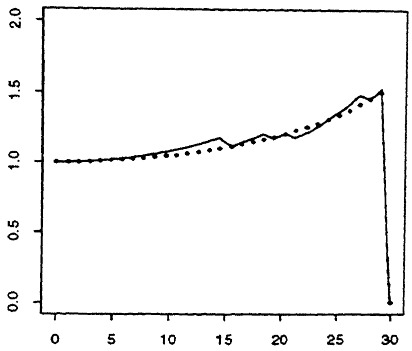
Figure captions

1. a) 32 Messor (+) and 15 Cataglyphis ($\cdot$) nests (originally studied in Isham and Harkness)
   b) 77 pyknotic cells ($\cdot$) and 266 nuclei in metaphase (+) in a hamster kidney (originally studied by Aherne and Diggle)
   c) 65 on (+) and 70 off ($\cdot$) ganglion beta cells in a cat retina. (originally studied in Wassle et al.)
   d) 221 healthy (+) and 106 diseased ($\cdot$) myrtles (originally studied by Kile)

2. empirical J-function (solid line) and J_c-function (dotted line) for a) ants, b) hamster, (c) retina and d) myrtles.

3. empirical I-function (dotted line) and envelope of 99 simulations of a random mark allocation for a) ants, b) hamster, (c) retina and d) myrtles.





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1.0
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0.0
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5
10
15
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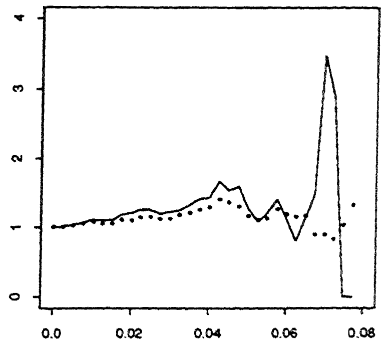
25
30



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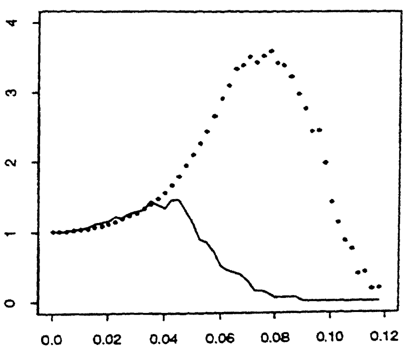

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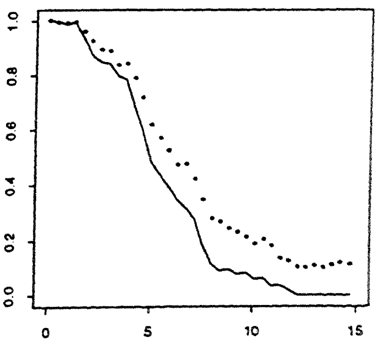

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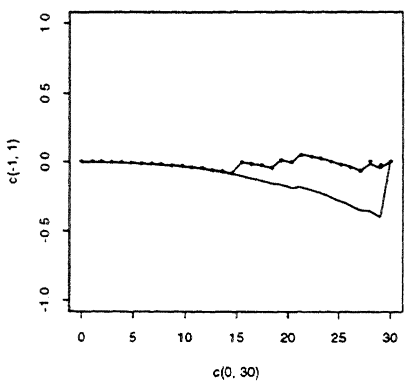
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