A nonparametric measure of spatial interaction in point patterns

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In memory of Philip Holgate

Abstract

The strength and range of interpoint interactions in a spatial point process can be quantified by the function $J = (1 - G)/(1 - F)$, where G is the nearest–neighbour distance distribution function and F the empty space function of the process. $J(r)$ is identically equal to 1 for a Poisson process; values of $J(r)$ smaller or larger than 1 indicate clustering or regularity, respectively. We show that, for a large class of point processes, $J(r)$ is *constant* for distances r greater than the range of spatial interaction. Hence both the range and type of interaction can be inferred from J without parametric model assumptions. It is also possible to evaluate $J(r)$ explicitly for many point process models, so that J is also useful for parameter estimation. Various properties are derived, including the fact that the J function of the superposition of independent point processes is a weighted mean of the J functions of the individual processes. Estimators of J can be constructed from standard estimators of F and G. We compute estimates of J for several standard point pattern datasets and conclude that it is a useful indicator of spatial interaction.

1. Introduction

The statistical analysis of a point pattern usually begins with the computation of estimates of the summary functions F (empty space function), G (nearest–neighbour distance distribution function) and K (reduced second moment function), defined e.g. in [11, 14, 34, 35]. While these are useful descriptions of spatial pattern, and can easily be estimated from data, there are very few stochastic models for which F, G or K is known analytically, so that parameter estimation and inference based on F, G, K are difficult.

Recall that, for a stationary point process, F is the distribution function of the distance from an arbitrary fixed point to the nearest random point of the process, and G of the distance from a point *of the process* to the nearest other point of the process. This paper advocates the use of

$$J(r) = \frac{1 - G(r)}{1 - F(r)}.$$

This is a nonparametric measure of the type of spatial interaction: the value 1 can be interpreted as indicating complete randomness or lack of interaction, while values less than 1 imply 'clustered' pattern and values greater than 1 imply 'ordered' or 'inhibitory' pattern.

We show that, for a very large class of point processes, the function J is *constant* for values of r larger than the effective range of spatial interaction. Hence J can be used to infer both the range and type of spatial interaction. Furthermore we are able to evaluate J explicitly for several stochastic models, so that it could be used directly for parameter estimation.

An appealing interpretation of J is that it compares the environment of a typical random point of the process with the environment of a fixed arbitrary point. $J(r)$ is the ratio of the probabilities, under these two situations, of the event that there are no points within a distance r of the given point. In terms of survival analysis, J is the ratio of the survival functions of the distance-to-nearest-point under these two situations; and our main result states that the hazard measures [25] of F and G are *equal* beyond the effective range of interaction r.

Special cases of these results are implicit in the literature. The forms of F and G for a Neyman-Scott cluster process were derived by Bartlett [9]; see [1, 31, 32], [10, pp. 8–9], and for detailed derivations [12, §8.3, p. 243 ff.], [41, p. 143]. For a general Poisson cluster process (Poisson parent points, i.i.d. offspring)

$$1 - G(r) = (1 - F(r))\, E(r)$$

where $E(r)$ is the probability that a randomly-chosen point in a typical cluster is more than r units distant from any other point belonging to the same cluster. Hence in particular if all offspring lie within a radius t of the parent point, we have $J(r) = 1$ for all $r > 2t$. Again, for a stationary, pairwise-interaction Gibbs process, Stoyan et al. [41, p. 159] exhibit a relationship between $1 - F(r)$ and $1 - G(r)$ when r is exactly equal to the interaction distance R. In this paper we extend the relationship to all $r \geq R$.

Statistical inference based on comparisons between F and G has occasionally been suggested. Diggle [13, (5.7)] proposed the statistic $D = \sup_r \mid \widehat{F}(r) - \widehat{G}(r) \mid$ as a measure of deviation from the Poisson process.

This paper is organised as follows. In Section 2 we review the main techniques from spatial statistics that are used in the sequel. Section 3 introduces the J-function; the main theorem states that $J(r)$ is constant beyond the effective range of interaction. We also examine the behaviour of J under the basic operations of superposition and thinning and show that the J-function of a superposition of independent processes is a convex combination of the J-functions of the components. The relationship between the J-function of a thinned process and that of the original process appears to be rather complex; in particular, in contrast to Ripley's K-function, the J-function is not invariant under thinning.

In Section 4 we show that the J-function can be computed explicitly for a large class of point process models, including Poisson processes, Markov point porcesses, Neyman-Scott and Cox processes. For these examples at least, the classification of patterns as 'clustered' or 'regular' on the basisof their J-function values agrees with similar classifications based on F, G and K.

In Section 5 we discuss briefly how the J-statistic can be used for parameter estimation, while Section 6 is a simple illustration on three standard data sets [14], representing regular, random and clustered patterns.

2. Background

Throughout this paper we consider a stationary point process X in $\mathbb{R}^k$, regarded as a random set of points. For details of the theory of point processes see [12] or [11, 41].

Define the empty space function F of X to be the distribution function

$$F(r) = \mathbb{P}\{\rho(y, X) \leq r\}$$

of

$$\rho(y, X) = \min\{||y - x|| : x \in X\},$$

the distance from an arbitrary fixed point $y \in \mathbb{R}^k$ to the nearest point of the process. By stationarity, the definition of F does not depend on y.

Write $B(y, r) = \{x \in \mathbb{R}^k : \rho(x, y) \leq r\}$ for the closed ball of radius $r > 0$ centred at y in $\mathbb{R}^k$. Then $1 - F(r)$ is the probability that X puts no points in $B(y, r)$:

$$1 - F(r) = \mathbb{P}\{X \cap B(y, r) = \emptyset\}.$$

For example, for a Poisson process of intensity λ in $\mathbb{R}^2$ we obtain $F(r) = 1 - \exp\{-\lambda\pi r^2\}$. F has been variously dubbed the 'empty space, 'point-event distance' and 'spherical contact' distribution function.

To define G we need the *Palm distribution* $\mathbb{P}^y$ of X at $y \in \mathbb{R}^k$, which can be regarded as the conditional distribution of the entire process given that there is a point of X at y [12, chap. 12], [11, pp. 630-631], [41, p. 110 ff.]. Then define

$$G(r) = \mathbb{P}^y\{\rho(y, X \setminus \{y\}) \leq r\};$$

again this does not depend on y, by stationarity. Thus G is the distribution function of the distance from a point of the process to the nearest other point, and is known variously as the 'nearest–neighbour' or 'event–event' distribution function.

It is convenient to use the *reduced Palm distribution* $\mathbb{P}^!_y$ defined as the distribution of $X \setminus \{y\}$ under $\mathbb{P}^y$, i.e. the conditional distribution of the *rest* of the process given that there is a point at y. Then the definition of G reads

$$G(r) = \mathbb{P}^!_y\{\rho(y, X) \leq r\}$$

in harmony with the definition of F. For example, for a stationary Poisson process of intensity λ, the reduced Palm distribution $\mathbb{P}^!_y$ is identical to $\mathbb{P}$, and $G \equiv F$.

Our main tool will be the *Takacs-Fiksel formula* which relates the reduced Palm distribution of X to its (ordinary) distribution:

$$\lambda \, \mathbb{E}_y^! f(X) = \mathbb{E}[\lambda(y; X) \, f(X)] \tag{2.1}$$

holding (under suitable conditions on X) for any bounded nonnegative measurable function on the space of realizations of X [22, 24, 42, 43] (see also [37], [29, 30], [35, p. 54–55], [15, §2.4]). Here λ is the intensity of X and $\lambda(y; X)$ is the *Papangelou conditional intensity* of X at y. In other words, (2.1) states that $\mathbb{P}_y^!$ is equivalent to the $\lambda(y; X)$-weighted distribution of X. In particular

$$\lambda = \mathbb{E}\lambda(0; X). \tag{2.2}$$

A necessary and sufficient condition (in the stationary case) for validity of (2.1) is that $\mathbb{P}_y^!$ be absolutely continuous with respect to $\mathbb{P}$, whereupon $\lambda(y; X)$ is uniquely defined by (2.1). The Takacs-Fiksel formula holds in particular for all stationary Gibbs point processes [33, 35] and for Poisson cluster processes when the cluster distribution is absolutely continuous. The corresponding expressions for $\lambda(y; X)$ are given in Section 4. Examples of processes which fail to satisfy (2.1) are randomly translated grids, and cluster processes consisting of pairs of points separated by a fixed distance.

Kallenberg [29, 30] gives a detailed explanation of the duality between the Palm distribution and Papangelou conditional intensity. The reduced Palm distribution is concerned with the remainder of the pattern given that a point falls at a particular location ('internal conditioning'), while the conditional intensity describes the behaviour of the process at a single point in space given the realisation everywhere else ('external conditioning').

3. The J-function

Definition 1 *For a stationary point process X define*

$$J(r) = \frac{1 - G(r)}{1 - F(r)} \tag{3.3}$$

for all $r \geq 0$ such that $F(r) < 1$.

For example, if X is a Poisson process then $F \equiv G$, so we obtain $J(r) \equiv 1$. Note that, even in a completely nonparametric context, the function J has an interpretation as the ratio of the survival functions of the distance to the nearest (other) point of X from (a) a point of the process, (b) a fixed arbitrary point. Values $J(r) < 1$ indicate that the survival function in situation (a) is smaller than that for (b), which may be interpreted as indicating 'clustered' pattern; values $J(r) > 1$ indicate 'ordered' pattern. In the examples in Section 4 we will reconcile this with other definitions of 'clustering' and 'ordering'.

Note that $J(0) = 1$ always. The denominator $1 - F$ is always absolutely continuous [3] but the numerator $1 - G$ need not be, so the discontinuity points of J are those of G. In general $1 - G(r)$ might be nonzero when $1 - F(r)$ is zero (e.g. for a randomly-translated unit square grid when $r = 1/\sqrt{2}$) but this does not occur for point processes of real interest.

Theorem 1 *Let X be a stationary point process with intensity λ whose Papangelou conditional intensity $\lambda(y; X)$ exists. Then $G(r) < 1$ implies $F(r) < 1$ and*

$$J(r) = \left(\mathbb{E}_0^! \left[\frac{\lambda}{\lambda(0;X)} \mid X \cap B(0,r) = \emptyset \right] \right)^{-1} \tag{3.4}$$

In particular, suppose X has 'interactions of finite range R' in the sense that $\lambda(0;X)$ is constant (and thus equal to $\lambda(0;\emptyset)$) for all point patterns X which contain no points in $B(0,R)$. Then

$$J(r) = \frac{\lambda(0;\emptyset)}{\lambda} \qquad \text{for } r \geq R. \tag{3.5}$$

Proof: Let A be the event $\{X \cap B(0,r) = \emptyset\}$, so that $1-F(r) = \mathbb{P}(A)$ and $1-G(r) = \mathbb{P}_0^!(A)$. Apply the Takacs-Fiksel formula (2.1) to

$$f(X) = \frac{\mathbb{1}_A}{\lambda(0;X)}, \quad r > 0$$

(cf. [41, (5.5.18), p. 159]). The right hand side of (2.1) is $\mathbb{E}[\lambda(0;X)\, f(X)] = 1 - F(r)$ giving

$$1 - F(r) = \lambda \mathbb{E}_0^! \left[\frac{\mathbb{1}_A}{\lambda(0;X)} \right]$$

Dividing this by $1 - G(r) = \mathbb{P}_0^!(A)$ gives the reciprocal of (3.4).

In the second case, if $\lambda(0;X) \equiv \lambda(0;\emptyset)$ on A then

$$f(X) \equiv \frac{\mathbb{1}_A}{\lambda(0;\emptyset)}$$

so that the left side of (2.1) is

$$\lambda \frac{1}{\lambda(0;\emptyset)} \mathbb{P}_0^!(A) = \frac{\lambda}{\lambda(0;\emptyset)}(1 - G(r))$$

yielding (3.5). □

Next we examine the behaviour of J under the basic point process operations of superposition and thinning.

Theorem 2 *Let X_1, X_2 be independent, stationary point processes with intensities λ_1, λ_2 and J-functions J_1, J_2 respectively. Then the J-function of the superposition $X = X_1 \cup X_2$ is a convex combination of the J-functions of the components:*

$$J(r) = \frac{\lambda_1}{\lambda_1 + \lambda_2} J_1(r) + \frac{\lambda_2}{\lambda_1 + \lambda_2} J_2(r). \tag{3.6}$$

Proof: By independence

$$1 - F(t) = (1 - F_1(t))(1 - F_2(t)).$$

Writing $P_{(i)}^0$ for the Palm distribution (on the entire probability space) with respect to X_i, $i = 1, 2$ and P^0 for the Palm distribution with respect to X, we have (e.g. [41, p. 116])

$$P^0 = \frac{\lambda_1}{\lambda_1+\lambda_2} P^0_{(1)} + \frac{\lambda_2}{\lambda_1+\lambda_2} P^0_{(2)}.$$

The joint distribution of X_1 and X_2 under $P^0_{(1)}$ is independent, with X_1 governed by its Palm distribution (the Palm distribution of its marginal distribution) and X_2 by its (ordinary) marginal distribution. Similarly for $P^0_{(2)}$. Hence

$$1 - G(t) = \frac{\lambda_1}{\lambda_1+\lambda_2}(1-G_1(t))(1-F_2(t)) + \frac{\lambda_2}{\lambda_1+\lambda_2}(1-F_1(t))(1-G_2(t)).$$

Dividing this by the identity for F gives (3.6). □

For comparison, the K-function of the superposition in the same situation is

$$\begin{aligned} K(t) &= \frac{\lambda_1}{\lambda_1+\lambda_2}\left[\frac{1}{\lambda_1+\lambda_2}(\lambda_2\pi t^2 + \lambda_1 K_1(t))\right] + \frac{\lambda_2}{\lambda_1+\lambda_2}\left[\frac{1}{\lambda_1+\lambda_2}(\lambda_1\pi t^2 + \lambda_2 K_2(t))\right] \\ &= \frac{1}{(\lambda_1+\lambda_2)^2}\left[2\lambda_1\lambda_2\pi t^2 + \lambda_1^2 K_1(t) + \lambda_2^2 K_2(t)\right]. \end{aligned}$$

Theorem 3 *Let X_p be the process obtained from a stationary point process X by randomly deleting or retaining each point independently of other points, with retention probability $p > 0$. Then the J-function of X_p is*

$$J_p(r) = \frac{Q^0_r(1-p)}{Q_r(1-p)} \tag{3.7}$$

where Q^0_r, Q_r are the generating functions of $n(X \cap B(0,r))$ under $\mathbb{P}^!_0$ and $\mathbb{P}$ respectively. [The J-function of X itself is the case $p = 1$.]

Proof : Let F_p, G_p be the F and G functions for X_p. Clearly $1 - F_p(r) = Q_r(1-p)$. To prove $1 - G_p(r) = Q^0_r(1-p)$ use the fact that the Palm distribution of X_p coincides with the effect of random p-thinning on the Palm distribution of X. □

Thus while the K-function is invariant under random thinning [14, p. 67], [41, p. 134], in general the J-function is not. There does not appear to be a simple general relationship between J_p and J.

4. Examples

4.1 Poisson process

For a stationary Poisson process of intensity λ we have $F \equiv G$ so that $J \equiv 1$. We could also derive this from Theorem 1 by observing that $\lambda(0; X) = \lambda$ for arbitrary X.

4.2 Pairwise-interaction Markov point process

For a pairwise interaction point process [36], [41, section 5.5] with activity constant β and interaction $\gamma(u,v)$ between points $u, v \in \mathbb{R}^k$,

$$\lambda(y; X) = \beta \prod_{x \in X} \gamma(x,y). \tag{4.8}$$

The process is *Markov* (in the Ripley-Kelly sense [36]) with interaction range R, if $\gamma(u,v) = 1$ when $||u - v|| \geq R$. Examples include the *hard core process* defined by

$$\gamma(u,v) = \begin{cases} 0 & \text{if } ||u-v|| < R \\ 1 & \text{otherwise} \end{cases} \tag{4.9}$$

and the *Strauss process* defined by replacing 0 in (4.9) by a constant $0 < \gamma < 1$.

Theorem 4 *For a Markov pairwise-interaction process with interaction range R,*

(a) $J(r)$ is defined for all r;

(b) $$J(r) = \frac{\beta}{\lambda} \quad \text{for } r \geq R; \tag{4.10}$$

(c) for 'purely inhibitive' interactions ($\gamma(u,v) \leq 1$ for all u,v) we have $J(r) \geq 1$ for all r;

(d) for the hard core process $J(r) = 1/(1-F(r))$ for $r < R$, and in particular J is continuous and monotone increasing for $r < R$. Furthermore $J(r) = 1/(1 - \lambda m(B(0,r)))$ for $r < R/2$.

Thus, the hard-core and Strauss processes yield values (for r outside the interaction range) indicating 'ordered' pattern in the sense defined below Definition 1. Equation (4.10) was implicitly computed in [41, (5.5.18), p. 159] for the value $r = R$ only.

Proof : To prove this we note that the product in (4.8) depends only on points $x \in X$ with $||x - y|| \leq R$, so $\lambda(y; X)$ depends only on $X \cap B(y, R)$. Hence X has finite range interaction in the sense of Theorem 1(b) with $\lambda(0; \emptyset) = \beta$, and we get (4.10).

Note that λ, the intensity of X, is determined by the parameters β and $\gamma(\cdot,\cdot)$ in a generally complex way. However for a purely inhibitive process we have $\lambda(0; X) \leq \beta$ a.s. so that $\lambda \leq \beta$ using (2.2). This gives $J(r) \geq 1$ for $r \geq R$.

For values $r < R$ it is again a complex task to compute $J(r)$, except that for purely inhibitive γ we can again show that $J(r) \geq 1$ for all r. For a hard core process, clearly $G(r) = 0$ for $r < R$, so $J(r) = 1/(1 - F(r))$ for $r < R$. In particular J is monotone nonincreasing. Furthermore since spheres of radius $r < R/2$ centred at the points of a hard core process do not overlap, we have $F(r) = \lambda m(B(0,r))$ for $r < R/2$, and hence $J(r) = 1/(1 - \lambda m(B(0,r)))$, for $r < R/2$. □

4.3 Markov point processes (general)

Many of the arguments in the preceding paragraph carry over to Markov point processes in general. A stationary process X is Markov with finite interaction range R if its conditional intensity $\lambda(0; X)$ at 0 exists and depends only on $X \cap B(0, R)$. It follows that for any X satisfying $X \cap B(0, R) = \emptyset$

$$\lambda(0; X) = \lambda(0; \emptyset)$$

so that Theorem 1(b) applies and $J(r)$ is constant for $r > R$.

An example of interest is the *area-interaction process* [4] for which

$$\lambda(0; X) = \beta\gamma^{-m(B(0,t)\setminus U(X))} \tag{4.11}$$

where m is Lebesgue measure and $U(X) = \bigcup_{x\in X} B(x,t)$ for a fixed $t > 0$. The process is defined for all finite γ, with $\gamma < 1$ generating 'ordered' patterns and $\gamma > 1$ 'clustered' patterns. For any X such that $X \cap B(0,2t) = \emptyset$ we have $U(X) \cap B(0,t) = \emptyset$ so that $\lambda(0,X) = \beta\gamma^{-m(B(0,t))} = \beta\eta$, say. Thus Theorem 1(b) applies with $R = 2t$, and

$$J(r) = \frac{\beta\eta}{\lambda} \qquad \text{for } r > 2t. \tag{4.12}$$

Since $0 \le m(B(0,t) \setminus U(X)) \le m(B(0,t))$ we have for $\gamma < 1$ that $\lambda(0;X) \le \beta\eta$ for all X so that $\lambda \le \beta\eta$ and hence $J(r) \ge 1$ for all r, i.e. this is also 'ordered' in terms of J. Similarly, for $\gamma > 1$ we have $\lambda(0;X) \ge \beta\eta$ a.s. so that $\lambda \ge \beta\eta$ and $J(r) \le 1$ for all r, i.e. this is 'clustered' in terms of J.

4.4 Poisson cluster processes

A stationary Poisson cluster process is constructed by generating a stationary Poisson process Y of 'parent points'; generating i.i.d. finite point processes ('clusters') Z_y for each $y \in Y$; and forming the superposition $X = \bigcup_{y\in Y}(y + Z_y)$ of the translated clusters. *Neyman-Scott* processes are the special case where the typical cluster Z consists of a random number N of i.i.d. points. The *Matérn* cluster process is the further special case of Neyman-Scott processes where N is a Poisson variable and the common distribution of the cluster points is uniform over the ball of radius t centred on the parent point.

Stoyan et al. [41, p. 142 ff.] (and Bartlett [10, p. 8–9]) show that for any stationary Poisson cluster process

$$1 - G(r) = [1 - F(r)]\, C_0\{Z \cap B(0,r) = \{0\}\}, \qquad r \ge 0$$

where C_0 is the Palm distribution of the typical cluster Z. This follows from a fundamental identity for the Palm distribution of a Poisson cluster process [41, (5.3.2), p. 142].

Since Z is a finite point process, C_0 can be interpreted as the $n(Z)$-weighted distribution of $Z - z$ where, given Z, z is one of the points of Z chosen with equal probability. Hence we may interpret $C_0\{Z \cap B(0,r) = \{0\}\}$ as the defective distribution function of the distance from a typical point of Z to the nearest other point of Z, if any [41, p. 143]. Hence we have the following result.

Theorem 5 *For any stationary Poisson cluster process, $J(r)$ is defined for all $r \ge 0$;*

$$J(r) = C_0\{Z \cap B(0,r) = \{0\}\}$$

is a monotone nonincreasing function, with values $J(r) \le 1$, determined only by the distribution of the clusters.

If the typical cluster Z is a.s. contained within the ball of radius t around the parent point, then $J(r)$ is constant for $r > 2t$ where it is equal to $\mathbb{P}\{n(Z) = 1\}/\mathbb{E}n(Z)$.

Hence all stationary Poisson cluster processes are 'clustered' with respect to J as defined below Definition 1.

For example, for the Matérn cluster process in $\mathbb{R}^2$ with $\mathbb{E}n(Z) = \mu$ points per cluster we find

$$\begin{aligned} J(r) &= \frac{1}{m(B(0,t))} \int_{B(0,t)} e^{-\mu V(x,r,t)} dx \\ &= e^{-\mu} \qquad \text{for } r > 2t \end{aligned}$$

where $V(x,r,t) = m(B(x,r) \cap B(0,t))/m(B(0,t))$.

Note that Theorem 5 is proved using the cluster formula [41, (5.3.2), p. 142] rather than Theorem 1, and holds even in cases when the Papangelou conditional intensity does not exist. However, if the Palm distribution of the typical cluster Z is absolutely continuous with respect to the distribution of Z, then Theorem 1 applies and yields the conclusions of Theorem 5.

This result is perhaps less surprising in view of the recent proof [5] that Poisson cluster processes with bounded clusters are nearest-neighbour Markov processes in the sense of [6].

4.5 Cox processes

Cox point processes are constructed by generating a random measure Λ and, conditional upon Λ, generating an inhomogeneous Poisson point process X with intensity measure Λ.

Theorem 6 *Let X be a Cox point process with driving random measure Λ which is stationary and a.s. nonatomic. Then the J-function of X is defined for all $r \geq 0$ and equals*

$$J(r) = \frac{\mathbb{E}^0 e^{-\Lambda(B(0,r))}}{\mathbb{E} e^{-\Lambda(B(0,r))}}$$

where $\mathbb{E}^0$ denotes expectation with respect to the Palm distribution of Λ.

This follows from the fact that the reduced Palm distribution of X is the distribution of a Cox process with driving measure distributed as the Palm distribution of Λ, cf. [41, p. 141].

For example, consider a mixed Poisson process, where Λ is a random constant multiple of Lebesgue measure, $\Lambda = \alpha m(\cdot)$ where α is any nonnegative random variable not identically equal to zero. Then the Palm distribution of Λ is simply the α-weighted distribution, and

$$J(r) = \frac{\mathbb{E}\left[\alpha e^{-\alpha \pi r^2}\right]}{\mathbb{E}\alpha \, \mathbb{E} e^{-\alpha \pi r^2}}.$$

5. Statistical aspects

5.1 Nonparametric estimation of J

Edge-corrected estimators for F and G based on observations of X within a bounded window $W \subset \mathbb{R}^k$ are reviewed in [35, chap. 3], [41, pp. 122–131], [11, chap. 8]. For recent variations see [3, 7, 8, 16, 17, 18, 19, 20, 21, 23, 39].

We propose estimating J by plugging into (3.3) estimates of F and G obtained by methods that are comparable to one another. For example one may estimate F by the standard 'border correction' estimator [35, chap. 3] and G by Hanisch's border correction estimator $\widehat{G}_4$ [26] (see [41, p. 128] where G is called D). These are Horvitz-Thompson type ratio estimators with comparable denominators, and are pointwise unbiased for F and pointwise approximately

unbiased for G, respectively. Alternatively the Kaplan-Meier style estimators of F and G proposed by Baddeley & Gill [3] could be used. These have the advantage of being proper distribution functions (possibly defective), and correspond to unbiased and approximately unbiased estimators of the hazard measures of F and G, respectively. Furthermore the estimator of F in [3] has the same continuity properties as F itself.

We know little about the sampling properties of either estimator of J. Clearly $\widehat{J}(0) = 1$ always. It seems plausible that the relative error of J will increase with r, since this is true of standard estimators of $F(r)$ and $G(r)$ [3, 18, 20, 21]. Central limit theorems have been proved for $\widehat{F}$ and $\widehat{G}$ of both the Horvitz-Thompson and Kaplan-Meier types under various regimes [2, 3, 27, 28, 40], [11, p. 480]; a joint CLT for $(\widehat{F}, \widehat{G})$, and hence for $\widehat{J}$, seems plausible but has not been established to the authors' knowledge.

Edge effects have a far greater influence on $\widehat{G}$ than on $\widehat{F}$ [35, chap. 3],[3]. The sampling properties of $\widehat{G}$ and therefore of $\widehat{J}$ may be particular cause for concern when the sampling window W is irregular, or in dimensions higher than two [3, 7].

5.2 Estimation and inference based on J

In section 4 we were able to calculate the J function (at least for $r > R$) for a number of parametric or semi-parametric stochastic models. One could use these results to estimate the parameters of a chosen model from values of $\widehat{J}$.

It should be stressed that this approach is merely a special case of the Takacs-Fiksel estimation method [22, 24, 42, 43], [35, p. 54–55], [15, §2.4], [37, 38] since the basic equations (3.4)–(3.5) are special cases of (2.1) with the choice of f given in the proof of Theorem 1.

For a Markov pairwise-interaction process, (4.10) gives the constant value of $J(r)$ for all $r > R$ in terms of the parameter β and the intensity λ. The intensity is determined by β and by the interaction function $\gamma(\cdot, \cdot)$ in a complex way. However λ may be estimated directly from the data, as $\widehat{\lambda} = n(X \cap W)/m(W)$ in the usual way. If R is assumed known then β can be estimated via (4.10). This is semi-parametric estimation, since γ is unknown apart from the constraint that $\gamma(u, v) = 1$ for $||u - v|| > R$.

Similarly, for an area-interaction process, (4.12) allows us to estimate the parameters β and $\eta = \gamma^{m(B(0,t))}$ given the interaction radius $R = 2t$.

Estimation of the interaction distance R, in any of the models studied, amounts to estimating the largest interval $[R, \infty)$ on which J is constant. At present we have only the *ad hoc* suggestion of taking

$$\widehat{R} = \inf \left\{ R : \sup_{r \geq R} \widehat{J}(r) - \inf_{r \geq R} \widehat{J}(r) < \epsilon \right\}$$

where ϵ is of order $n(X \cap W)^{-1/2}$.

6. Examples

We have taken three standard point pattern datasets discussed at length by Diggle [14], entitled **pines** ('Japanese pine saplings'), **redwood** ('Californian redwood seedlings') and **cells** ('biological cells'). These were exhibited as typical examples of random, clustered, and ordered patterns respectively.

Figures 1–3 show the data and corresponding estimates $\widehat{J}$ obtained using the Kaplan-Meier estimators of F and G [3]. For **pines** the value of J is close to 1 for almost the entire range of r

values expect at high r values; for `redwood` it is below 1 and monotonically decreasing except for small fluctuations; and for `cells` it is above 1 for the entire range and is monotonically increasing. These results are consistent with our expectations.

We may conclude provisionally that the J-statistic is a useful indicator of the type of spatial pattern. Further numerical experiments will be described elsewhere.

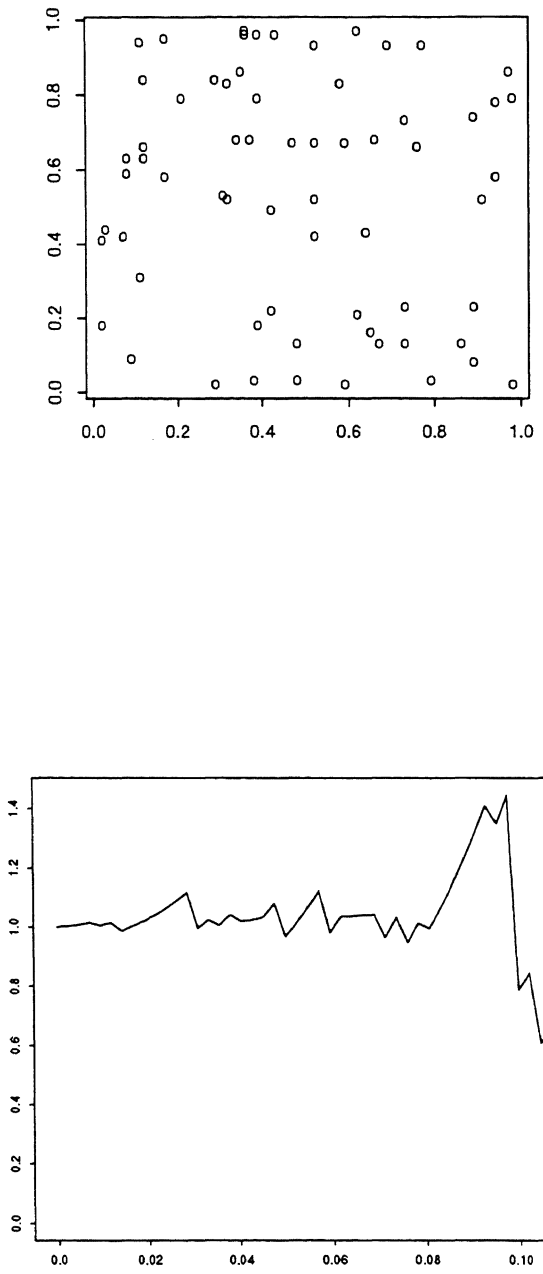


Figure 1: **pines** data (above) and estimate of J (below).

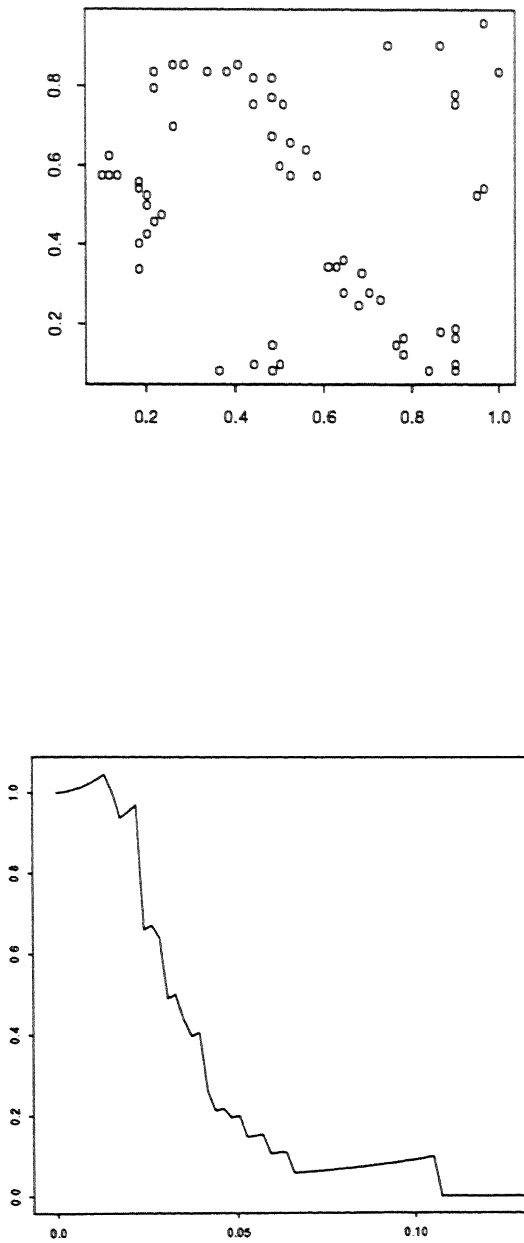
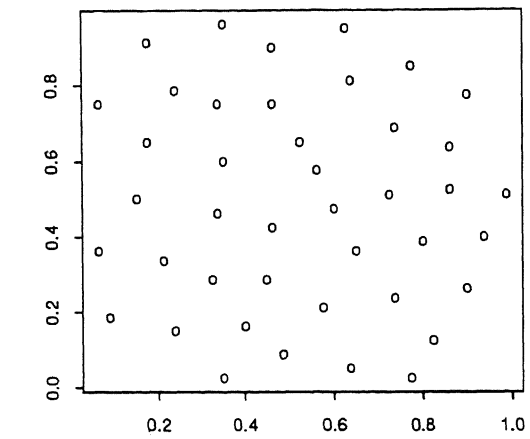


Figure 2: **redwood** data (above) and estimate of J (below).



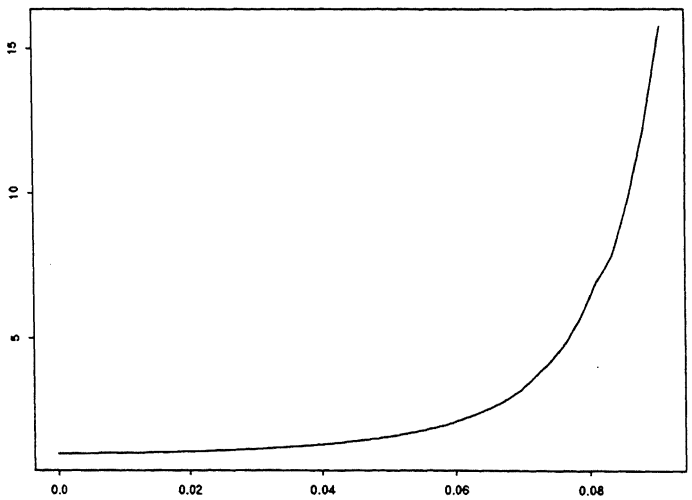


Figure 3: `cells` data (above) and estimate of J (below).

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