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INHOMOGENEOUS SPATIAL POINT PROCESSES BY LOCATION DEPENDENT SCALING

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Abstract

A new class of models for inhomogeneous spatial point processes is introduced. These locally scaled point processes are modifications of homogeneous template point processes, having the property that regions with different intensity differ only by a scale factor, i.e. appear to be scaled versions of the template point process. This is achieved by replacing volume measures used in the density with locally scaled analogues defined by a location dependent scaling function. If the scaling function is constant, then local scaling coincides with global scaling by a constant factor. The new approach is particularly appealing for modelling inhomogeneous Markov point processes. Distance-interaction and shot noise Markov point processes are discussed in detail. It is shown that the locally scaled versions are again Markov and that locally the Papangelou conditional intensity of the new process behaves like that of a global scaling of the homogeneous process. Approximations are suggested that simplify calculation of the density e.g. in simulation. For sequential point processes, an alternative and simpler definition of local scaling is proposed.

Keywords: Point processes; inhomogeneity; local scaling; sequential point processes

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1. Introduction

Point patterns with non-homogeneous intensity are observed quite frequently in nature and technology. For example, the number of trees per unit area in a forest depends on environmental conditions and therefore maps showing tree locations usually look inhomogeneous. In plant and animal tissue, cell size and, correspondingly, cell number often depend on the distance to the boundary of an organ. Many modern materials are designed with structural inhomogeneity, imitating natural structures in order to improve functional properties. An example is the bronze sinter filter shown in Fig. 1. The data have earlier been analysed in Hahn et al. [5]. The filter consists of almost spherical bronze particles with diameters that decrease along an axis which marks the filtering direction. Since the particles are densely packed, the number of particles per unit volume increases as the diameters decrease. This is also observable on sections parallel to the directions of inhomogeneity: The centers of the particle section profiles form an inhomogeneous point pattern.

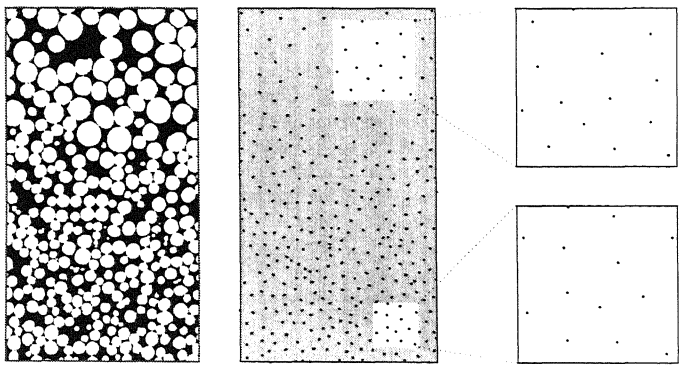

FIGURE 1: Left part: Section of a bronze sinter filter with a gradient in particle size and number. Right part: centers of the particle profiles. Two enlargements from top and bottom part, containing about the same number of points, show similar geometry.

While it is easy to model inhomogeneous point patterns with independently positioned points by inhomogeneous Poisson point processes, situations as shown in Fig. 1 require more sophisticated approaches. This pattern is characterized by repulsive interaction between the points due to the fact that it

results from a packing of spheres. The packing is of similar volume fraction, and similar geometry in regions with larger and with smaller sphere diameters. Therefore, regions with large sphere diameters look like scaled versions of regions with small diameters and vice versa. A similar effect can often be observed in nature, e.g. in plant communities where number density is governed by environmental conditions. For example desert plants tend to form regular patterns with varying scale, such that distances between plants are smaller in densely covered regions. Such point patterns appear also homogeneous up to a local scale factor.

In recent years, various models have been suggested for inhomogeneous point processes with interaction. Since Markov point processes are very useful for modelling interaction in homogeneous point patterns it is natural that they are used as starting points for inhomogeneous models. The survey by Jensen and Nielsen [8] discusses three ways of introducing inhomogeneity into a Markov model. As will be explained in more detail in Section 2, homogeneous Markov point processes are defined by a density with respect to the unit rate Poisson point process. A straightforward idea is therefore to define an inhomogeneous process by the same density (up to a constant factor) but with respect to an inhomogeneous Poisson point process (Stoyan and Stoyan [15], Ogata and Tanemura [13]). Inhomogeneity can also be obtained by location dependent thinning (Baddeley et al. [1]), or by transformation of a homogeneous Markov point process (Jensen and Nielsen [7]).

In these three cases, the local geometry of the point pattern changes with intensity. This is illustrated in Fig. 2, which shows realizations of inhomogeneous hard core point processes obtained by the three methods. In order to obtain patterns that appear homogeneous up to a scale factor, range and strength of interaction have to be adapted to intensity. However this is not accomplished by the first approach where the interaction between points does not depend on their locations, cf. Fig. 2a. Thinning on the other hand in general destroys the interaction structure. This leads to a Poisson like appearance of sparse regions, see Fig. 2b. Transformation of coordinates finally not only introduces inhomogeneity but also local anisotropy, as shown in Fig. 2c. Therefore these three approaches are not suitable for modelling situations as given in Fig. 1.

In the present paper we propose alternative inhomogeneous point process models that aim to preserve local geometry. As in the three approaches discussed above, the inhomogeneous model is obtained by modifying a homogeneous "template" process that yields the interaction. The idea is that

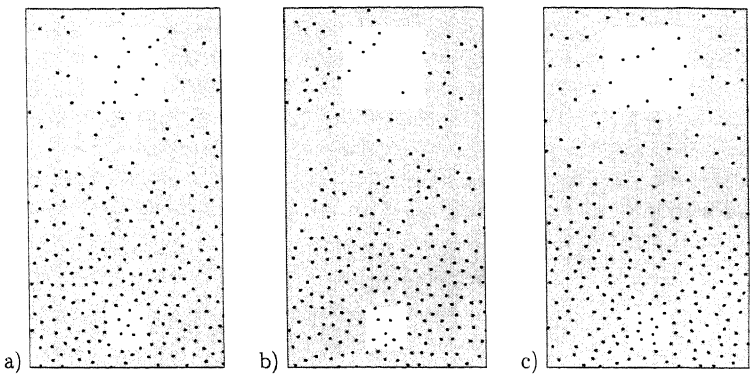

FIGURE 2: Inhomogeneous hard core point patterns obtained by a) defining the density with respect to an inhomogeneous Poisson point process, b) inhomogeneous independent thinning, c) transformation of coordinates. Note that dense and sparse regions differ in geometry. The parameters were chosen such that the processes have similar intensity as the example of Figure 1.

inhomogeneity is obtained by scaling the template process with a location dependent scaling factor. A large scaling factor hereby results in low intensity and large interaction distances, whereas a small scaling factor yields high intensity and small interaction range. In regions with constant scaling factor the point process should locally behave like a scaled version of the template, see Fig. 3.

The method and results presented in this paper are applicable to homogeneous template processes that are given by a density with respect to a homogeneous Poisson point process; however the main emphasis will be on Markov point processes. The definition of Markov point processes and other prerequisites are recalled in Section 2.

Calculating the density function of a point process for a given point pattern usually implies evaluating distances, areas, etc. The local scaling model proposed in Section 3 changes the way such quantities are measured according to a location dependent scaling function.

Sections 4 and 5 give a closer look on the important classes of distance-interaction and shot noise processes. In particular we show that locally scaled Markov point processes are again Markov, now with respect to a location

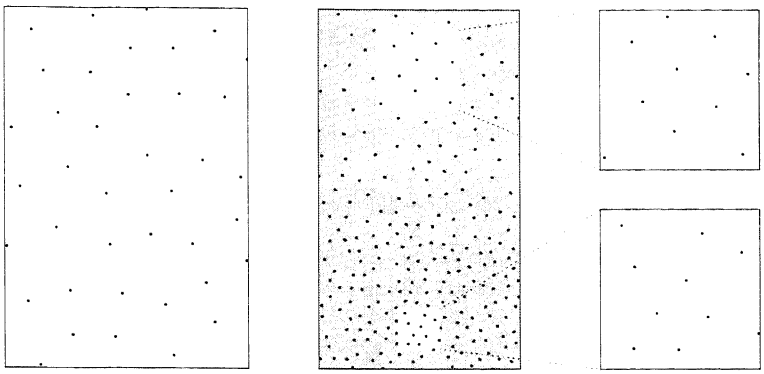

FIGURE 3: Homogeneous template hard core process (left part) and inhomogeneous process obtained by local scaling. Enlargements from dense and sparse regions of the inhomogeneous pattern look similar to the template pattern.

dependent relation. Useful approximations of local scaling simplifying calculations e.g. in simulation are presented in Section 6.

For the class of sequential point processes, another approach to obtain local scaling by means of conditional intensities is suggested in Section 7.

The paper concludes with a critical discussion.

2. Prerequisites

Let $\mathbb{B}^k$ denote the set of all full-dimensional bounded subsets of $\mathbb{R}^k$ and write $\mathcal{B}_k$ for the Borel σ-algebra on $\mathbb{R}^k$. We consider finite point processes X on sets $\mathcal{X} \in \mathbb{B}^k$. A point process X on $\mathcal{X}$ is a random variable taking values in $\Omega_{\mathcal{X}}$, the set of all finite subsets $\mathbf{x} = \{x_1, \cdots, x_n\}$ of $\mathcal{X}$, equipped with the smallest σ-algebra for which the number of points placed in a Borel set $B \subseteq \mathcal{X}$ is a random variable.

We will concentrate on point processes X that have a density f_X with respect to the restriction of the unit rate Poisson point process Π to $\mathcal{X}$. A point process X on $\mathcal{X}$ is called *homogeneous* if f_X is the restriction to $\mathcal{X}$ of a translation invariant function defined on all finite subsets of $\mathbb{R}^k$, see [12].

Markov point processes in the sense of Ripley and Kelly [14] are particularly useful for modelling point patterns with interaction. They are defined with respect to a symmetric and reflexive relation $\sim$ on $\mathcal{X}$. Two points $x_1, x_2 \in \mathcal{X}$

are said to be neighbours if $x_1 \sim x_2$, and a finite subset $\mathbf{x} \subset \mathcal{X}$ is called a clique if all points in $\mathbf{x}$ are neighbours.

By the Hammersley-Clifford theorem (Ripley and Kelly [14]), a Ripley-Kelly Markov point process X has density with respect to the unit rate Poisson point process of the form

$$f_X(\mathbf{x}) = \prod_{\mathbf{y} \subseteq \mathbf{x}} \varphi(\mathbf{y}), \quad \mathbf{x} \in \Omega_{\mathcal{X}}, \tag{1}$$

where φ is an interaction function, i.e. $\varphi(\mathbf{y}) = 1$ when the set $\mathbf{y}$ is not a clique. We will always assume that the interaction function φ is defined on *all* finite subsets of $\mathbb{R}^k$. A Markov point process X is thereby homogeneous if φ is translation invariant (for a proof see Nielsen [11, p. 29]).

For a Markov point process, the Papangelou conditional intensity

$$\lambda_X(x \mid \mathbf{x}) = \begin{cases} \frac{f_X(\mathbf{x} \cup \{x\})}{f_X(\mathbf{x})}, & f_X(\mathbf{x}) > 0, \qquad x \notin \mathbf{x}, \\ 0 & \text{otherwise}, \end{cases}$$

depends only on those points in $\mathbf{x}$ which are neighbours of x. If we let dx be an infinitesimal region around x and $\nu^k(dx)$ the k-dimensional volume (Lebesgue measure) of dx, then $\lambda(x \mid \mathbf{x})\nu^k(dx)$ can be interpreted as the conditional probability of finding a point from the process in dx given the configuration elsewhere is $\mathbf{x}$, cf. e.g. Van Lieshout [9].

Before defining local scaling of point processes let us consider global scaling with a constant factor $c > 0$. As a transformation of coordinates, global scaling maps a point process X on $\mathcal{X}$ to a process $X_c = cX$ on the set $c\mathcal{X} = \{x : x/c \in \mathcal{X}\}$, see also Jensen and Nielsen [7].

The unit rate Poisson point process Π on $\mathcal{X}$ with intensity measure ν^k is transformed into a Poisson point process Π_c on $c\mathcal{X}$ with intensity measure $c^{-k}\nu^k$.

Let f_X be the density of the original process with respect to Π. Then the scaled process X_c has density $f_{X_c}^{(c)}$ with respect to Π_c,

$$f_{X_c}^{(c)}(\mathbf{x}) = f_X(\mathbf{x}/c). \tag{2}$$

(The superscript (c) in $f_{X_c}^{(c)}$ is used to indicate that the density is with respect to Π_c instead of with respect to Π.) The conditional intensity associated with $f_{X_c}^{(c)}$ is

$$\lambda_{X_c}^{(c)}(x \mid \mathbf{x}) = \lambda_X\left(\frac{x}{c} \mid \frac{\mathbf{x}}{c}\right). \tag{3}$$

The density of X_c with respect to Π is

$$f_{X_c}(\mathbf{x}) = e^{(1-c^{-k})\nu^k(c\mathcal{X})} c^{-kn(\mathbf{x})} f_{X_c}^{(c)}(\mathbf{x}),$$

where $n(\mathbf{x})$ is the number of points in $\mathbf{x}$.

3. Local scaling of homogeneous point processes

In this section, we give a general definition of a locally scaled version of a homogeneous template process.

The concept of scale invariance plays a crucial role in the definition. This concept relates to global scaling with a constant factor $c > 0$. Note that under scaling with factor c, a measure μ on $(\mathbb{R}^k, \mathcal{B}_k)$ is transformed into μ_c where $\mu_c(A) = \mu(c^{-1}A), A \in \mathcal{B}_k$.

Definition 1. Let $g(\mathbf{x}; \mu^*)$ be a real-valued measurable function defined on $\Omega_{\mathbb{R}^k}$, depending on a set $\mu^* = (\mu^1, \cdots, \mu^m)$ of measures on $(\mathbb{R}^k, \mathcal{B}_k)$. The function g is called *scale invariant* if for all $\mathbf{x} \in \Omega_{\mathbb{R}^k}$ and all $c > 0$

$$g(c\mathbf{x}; \mu_c^*) = g(\mathbf{x}; \mu^*), \tag{4}$$

where $\mu_c^* = (\mu_c^1, \cdots, \mu_c^m)$.

The classical homogeneous point process models that appear in the spatial statistics literature have a density which is the restriction to $\Omega_{\mathcal{X}}$ of a scale-invariant function $g(\cdot; \mu^*)$, where $\mu^* = \nu^* = (\nu^0, \cdots, \nu^k)$ is the set of d-dimensional volume (Hausdorff) measures ν^d in $\mathbb{R}^k$, $d = 0, 1, \cdots, k$. A comprehensive set of examples will be given in the sections to follow. Recall that ν^0 is the counting measure, thus $\nu^0(\mathbf{x}) = n(\mathbf{x})$, and ν^1 is the length measure in $\mathbb{R}^k$. Note also that $\nu_c^d(A) = \nu^d(c^{-1}A) = c^{-d}\nu^d(A)$, $A \in \mathcal{B}_k$.

Under local scaling, the constant scaling factor c is replaced by a non constant location dependent scaling function $c : \mathbb{R}^k \to \mathbb{R}_+$. The globally scaled measures ν_c^d can easily be extended to this case.

Definition 2. Let c be a positive Borel measurable function on $\mathbb{R}^k$. Then the *locally scaled d-dimensional volume measure* ν_c^d is defined by

$$\nu_c^d(A) = \int_A c(u)^{-d} \nu^d(du), \tag{5}$$

for all $A \in \mathcal{B}_k$.

In the following, we will assume that c is bounded from below and above, i.e. there exist $0 < \underline{c} \leq \bar{c} < \infty$ such that

$$\underline{c} \leq c(u) \leq \bar{c}, \ u \in \mathbb{R}^k.$$

This assumption implies in particular that $\nu_c^d(A) < \infty$ whenever $\nu^d(A) < \infty$.

We can now present the definition of locally scaled point processes.

Definition 3. Let X be a homogeneous point process on $\mathcal{X}$, with density f_X with respect to Π of the form

$$f_X(\mathbf{x}) \propto g(\mathbf{x}; \nu^*), \ \mathbf{x} \in \Omega_{\mathcal{X}},$$

where g is scale invariant. Let c be a positive, Borel measurable function in $\mathbb{R}^k$ and let Π_c be the Poisson point process with the locally scaled volume measure ν_c^k as intensity measure. Let $\mathcal{X}' \in \mathbb{B}^k$ be arbitrary and suppose that $g(\cdot; \nu_c^*)$ is integrable on $\Omega_{\mathcal{X}'}$ with respect to Π_c. A *locally scaled point process* X_c on $\mathcal{X}'$ with template X is defined by the following density with respect to Π_c,

$$f_{X_c}^{(c)}(\mathbf{x}) \propto g(\mathbf{x}; \nu_c^*), \quad \mathbf{x} \in \Omega_{\mathcal{X}'}, \tag{6}$$

where ν_c^* is the set of locally scaled volume measures.

If $c : \mathbb{R}^k \to \mathbb{R}_+$ is constant, $c(u) \equiv c$, say, then the density with respect to Π_c of the scaled process on $\mathcal{X}' = c\mathcal{X}$ becomes

$$f_{X_c}^{(c)}(\mathbf{x}) \propto g(\mathbf{x}; \nu_c^*) = g(c^{-1}\mathbf{x}; \nu^*), \quad \mathbf{x} \in \Omega_{\mathcal{X}'}.$$

Local scaling with a constant scaling function is thereby equivalent to global scaling. In the general case where c is non constant, local scaling does not necessarily correspond to a mapping. Therefore there is no natural choice of $\mathcal{X}'$ which is related to $\mathcal{X}$, and the set $\mathcal{X}'$ can be arbitrary. In particular, we may choose $\mathcal{X}' = \mathcal{X}$. Note that the density of the locally scaled process X_c with respect to the unit rate Poisson point process Π is

$$f_{X_c}(\mathbf{x}) \propto \prod_{x \in \mathbf{x}} c(x)^{-k} \cdot f_{X_c}^{(c)}(\mathbf{x}). \tag{7}$$

The locally scaled processes are Markov with respect to a suitably chosen relation $\sim_c$, which in general is different from the template relation $\sim$, see the Appendix A. Local scaling of two general Markov model classes, distance-interaction processes and shot noise processes, is discussed in detail in Sections

4 and 5. For these classes, conditions on the scaling function which ensure integrability of $g(\cdot, \nu_c^*)$ will be given, and it will be shown that the Papangelou conditional intensity of the locally scaled process,

$$\lambda_{X_c}^{(c)}(x \mid \mathbf{x}) = \begin{cases} \dfrac{f_{X_c}^{(c)}(\mathbf{x} \cup \{x\})}{f_{X_c}^{(c)}(\mathbf{x})}, & f_{X_c}^{(c)}(\mathbf{x}) > 0, \\ 0 & \text{otherwise}, \end{cases} \tag{8}$$

satisfies a local analogue of (3),

$$\lambda_{X_c}^{(c)}(x \mid \mathbf{x}) = \lambda_X\Big(\frac{x}{c(x)} \,\Big|\, \frac{\mathbf{x}}{c(x)}\Big), \tag{9}$$

if c is constant in a $\sim_c$-neighbourhood of x. The locally scaled processes thereby behave locally like a scaled version of the template process and the local geometry is preserved. In particular, if the template is locally isotropic in the sense that $\lambda_X(x \mid \cdot)$ is invariant under rotations around x, then so is the locally scaled process. Indeed, let R_x be a rotation around x. Then

$$\begin{aligned} \lambda_{X_c}^{(c)}(x \mid R_x\mathbf{x}) &= \lambda_X\left(\frac{x}{c(x)} \,\middle|\, \frac{R_x\mathbf{x}}{c(x)}\right) = \lambda_X\left(\frac{x}{c(x)} \,\middle|\, R_{\frac{x}{c(x)}}\left(\frac{\mathbf{x}}{c(x)}\right)\right) = \\ &= \lambda_X\left(\frac{x}{c(x)} \,\middle|\, \frac{\mathbf{x}}{c(x)}\right) = \lambda_{X_c}^{(c)}(x \mid \mathbf{x}). \end{aligned} \tag{10}$$

4. Distance-interaction processes

Consider the important class of *distance-interaction processes* X, where higher order interactions are functions of pairwise distances. Thus these point processes have densities of the form

$$f_X(\mathbf{x}) = \prod_{\mathbf{y} \subseteq \mathbf{x}} \varphi\big(D(\mathbf{y})\big), \tag{11}$$

where $D(\mathbf{y}) = \mathbf{y}$ if $n(\mathbf{y}) < 2$, and for $n(\mathbf{y}) \geq 2$

$$D(\mathbf{y}) = \{\nu^1([u, v]) : \{u, v\} \subseteq \mathbf{y},\ u \neq v\}$$

denotes the set of all pairwise distances of points in $\mathbf{y}$, and $[u, v]$ is the line segment connecting the points u and v. Distance-interaction processes include pairwise interaction processes, such as the hard-core process and the Strauss process, as well as higher order processes, e.g. the triplets process (Geyer [4]).

Assume that $\varphi(\{x\}) = \beta$ and that $\varphi\big(D(\mathbf{y})\big) = 1$ for $n(\mathbf{y}) \geq 2$ unless $\nu^1([u,v]) \leq r$ for all $\{u,v\} \subseteq \mathbf{y}$. The process X is thereby homogeneous and Markov with respect to the relation

$$u \sim v \Longleftrightarrow \nu^1([u,v]) \leq r.$$

According to (11),

$$f_X(\mathbf{x}) \propto g(\mathbf{x};\nu^*), \quad \mathbf{x} \in \Omega_{\mathcal{X}},$$

where

$$g(\mathbf{x};\nu^*) = \beta^{\nu^0(\mathbf{x})} \prod_{\mathbf{y} \subseteq_2 \mathbf{x}} \varphi\big(\{\nu^1([u,v]) : \{u,v\} \subseteq \mathbf{y}, u \neq v\}\big)$$

and $\mathbf{y} \subseteq_2 \mathbf{x}$ is short for $\{\mathbf{y} \subseteq \mathbf{x} : \nu^0(\mathbf{y}) \geq 2\}$. The function g is clearly scale invariant since for any constant $c > 0$

$$\begin{aligned}
&\prod_{\mathbf{y} \subseteq_2 c\mathbf{x}} \varphi\big(\{\nu_c^1([u,v]) : \{u,v\} \subseteq \mathbf{y}\}\big) \\
&= \prod_{c^{-1}\mathbf{y} \subseteq_2 \mathbf{x}} \varphi\big(\{\nu^1([c^{-1}u, c^{-1}v]) : \{u,v\} \subseteq \mathbf{y}\}\big) \\
&= \prod_{\mathbf{y} \subseteq_2 \mathbf{x}} \varphi\big(\{\nu^1([u,v]) : \{u,v\} \subseteq \mathbf{y}\}\big).
\end{aligned}$$

If $\varphi\big(D(\mathbf{y})\big) \leq 1$ for $n(\mathbf{y}) \geq 2$, then X is repulsive since each clique $\mathbf{y} \subseteq \mathbf{x}$ contributes a penalty $\varphi\big(D(\mathbf{y})\big)$ to the density. In this case,

$$g(\mathbf{x};\nu_c^*) \leq \beta^{n(\mathbf{x})}$$

for any scaling function $c : \mathbb{R}^k \to \mathbb{R}_+$. Hence, $g(\cdot;\nu_c^*)$ is integrable on $\Omega_{\mathcal{X}'}$ for any $\mathcal{X}' \in \mathbb{B}^k$, and therefore, locally scaled versions of such processes do exist. Otherwise, integrability has to be proved case by case and may require certain restrictions on the scaling function c. The locally scaled process has a density of the form

$$f_{X_c}^{(c)}(\mathbf{x}) \propto \beta^{n(\mathbf{x})} \prod_{\mathbf{y} \subseteq_2 \mathbf{x}} \varphi\big(D_c(\mathbf{y})\big), \quad \mathbf{x} \in \Omega_{\mathcal{X}'},$$

where

$$D_c(\mathbf{y}) = \{\nu_c^1([u,v]) : \{u,v\} \subseteq \mathbf{y} \;\; u \neq v\}.$$

Example (Strauss process). A Strauss process X on $\mathcal{X} \subseteq \mathbb{R}^k$ with intensity parameter $\beta > 0$, interaction parameter $\gamma \in [0,1]$ and interaction distance r

has density

$$f_X(\mathbf{x}) \propto \beta^{n(\mathbf{x})}\gamma^{s(\mathbf{x})}, \quad s(\mathbf{x}) = \sum_{\{u,v\}\subseteq\mathbf{x}}^{\neq} \mathbf{1}\left(\nu^1([u,v]) \leq r\right), \quad \mathbf{x} \in \Omega_{\mathcal{X}},$$

where $s(\mathbf{x})$ is the number of r-close pairs in $\mathbf{x}$ (Strauss [16]). (The superscript $\neq$ in the summation indicates that u and v are different.) For $\gamma = 0$ we obtain the hard core process, for $\gamma = 1$ the Poisson point process with intensity β. The locally scaled Strauss process has density

$$f_{X_c}^{(c)}(\mathbf{x}) \propto \beta^{n(\mathbf{x})}\gamma^{s_c(\mathbf{x})}, \quad s_c(\mathbf{x}) = \sum_{\{u,v\}\subseteq\mathbf{x}}^{\neq} \mathbf{1}\left(\nu_c^1([u,v]) \leq r\right), \quad \mathbf{x} \in \Omega_{\mathcal{X}'}.$$

Figure 4 shows a realization; for details see the figure caption.

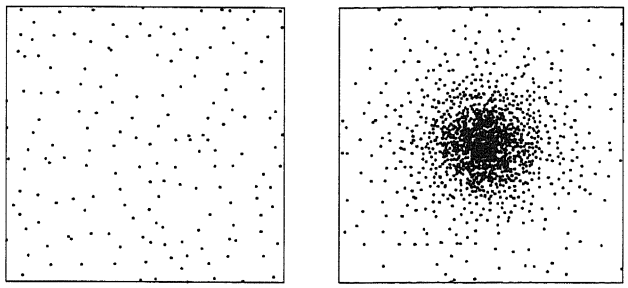

FIGURE 4: Left: Homogeneous template Strauss process X on $\mathcal{X} = [-1,1]^2$ with parameters $\beta = 200, \gamma = 0.1, r = 0.1$. Right: Inhomogeneous Strauss process X_c on $\mathcal{X}' = [-1,1]^2$ obtained by local scaling of X with $c(u) = 0.1 + \|u\|^2$.

The locally scaled process is Markov with respect to the relation

$$u \sim v \iff \nu_c^1([u,v]) \leq r.$$

Thus, the shape of the neighbourhood,

$$\partial_c(x) = \{y : \nu_c^1([x,y]) \leq r\} = b_c(x,r),$$

in the locally scaled process depends on the scaling function c. It is not necessarily convex but always star shaped, cf. Figure 5. A neighbourhood

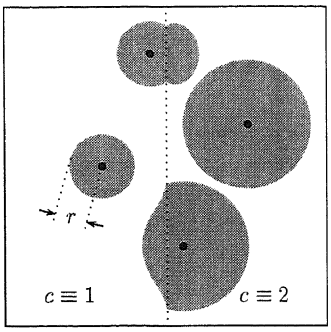

FIGURE 5: Neighbourhoods of four points in a locally scaled distance-interaction process, with scale factor $c \equiv 1$ in the left half and $c \equiv 2$ in the right half of the domain.

$\partial_c(x)$ is called star-shaped if it is star-shaped with respect to x, which means that $u \in \partial_c(x) \Longrightarrow [x, u] \subseteq \partial_c(x)$.

In regions where c is constant, the neighbourhood $\partial_c(x) = b_c(x, r)$ is ball shaped and thus similar to the neighbourhood $\partial(x) = b(x, r)$ in the homogeneous and isotropic template, cf. Figure 5. More precisely, we have

Proposition 1. *If $c(u) = \tilde{c}$ for all $u \in b(x, \tilde{c}r)$, then*

$$b_c(x, r) = b(x, \tilde{c}r).$$

Proof. With straightforward calculations it can be shown that $v \in b(x, \tilde{c}r)$ implies $v \in b_c(x, r)$ and $v \notin b(x, \tilde{c}r)$ implies $v \notin b_c(x, r)$.

Locally scaled distance-interaction processes have the desired property that in regions where c is constant the process behaves like a scaled version of the template process.

Proposition 2. *Let X be a distance-interaction point process with conditional intensity λ_X. Suppose $c(u) = \tilde{c}$ for all $u \in b(x, \tilde{c}r)$. Then the conditional intensity of the locally scaled process X_c is given by*

$$\lambda_{X_c}^{(c)}(x \mid \mathbf{x}) = \lambda_X\big(x/\tilde{c} \,\big|\, \mathbf{x}/\tilde{c}\big).$$

Note that $c(x) = \tilde{c}$ under the assumption.

Proof. First notice that the assumption implies that $b(x, \tilde{c}r) = b_c(x, r)$, see Proposition 1.

The conditional intensity of the locally scaled process is of the form

$$\lambda_{X_c}^{(c)}(x \mid \mathbf{x}) = \beta \prod_{\mathbf{y} \subseteq_1 \mathbf{x}} \varphi\big(D_c(\mathbf{y} \cup \{x\})\big),$$

where $\mathbf{y} \subseteq_1 \mathbf{x}$ is short for $\{\mathbf{y} \subseteq \mathbf{x} : n(\mathbf{y}) \geq 1\}$. Suppose that $\mathbf{y} \subseteq b(x, \tilde{c}r)$. Since

$$\nu_c^1([u,v]) = \tilde{c}^{-1}\nu^1([u,v]) = \nu^1([u/\tilde{c}, v/\tilde{c}]) \quad \text{for any} \quad u, v \in b(x, \tilde{c}r),$$

we get

$$D_c(\mathbf{y} \cup \{x\}) = D\big(\mathbf{y}/\tilde{c} \cup \{x/\tilde{c}\}\big).$$

On the other hand, suppose that there exists $u \in \mathbf{y}$ such that $u \notin b(x, \tilde{c}r)$. Thus, $\nu_c^1([u,x]) > r$ and therefore

$$\varphi\big(D_c(\mathbf{y} \cup \{x\})\big) = 1 = \varphi\Big(D\big(\mathbf{y}/\tilde{c} \cup \{x/\tilde{c}\}\big)\Big).$$

It follows that

$$\lambda_{X_c}^{(c)}(x \mid \mathbf{x}) = \beta \prod_{\mathbf{y} \subseteq_1 \mathbf{x}} \varphi\Big(D\big(\mathbf{y}/\tilde{c} \cup \{x/\tilde{c}\}\big)\Big) = \lambda_X\big(x/\tilde{c} \,\big|\, \mathbf{x}/\tilde{c}\big).$$

5. Shot noise processes

Shot noise processes are based on geometric quantities other than pairwise distances. Write

$$C_{\mathbf{x}}(u) = \sum_{x \in \mathbf{x}} \mathbf{1}\,(u \in b(x,r))$$

for the template coverage function. Then a shot noise weighted point process with potential function $p(\cdot)$ is defined by

$$f_X(\mathbf{x}) \propto \beta^{n(\mathbf{x})} \gamma^{-\int p(C_{\mathbf{x}}(u))\nu^k(du)}, \quad \mathbf{x} \in \Omega_{\mathcal{X}}, \tag{12}$$

where $\gamma > 0$ and p is a function on the non-negative integers $\mathbb{N}_0$ with $p(0) = 0$. The integral $\int p(C_{\mathbf{x}}(u))\nu^k(du)$ is taken on all $\mathbb{R}^k$. The special case $p(n) = \mathbf{1}\,(n \geq 1)$ is known as area-interaction.

The interaction functions of a shot noise process are

$$\begin{aligned} \varphi(\mathbf{y}) &= \beta\gamma^{-m(\mathbf{y})}, \quad n(\mathbf{y}) = 1, \\ \varphi(\mathbf{y}) &= \gamma^{-m(\mathbf{y})}, \quad n(\mathbf{y}) > 1, \end{aligned}$$

where

$$m(\mathbf{y}) = \nu^k\Big(\bigcap_{y\in\mathbf{y}} b(y,r)\Big)\sum_{l=1}^{n(\mathbf{y})}\binom{n(\mathbf{y})}{l}(-1)^{n(\mathbf{y})-l}p(l),$$

compare with Theorem 3.3 in Van Lieshout and Molchanov [10]. As usual, $\varphi(\emptyset)$ is the normalizing constant. Homogeneous shot noise processes are Markov with respect to the overlapping objects relation

$$u \sim v \iff b(u,r)\cap b(v,r)\neq\emptyset \iff \|u-v\|\leq 2r, \tag{13}$$

that means, the neighbourhood of a point is a ball with radius $R=2r$.

It is easy to show that

$$g(\mathbf{x};\nu^*) = \beta^{\nu^0(\mathbf{x})}\gamma^{-\int p(\sum_{x\in\mathbf{x}}\mathbf{1}(\nu^1([x,u])\leq r))\nu^k(du)}$$

is scale invariant. The locally scaled shot noise process has density

$$f_{X_c}^{(c)}(\mathbf{x}) \propto g(\mathbf{x};\nu_c^*) = \beta^{n(\mathbf{x})}\gamma^{-\int p(C_{c,\mathbf{x}}(u))\nu_c^k(du)}, \quad \mathbf{x}\in\Omega_{\mathcal{X}'} \tag{14}$$

with scaled coverage function

$$C_{c,\mathbf{x}}(u) = \sum_{x\in\mathbf{x}}\mathbf{1}\left(u\in b_c(x,r)\right).$$

Van Lieshout and Molchanov (1998) show that (12) is integrable if there exists some constant $0<C<\infty$ with

$$|p(n)|\leq Cn \quad \forall\, n\in\mathbb{N}_0. \tag{15}$$

A similar result holds for the scaled process.

Proposition 3. *Under condition* (15), $g(\cdot;\nu_c^*)$ *is integrable on* $\Omega_{\mathcal{X}'}$ *for any* $\mathcal{X}'\in\mathbb{B}^k$ *and hence the locally scaled process defined by* (14) *does exist.*

Proof. We show that there exists $M>0$ such that

$$g(\mathbf{x};\nu_c^*)\leq M^{n(\mathbf{x})}, \quad \mathbf{x}\in\Omega_{\mathcal{X}'}.$$

This is fulfilled if

$$\Big|\int_{\mathbb{R}^k} p(C_{c,\mathbf{x}}(u))\nu_c^k(du)\Big| \leq M'n(\mathbf{x}) \tag{16}$$

for some $0<M'<\infty$ and all $\mathbf{x}\in\Omega_{\mathcal{X}'}$.

Let $S_C(\mathbf{x})$ denote the support of $C_{c,\mathbf{x}}$. As $C_{c,\mathbf{x}}(u)\leq n(\mathbf{x})$, we have

$$\Big|\int_{\mathbb{R}^k} p(C_{c,\mathbf{x}}(u))\nu_c^k(du)\Big| \leq Cn(\mathbf{x})\nu_c^k\big(S_C(\mathbf{x})\big)$$

with C as in (15). Since

$$S_C(\mathbf{x}) = \bigcup_{x \in \mathbf{x}} b_c(x, r) \subseteq \mathcal{X}' \oplus b(0, \bar{c}r),$$

where $\oplus$ denotes Minkowski addition, and

$$\nu_c^k(B) \leq \underline{c}^{-k} \nu^k(B) \quad \forall\ B \in \mathcal{B}_k,$$

(16) holds with

$$M' = C\, \underline{c}^{-k} \nu^k\big(\mathcal{X}' \oplus b(0, \bar{c}r)\big).$$

The locally scaled shot noise process has interaction functions

$$\begin{aligned} \varphi_c(\mathbf{y}) &= \beta \gamma^{-m_c(\mathbf{y})}, \quad n(\mathbf{y}) = 1, \\ \varphi_c(\mathbf{y}) &= \gamma^{-m_c(\mathbf{y})}, \quad n(\mathbf{y}) > 1, \end{aligned}$$

where

$$m_c(\mathbf{y}) = \nu_c^k\left(\bigcap_{y \in \mathbf{y}} b_c(y, r)\right) \sum_{l=1}^{n(\mathbf{y})} \binom{n(\mathbf{y})}{l} (-1)^{n(\mathbf{y})-l} p(l).$$

It follows that X_c is Markov with respect to the relation

$$\begin{aligned} u \sim_c v \quad &\Longleftrightarrow \quad b_c(u, r) \cap b_c(v, r) \neq \emptyset \\ &\Longleftrightarrow \quad \exists w : \nu_c^1([u, w]) \leq r \wedge \nu_c^1([w, v]) \leq r. \end{aligned} \tag{17}$$

The neighbourhood of a point x is

$$\partial_c(x) = \bigcup_{w \in b_c(x,r)} b_c(w, r), \tag{18}$$

which in general is not ball shaped, but contains all points that are $2r$-close to x with respect to ν_c^1. Additionally, it is possible that two points are neighbours in X_c although their scaled distance is larger than $2r$, since the triangular inequality does not necessarily hold for scaled distances defined by ν_c^1 . However, in analogy with the results obtained for the distance-interaction processes, the following proposition holds.

Proposition 4. *For a shot noise process,*

$$c(u) = \tilde{c}\ \forall\ u \in b(x, 2\tilde{c}r) \quad \Longrightarrow \quad \partial_c(x) = b(x, 2\tilde{c}r).$$

Proof. Consider $w \in b_c(x,r)$. Then $b(w,\tilde{c}r) \subset b(x,2\tilde{c}r)$ and thus $c(u) = \tilde{c}$ for all $u \in b(w,\tilde{c}r)$. Therefore Proposition 1 yields

$$b_c(x,r) = b(x,\tilde{c}r) \text{ and } b_c(w,r) = b(w,\tilde{c}r),$$

and hence

$$\partial_c(x) = \bigcup_{w \in b_c(x,r)} b_c(w,r) = \bigcup_{w \in b(x,\tilde{c}r)} b(w,\tilde{c}r) = b(x,2\tilde{c}r).$$

If the scaling function is constant in a neighbourhood of a point x, the conditional intensity of a locally scaled shot noise process again behaves like under global scaling.

Proposition 5. *Let X be a shot noise point process with conditional intensity λ_X. Suppose $c(u) = \tilde{c}$ for all $u \in b(x,2\tilde{c}r)$. Then the conditional intensity of the locally scaled process X_c is given by*

$$\lambda_{X_c}^{(c)}(x \,|\, \mathbf{x}) = \lambda_X\big(x/\tilde{c} \,\big|\, \mathbf{x}/\tilde{c}\big).$$

Proof. Since the conditional intensity can be written as a product of interactions, $\lambda_X(x \,|\, \mathbf{x}) = \prod_{\mathbf{y} \subseteq \mathbf{x} \cap \partial(x)} \varphi(\mathbf{y} \cup \{x\})$, we only need to show that

$$\prod_{\mathbf{y} \subseteq \mathbf{x} \cap \partial_c(x)} \varphi_c(\mathbf{y} \cup \{x\}) = \prod_{\frac{\mathbf{y}}{\tilde{c}} \subseteq \frac{\mathbf{x}}{\tilde{c}} \cap \partial(\frac{x}{\tilde{c}})} \varphi\Big(\frac{\mathbf{y} \cup \{x\}}{\tilde{c}}\Big).$$

This is fulfilled if

$$m_c(\mathbf{y} \cup \{x\}) = m\left(\frac{\mathbf{y} \cup \{x\}}{\tilde{c}}\right) \quad \text{for finite } \mathbf{y} \subset \partial_c(x),\ \mathbf{y} \neq \emptyset,$$

i.e. if

$$\nu_c^k\Big(\bigcap_{z \in \mathbf{y} \cup \{x\}} b_c(z,r)\Big) = \nu^k\Big(\bigcap_{z \in (\mathbf{y} \cup \{x\})/\tilde{c}} b(z,r)\Big). \tag{19}$$

By the assumption, $\nu_c([z,w]) = \tilde{c}^{-1}\nu([z,w])$ for any $w, z \in b(x,2\tilde{c}r)$. Thus, for all $z \in \partial_c(x) = b(x,2\tilde{c}r)$ (Proposition 4), we have

$$w \in b_c(z,r) \cap b_c(x,r) \iff w \in b(z,\tilde{c}r) \cap b(x,\tilde{c}r).$$

Therefore, $b_c(z,r) \cap b_c(x,r) = b(z,\tilde{c}r) \cap b(x,\tilde{c}r)$, and

$$\bigcap_{z \in \mathbf{y} \cup \{x\}} b_c(z,r) = \bigcap_{z \in \mathbf{y} \cup \{x\}} b(z,\tilde{c}r) = \bigcap_{z \in (\mathbf{y} \cup \{x\})/\tilde{c}} \tilde{c}\, b(z,r),$$

which immediately leads to (19).

Remark 1. In the present section, the focus has been on local scaling of homogeneous shot noise processes defined using balls $b(x,r)$. It is possible to define a more general type of homogeneous shot noise process with $b(x,r)$ replaced by $x+Z$ where Z is an arbitrary bounded subset of $\mathbb{R}^k$. The process has density of the form (12) with $C_{\mathbf{x}}(u)$ replaced by

$$\tilde{C}_{\mathbf{x}}(u) = \sum_{x \in \mathbf{x}} \mathbf{1}\left(u \in x + Z\right), u \in \mathbb{R}^k.$$

If we for $v \in \mathbb{R}^k$ let $\theta(v) = v/\mid v \mid$, we can write

$$\tilde{C}_{\mathbf{x}}(u) = \sum_{x \in \mathbf{x}} \mathbf{1}\left(\nu^1([x,u])\,\theta(u-x) \in Z\right).$$

Using this expression instead of $C_{\mathbf{x}}(u)$ in (12) it is easy to see that the resulting density becomes scale invariant. It is therefore also possible to define local scaling of these more general processes.

6. Approximation of local scaling

For simulation of locally scaled Markov point processes using e.g. the Metropolis-Hastings algorithm (Geyer [4]), one has to evaluate expressions of the form $g(\mathbf{x};\nu_c^*)$. This usually involves integration with respect to scaled d-dimensional volume measures ν_c^d. In the locally scaled distance-interaction processes introduced in Section 4, for example, we deal with scaled distances

$$\begin{aligned}\nu_c^1([u,v]) &= \int_{[u,v]} c(w)^{-1} dw = \|u-v\| \int_0^1 c\left(u + t(v-u)\right)^{-1} dt \\ &= \|u-v\|\,\overline{c^{-1}}(u,v),\end{aligned} \tag{20}$$

where $\overline{c^{-1}}(u,v)$ is the integral mean of the inverse scaling function $w \to 1/c(w)$ on the segment $[u,v]$.

For certain scaling functions c, such integrals can be expressed explicitly. However, if one strives to design programs that handle arbitrary scaling functions, one would have to resort to numeric algorithms. Time consuming calculations can be avoided by defining *approximately scaled* processes that require only pointwise evaluation of the scaling function.

Markov point processes comprise a large variety of models that are essentially different to each other e.g. with respect to the order of interaction, or to the dimensionality of volume measures involved in the definition of their density. There is no best recipe for approximate local scaling of all possible models. A

general method for pointwise local scaling of Markov point processes, based on averaging the interaction functions, is presented in the first subsection. Tailor-made, more intuitive approaches for distance-interaction processes and shot noise processes are suggested in the following subsections.

Note that the methods presented in the following do not require scale invariant density of the process.

6.1. Local scaling by φ-averaging for Markov point processes

In order to restrict the evaluation of c to the points in the pattern $\mathbf{x} = \{x_1, \cdots, x_n\}$, one could resort to interpreting local scaling as an average of global scalings, with scaling factors $c(x_1), \cdots, c(x_n)$, i.e. defining

$$\hat{f}^{(c)}_{X_c}(\mathbf{x}) = \text{Average}\big(f_X(\mathbf{x}/c(x_1)), \cdots, f_X(\mathbf{x}/c(x_n))\big).$$

This would however invalidate the paradigm of locally defined interaction, since interaction in a subset $\mathbf{y} \subset \mathbf{x}$ would then also be modified by scale factors taken from points outside $\mathbf{y}$, in $\mathbf{x} \setminus \mathbf{y}$.

In cases where the homogeneous process is Markov with interactions of finite order k, i.e. where $\varphi(\mathbf{y}) \equiv 1$ if $n(\mathbf{y}) > k$, a feasible concept of local averaging may be based on averaging interaction functions. We propose to define locally scaled interaction functions by the geometric mean

$$\hat{\varphi}_c(\mathbf{y}) = \left(\prod_{y \in \mathbf{y}} \varphi\big(\mathbf{y}/c(y)\big)\right)^{1/n(\mathbf{y})}, \quad n(\mathbf{y}) \geq 1. \tag{21}$$

Thus we obtain the density $\hat{f}^{(c)}_{X_c}$ *by local φ-averaging* as

$$\hat{f}^{(c)}_{X_c}(\mathbf{x}) \propto \prod_{\mathbf{y} \subseteq_1 \mathbf{x}} \hat{\varphi}_c(\mathbf{y}),$$

or, equivalently,

$$\hat{f}_{X_c}(\mathbf{x}) \propto \prod_{x \in \mathbf{x}} c(x)^{-k} \prod_{\mathbf{y} \subseteq_1 \mathbf{x}} \hat{\varphi}_c(\mathbf{y}),$$

where $\mathbf{y} \subseteq_1 \mathbf{x}$ is short for $\{\mathbf{y} \subseteq \mathbf{x} : n(\mathbf{y}) \geq 1\}$. Using the geometric mean in (21) is motivated by the fact that interaction functions are usually of the form

$$\varphi(\mathbf{y}) = \exp(p(\mathbf{y})),$$

where $p(\cdot)$ is the so-called potential function. This notion stems from statistical physics, where Markov point processes were first described as Gibbs processes.

Taking the geometric mean of φ means taking the arithmetic mean of the potential function,

$$\hat{\varphi}_c(\mathbf{y}) = \exp\left(\frac{1}{n(\mathbf{y})}\sum_{y\in\mathbf{y}} p\big(\mathbf{y}/c(y)\big)\right), \quad n(\mathbf{y}) \geq 1.$$

The following example of a Strauss process shows how this concept is applied to a distance-interaction process.

Example (Strauss process, continued). An approximately locally scaled Strauss process X_c obtained by φ-averaging has second-order interaction

$$\hat{\varphi}_c(\{u,v\}) = \gamma^{\frac{1}{2}\left(\mathbf{1}(\|u-v\|\leq c(u)r)+\mathbf{1}(\|u-v\|\leq c(v)r)\right)}.$$

Therefore the density takes the form

$$\hat{f}^{(c)}_{X_c}(\mathbf{x}) \propto \beta^{n(\mathbf{x})}\,\gamma^{\hat{s}_c(\mathbf{x})},$$

where $\hat{s}_c(\mathbf{x})$ is calculated from the number of directed neighbours given by the relation

$$u \rightsquigarrow v \iff \|u-v\| \leq c(u)r.$$

The number of directed neighbours divided by two,

$$\hat{s}_c(\mathbf{x}) = \sum_{\{u,v\}\subseteq\mathbf{x}}^{\neq} \frac{1}{2}\big(\mathbf{1}\,(u \rightsquigarrow v) + \mathbf{1}\,(v \rightsquigarrow u)\big),$$

can be considered as an approximation of the true number $s_c(\mathbf{x})$ of neighbours under local scaling, compare with Section 4.

6.2. Local scaling by c-averaging for distance-interaction processes

Locally scaled distance-interaction processes as introduced in Section 4 require only the calculation of scaled pairwise distances

$$\nu^1_c([u,v]) = \overline{c^{-1}}(u,v)\|u-v\|,$$

where $\overline{c^{-1}}(u,v)$ denotes the integral mean of c^{-1} over the segment $[u,v]$, cf. (20). A natural idea is to replace the integral mean by a simpler mean $\widehat{c^{-1}}(u,v)$ of the inverse scaling factors $c(u)^{-1}$ and $c(v)^{-1}$. We propose to use the harmonic mean

$$\widehat{c^{-1}}(u,v) = \frac{2}{c(u)+c(v)}.$$

The original neighbour relation $u \sim_c v \iff \nu_c^1([u,v]) \leq r$ is thus approximated by

$$u \sim_{\hat{c}} v \quad \iff \quad \|u - v\| \leq \frac{1}{2}\big(c(u) + c(v)\big)r. \tag{22}$$

This relation allows for a nice geometric interpretation. Two points u, v are neighbours iff the balls $b(u, \frac{1}{2}c(u)r)$ and $b(v, \frac{1}{2}c(v)r)$ overlap. Note that (22) actually means that the scaling function c itself is locally replaced by the arithmetic mean $\frac{1}{2}\big(c(u) + c(v)\big)$. Therefore we call this approach *local scaling by c-averaging.*

In proper locally scaled distance-interaction processes, neighbourhoods are always star shaped, i.e.

$$u \sim_c v \implies u \sim_c w \quad \text{for all } w \in [u,v]. \tag{23}$$

Conversely, neighbourhoods in distance-interaction processes obtained by c-averaging are not necessarily star shaped, as the example given in Figure 6 shows.

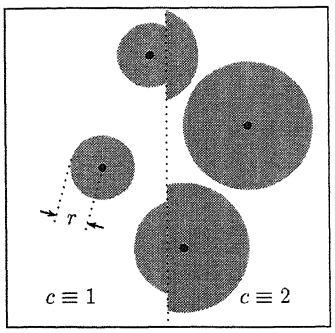

FIGURE 6: Neighbourhoods of four points in an approximately locally scaled distance-interaction process, obtained by c-averaging with scale factor $c \equiv 1$ in the left half and $c \equiv 2$ in the right half of the domain. Compare with Figure 5, where the neighbourhoods of the same four points are shown in the genuine locally scaled process.

However, a relatively weak Lipschitz condition on the scaling function ensures that the inhomogeneous point process has star shaped neighbourhoods.

Proposition 6. *Let $X_{\hat{c}}$ be an inhomogeneous point process on $\mathcal{X}' \in \mathbb{B}^k$ obtained by c-averaging from a homogeneous distance-interaction process with*

neighbour relation $u \sim v \Leftrightarrow \|u - v\| \leq r$. *Then* $X_{\hat{c}}$ *has star shaped neighbourhoods if the scaling function* c *satisfies*

$$|c(u) - c(v)| \leq \frac{2}{r}\|u - v\| \text{ for all } u, v \in \mathcal{X}'. \tag{24}$$

Proof. Let $u, v \in \mathcal{X}'$ be related with respect to the approximated relation given by (22), $u \sim_{\hat{c}} v$, and let $w \in [u, v]$ be any point on the line segment between u and v. Then we need to show that $w \sim_{\hat{c}} u$ under the Lipschitz condition (24). This is trivially fulfilled if $c(w) \geq c(v)$. Otherwise, by (24),

$$c(v) - c(w) \leq \frac{2}{r}\|v - w\|.$$

Recall that under c-averaging,

$$u \sim_{\hat{c}} v \Longleftrightarrow \|u - v\| \leq \tfrac{1}{2}\big(c(u) + c(v)\big)r \Longleftrightarrow c(u) + c(v) \geq \frac{2}{r}\|v - u\|.$$

Subtracting the two inequalities, we find that

$$c(u) + c(w) \geq \frac{2}{r}\big(\|v - u\| - \|v - w\|\big) = \frac{2}{r}\,\|u - w\|,$$

thus $u \sim_{\hat{c}} w$.

Example (Strauss process, continued). As for φ-averaging, the density of an approximately locally scaled Strauss process obtained by $c - averaging$ is of the form

$$\hat{f}^{(c)}_{X_c}(\mathbf{x}) \propto \beta^{n(\mathbf{x})}\,\gamma^{\hat{s}_c(\mathbf{x})}.$$

Now,

$$\hat{s}_c(\mathbf{x}) = \sum_{\{u,v\} \subseteq \mathbf{x}}^{\neq} \mathbf{1}\,(u \sim_{\hat{c}} v)$$

is the number of neighbour pairs with respect to the approximate neighbour relation $\sim_{\hat{c}}$ given by (22).

6.3. Local scaling by influence zones for shot noise processes

Shot noise processes as defined in Section 5 require the evaluation of the coverage function

$$C_{c,\mathbf{x}}(u) = \sum_{x \in \mathbf{x}} \mathbf{1}\,(u \in b_c(x, r))$$

which gives the number of "influence zones" $b_c(x, r)$ covering a point u. In Proposition 1 we saw that $b_c(x, r) = b(x, \tilde{c}r)$ if $c(u) = \tilde{c} = c(x)$ for all $u \in$

$b(x, \tilde{c}r) = b(x, c(x)r)$. Assuming that c does not vary very much in $b(x, c(x)r)$, we can use this result to approximate the influence zones by

$$b_c(x, r) \approx b(x, c(x)r)$$

and thus obtain the coverage function

$$\hat{C}_{c,\mathbf{x}}(u) = \sum_{x \in \mathbf{x}} \mathbf{1}\left(u \in b(x, c(x)r)\right).$$

Calculating the density function

$$\hat{f}^{(c)}_{X_c}(\mathbf{x}) \propto \beta^{n(\mathbf{x})} \gamma^{-\int p(\hat{C}_{c,\mathbf{x}}(u))\nu^k_c(du)}$$

still requires integration with respect to the locally scaled measure ν^k_c. But even when dealing with homogeneous processes, the integral $\int p(C_{c,\mathbf{x}}(u))du$ is usually approximated by grid methods. Once the coverage function $\hat{C}_{c,\mathbf{x}}(\cdot)$ is known, evaluating

$$\int_{\mathbb{R}^k} p(\hat{C}_{c,\mathbf{x}}(u))\nu^k_c(du) = \int_{\mathbb{R}^k} p(\hat{C}_{c,\mathbf{x}}(u))c(u)^{-k}\nu^k(du)$$

is therefore no bigger a problem than evaluating the corresponding integral in a homogeneous (template) setting.

7. Sequential local scaling

Sometimes it is more appropriate to consider a given point pattern as an ordered sequence of points rather than merely as an unordered set. For example, trees in a forest stand can clearly be ordered by age which corresponds to a natural sequence of emergence. In point pattern theory, the simple sequential inhibition (SSI) point process results from successively adding points to a bounded window such that a new point keeps a minimum distance to the points that have been added before.

Since sequential point processes have been quite rarely considered in the literature so far, we will first recall a few facts before presenting a general approach to define locally scaled versions of homogeneous Markov sequential point processes.

A sequential point process $\vec{X}$ on $\mathcal{X} \in \mathbb{B}^k$ is a random variable taking values in $\Omega_{s,\mathcal{X}}$, the set of all finite sequences $\vec{\mathbf{x}} = (x_1, \cdots, x_n)$ of points in $\mathcal{X}$. The set $\Omega_{s,\mathcal{X}}$ is equipped with the σ-algebra $\mathcal{F}_{s,\mathcal{X}}$ generated by the Borel σ-algebras $\mathcal{B}_n$ on $\mathcal{X}^n$. In the following, we will consider sequential point processes that have

a density $f_{\vec{X}}$ with respect to the probability measure μ on $\Omega_{s,\mathcal{X}}$, where for all $F \in \mathcal{F}_{S,\mathcal{X}}$

$$\mu(F) = \sum_{n=0}^{\infty} \frac{e^{-\nu^k(\mathcal{X})}}{n!} \int_{\mathcal{X}} \cdots \int_{\mathcal{X}} \mathbf{1}\left((x_1, \cdots, x_n) \in F\right) d\nu^k(x_1) \cdots d\nu^k(x_n).$$

This measure on the space $[\Omega_{s,\mathcal{X}}, \mathcal{F}_{s,\mathcal{X}}]$ of sequential point sets corresponds to the unit rate Poisson point process for unordered (non sequential) point processes on $[\Omega_{\mathcal{X}}, \mathcal{F}_{s,\mathcal{X}}]$.

For sequential point processes, the Papangelou conditional intensity,

$$\lambda_{\vec{X}}(x \mid (x_1, \cdots, x_n)) = \begin{cases} \frac{f_{\vec{X}}(x_1, \cdots, x_n, x)}{f_{\vec{X}}(x_1, \cdots, x_n)}, & f_{\vec{X}}(x_1, \cdots, x_n) > 0, \\ 0 & \text{otherwise}, \end{cases} \tag{25}$$

gains a particularly intuitive meaning, since $\lambda_{\vec{X}}(x \mid \vec{\mathbf{x}})\nu^k(dx)$ relates to the conditional probability of adding a new point x in a region dx to an existing sequence $\vec{\mathbf{x}} = (x_1, \cdots, x_n)$.

In analogy to hereditary (unordered) point processes, we define hereditary sequential point processes as follows.

Definition 4. A sequential point process $\vec{X}$ with density $f_{\vec{X}}$ is called *hereditary* if $f_{\vec{X}}(x_1, \cdots, x_n) > 0$ implies $f_{\vec{X}}(x_1, \cdots, x_m) > 0$ for all $m < n$.

If a sequential point process is hereditary, then, as a consequence of (25), the density can be written as

$$f_{\vec{X}}(x_1, \cdots, x_n) \propto \prod_{j=1}^{n} \lambda_{\vec{X}}(x_j \mid \vec{\mathbf{x}}_{<j}), \qquad \vec{\mathbf{x}}_{<j} = (x_1, \cdots, x_{j-1}). \tag{26}$$

This gives rise to a straightforward idea of defining local scaling in the sequential context by means of locally scaled conditional intensities. We start again with a homogeneous, now sequential, template process $\vec{X}$, which means that $f_{\vec{X}}$ and hence $\lambda_{\vec{X}}$ are translation invariant and in principle defined for any finite point sequence in $\mathbb{R}^k$. Motivated by the effect of global scaling on the conditional intensity, cf. (3), we require that the Papangelou conditional intensity of the scaled sequential process $\vec{\mathcal{X}}_c$ fulfils

$$\lambda^{(c)}_{\vec{X}_c}(x \mid \vec{\mathbf{x}}) = \lambda_{\vec{X}}\Big(\frac{x}{c(x)} \,\Big|\, \frac{\vec{\mathbf{x}}}{c(x)}\Big), \tag{27}$$

where c as before is a measurable, bounded scaling function that is bounded away from 0. Note that for locally scaled unordered point processes, the

corresponding property (9) is only fulfilled in regions where the scaling function c is constant.

Definition 5. Suppose $\vec{X}$ is a homogeneous hereditary sequential point process on $\mathcal{X}$ with Papangelou conditional intensity $\lambda_{\vec{X}}$ given by (25). Let μ_c be the distribution of a sequential Poisson point process with the locally scaled volume measure ν_c^k as intensity measure, and let $\mathcal{X}' \in \mathbb{B}^k$ be arbitrary.
Then the *locally scaled sequential point process* $\vec{X}_c$ on $\mathcal{X}'$ with template $\vec{X}$ is defined by the density

$$f^{(c)}_{\vec{X}_c}(x_1, \cdots, x_n) \propto \prod_{j=1}^{n} \lambda_{\vec{X}}\left(\frac{x_j}{c(x_j)} \,\middle|\, \frac{\vec{\mathbf{x}}_{<j}}{c(x_j)}\right) \tag{28}$$

with respect to μ_c, provided that $f_{\vec{X}_c}$ is integrable on $\Omega_{s,\mathcal{X}'}$.

Sequential templates can easily be obtained from homogeneous unordered point processes, since any unordered point process X on $\mathcal{X}$ with a density f_X with respect to the unit rate Poisson point process can be converted into a corresponding ordered point process $\vec{X}$ by defining the density $f_{\vec{X}}$ with respect to μ as

$$f_{\vec{X}}(x_1, \cdots, x_n) = f_X\big(\{x_1, \cdots, x_n\}\big). \tag{29}$$

This means every permutation of $\vec{\mathbf{x}}$ has the same density, see also Daley and Vere-Jones [3, p. 122]. We shall refer to the process $\vec{X}$ with density (29) as the *sequentialized version* of the process X.

Clearly, the sequentialized process $\vec{X}$ is hereditary if the unordered process X is hereditary. Combining (25) and (29), the Papangelou conditional intensity of a sequentialized process becomes

$$\lambda_{\vec{X}}(x \mid x_1, \cdots, x_n) = \lambda_X(x \mid \{x_1, \cdots, x_n\}).$$

Thereby, the Papangelou conditional intensity of the corresponding locally scaled point process $\vec{X}_c$ is

$$\lambda^{(c)}_{\vec{X}_c}(x \mid \vec{\mathbf{x}}) = \lambda_X\left(\frac{x}{c(x)} \,\middle|\, \frac{\mathbf{x}}{c(x)}\right), \tag{30}$$

cf. (27), and the density (28) of $\vec{X}_c$ becomes

$$f^{(c)}_{\vec{X}_c}(x_1, \cdots, x_n) \propto \prod_{j=1}^{n} \lambda_X\left(\frac{x_j}{c(x_j)} \,\middle|\, \frac{\mathbf{x}_{<j}}{c(x_j)}\right). \tag{31}$$

The locally scaled process $\vec{X}_c$ is locally stable, i.e. $\lambda^{(c)}_{\vec{X}_c}$ is bounded above, if the homogeneous unordered template X is locally stable, i.e. if λ_X is bounded above. Then (31) is integrable. As before, a constant scaling function simply yields a globally scaled sequential point process. Note in particular that a locally scaled sequential point process is isotropic in the sense of (10) if the template is isotropic.

7.1. Sequential Markov point processes

As a special class of hereditary processes, we define sequential Markov point processes with respect to a directed relation.

Definition 6. A sequential point process $\vec{X}$ with density $f_{\vec{X}}$ is said to be *Markov with respect to a directed relation* $\rightsquigarrow$, if $\vec{X}$ is hereditary and if the conditional intensity $\lambda_{\vec{X}}(x \mid \vec{\mathbf{x}})$ depends only on the point x and on its set of directed neighbours in $\vec{\mathbf{x}}$, $\partial(\vec{x}) \cap \mathbf{x} = \{x_i \in \vec{\mathbf{x}} : x \rightsquigarrow x_i\}$.

The following proposition suggests a way to construct locally scaled sequential Markov point processes from unordered Markov templates.

Proposition 7. *Let X be a locally stable, homogeneous (unordered) point process with density f_X on $\mathcal{X} \in \mathbb{B}^k$. If X is Markov with respect to some symmetric relation $\sim$, then the locally scaled sequential process $\vec{X}_c$ defined by (31) is Markov with respect to the directed relation*

$$u \rightsquigarrow_c v \iff \frac{u}{c(u)} \sim \frac{v}{c(u)}. \tag{32}$$

Proof. First we show that $\vec{X}_c$ is hereditary. By equation (31),

$$\begin{aligned} f^{(c)}_{\vec{X}_c}(x_1, \cdots, x_n) &= \alpha \prod_{j=1}^{n} \lambda_X\left(\frac{x_j}{c(x_j)} \,\middle|\, \frac{\mathbf{x}_{<j}}{c(x_j)}\right) \\ &= \alpha f^{(c)}_{\vec{X}_c}(x_1, \cdots, x_m) \prod_{k=m+1}^{n} \lambda_X\left(\frac{x_k}{c(x_k)} \,\middle|\, \frac{\mathbf{x}_{<k}}{c(x_k)}\right), \end{aligned}$$

where α is the normalizing constant. Therefore, $f^{(c)}_{\vec{X}_c}(x_1, \cdots, x_m) = 0$ implies $f^{(c)}_{\vec{X}_c}(x_1, \cdots, x_n) = 0$ if $m \leq n$. Secondly, by (30) and the fact that X is Markov, the sequential conditional intensity $\lambda^{(c)}_{\vec{X}_c}(x \mid \vec{\mathbf{x}})$ depends only on those sequence elements x_i for which $x/c(x) \sim x_i/c(x)$ and the proof is complete.

Note that the result reflects a similar proposition for global scaling: if the template X is Markov with respect to some relation $\sim$, its global scaling cX

by a factor $c > 0$ is Markov with respect to

$$u \sim_c v \iff u/c \sim v/c,$$

see Jensen and Nielsen [7].

Example (Strauss process, continued). For a Strauss process X, $\lambda_X(x \mid \mathbf{x}) = \beta\gamma^{s(x;\mathbf{x})}$, where $s(x;\mathbf{x}) = s(\mathbf{x} \cup \{x\}) - s(\mathbf{x})$ denotes the number of points in $\mathbf{x}$ that are closer than r to the new point $x \notin \mathbf{x}$. Here

$$f^{(c)}_{\tilde{X}_c}(x_1, \cdots, x_n) \propto \beta^n \gamma^{s_c(x_1,\cdots,x_n)},$$

with

$$s_c(x_1, \cdots, x_n) = \sum_{i<j} \mathbf{1}\left(\|x_j/c(x_j) - x_i/c(x_j)\| \le r\right) = \sum_{i<j} \mathbf{1}\left(x_j \rightsquigarrow_c x_i\right),$$

where

$$x_j \rightsquigarrow_c x_i \iff \|x_j - x_i\| \le c(x_j)r.$$

This is the same directed neighbour relation as in φ-averaging, see page 18.

Specializing to the hard core model ($\gamma = 0$), we obtain a sequential inhibition model in which each point upon arrival keeps a distance $c(x_j)r$ away from previously arrived points, cf. also Clausen et al [2].

The $\rightsquigarrow_c$-neighbourhood of a point $u \in \mathcal{X}'$,

$$\partial_c(u) = \{x \in \mathcal{X}' : u \rightsquigarrow_c x\} = c(u)\partial(u/c(u))$$

is equal to the neighbourhood in the template process scaled by a factor $c(u)$. In particular, the neighbourhood system $\partial_c(\cdot)$ inherits from $\partial(\cdot)$ geometric properties such as convexity that are invariant under scaling.

To the fixed range relation $u \sim v \iff \nu^1([u,v]) \le r$ in distance-interaction processes corresponds the directed relation under local scaling,

$$u \rightsquigarrow_c v \iff \nu^1([u,v]) \le c(u)\, r,$$

since $\lambda^{(c)}_{X_c}(x|(x_1, \ldots, x_n))$ depends only on those x_i that are closer than $c(x)r$ to x, cf. (32).

In the context of shot noise processes (Section 5), the overlapping objects relation $u \sim v \iff b(u,r) \cap b(v,r) \neq \emptyset$ transforms into the directed relation

$$u \rightsquigarrow_c v \iff b(u/c(u), r) \cap b(v/c(u), r) \neq \emptyset \iff b(u, c(u)r) \cap b(v, c(u)r) \neq \emptyset.$$

In other words, the influence zone at u associated with X_c is the influence zone of X scaled by $c(u)$.

An analogue of the Hammersley-Clifford factorization can be given for densities of locally scaled sequential Markov point processes, see the Appendix. Here we also show how these sequential point processes are related to unordered inhomogeneous point processes obtained by φ-averaging.

Note that it is always possible to map a sequential process to a classical point process by averaging its density over permutations. If we use the geometric average instead of the usual arithmetic average, the resulting process is identical to that obtained by φ-averaging in Section 6.1.

8. Discussion

Inhomogeneity in natural structures may be caused by very different mechanisms. Correspondingly, there is a myriad of ways to define inhomogeneous models. Therefore some restrictions have to be introduced that replace the usual homogeneity condition. The three models for inhomogeneous point processes described in the introduction stand for three different situations. In the first model, the interaction between points is independent of location. In the second model, inhomogeneity results from a (physical) location dependent thinning, and in the third from (physical) deformation of the matrix. In this paper we introduced a yet fourth approach which yields models for patterns that are homogeneous up to a local scale factor. Such point processes may describe packings of spheres with diameters that vary with location, cf. Figure 1, or situations where both intensity and interaction are governed by the same external factor, such as desert plant communities that are ruled by water supply.

When it comes to choosing an appropriate model for a given situation, there will sometimes be prior information about the physical genesis of the patterns that strongly suggests one of the approaches. In general however it will be necessary to define criteria for the best choice which then can be used to develop model tests. A test on the local scaling assumption could exploit scale invariant local geometric properties, as e.g. shape factors of corresponding Voronoi cells.

The intensity $\lambda_c(x)$ of a globally scaled point process is proportional to the intensity λ of the template, $\lambda_c(x) = c^{-k}\lambda(x/c) \approx c^{-k}\lambda$, since the intensity of a homogeneous template is approximately constant. Analogously, the intensity of a locally scaled process is (approximately)

$$\lambda_c(x) \approx c(x)^{-k}\lambda. \tag{33}$$

This allows firstly to model practically any inhomogeneous intensity and, secondly, to retrieve the scaling function (up to a proportionality constant) from a given or estimated density. In this aspect the scaling function plays a similar role as the survival probability of the thinning model.

Once the scaling function has been estimated, it can be used to subsequently fit the parameters of the template process and thus to complete the model specification. A similar approach has been followed by Nielsen and Jensen [12] for fitting the transformation model. Furthermore possible empirical relations between an estimated scaling function and explanatory variables such as water supply in the desert vegetation case can be used for prediction purposes.

Future work will concentrate on validating the approximation (33) as well as on development of model tests and other statistical methods.

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Appendix A. Appendix: Local scaling of Markov point processes, general case

The density $f_X(\cdot) = g(\cdot; \nu^*)$ of a homogeneous Markov point process is scale invariant iff the interaction function $\varphi(\cdot) = h(\cdot; \nu^*)$ is scale invariant. Then the Hammersley-Clifford factorization of f_X reads

$$g(\mathbf{x}; \nu^*) = \prod_{\mathbf{y} \subseteq \mathbf{x}} h(\mathbf{y}; \nu^*),$$

which allows to factorize the density of the locally scaled equivalent X_c as

$$f_{X_c}^{(c)}(\mathbf{x}) = \prod_{\mathbf{y} \subseteq \mathbf{x}} \varphi_c(\mathbf{y}), \qquad \varphi_c(\mathbf{y}) = h(\mathbf{y}; \nu_c^*). \tag{34}$$

It is indeed possible to define a neighbour relation $\sim_c$ such that φ_c is a proper interaction function. Hence the locally scaled process is Markov, too. The following definition which may look a little awkward at the first glance is in fact consistent with the simple neighbour relations derived earlier for locally scaled distance-interaction and shot noise processes.

Definition A.1. Consider a function φ_c on $\Omega_{\mathcal{X}'}$ as given by (34). Two points u and v are called *neighbours with respect to the scaled relation* $\sim_c$, iff there exists a finite point configuration $\mathbf{y} \in \Omega_{\mathcal{X}'}$ such that

$$\varphi_c(\mathbf{y} \cup \{u, v\}) \neq 1.$$

Proposition A.1. *Let X_c be an inhomogeneous point process obtained by local scaling of a homogeneous Markov template with scale invariant interaction function, such that the density $f_{X_c}^{(c)}$ can be factorized as in* (34). *Then X_c is Markov with respect to the neighbour relation $\sim_c$ given in Definition A.1.*

Proof. The proposition follows from the Hammersley-Clifford theorem, if the function φ_c is a proper interaction function with respect to the relation $\sim_c$, i.e. if $\varphi_c(\mathbf{y}) \neq 1$ implies that $\mathbf{y}$ is a $\sim_c$ clique. To see this, consider a set $\mathbf{y}$ with $n(\mathbf{y}) \geq 2$ and $\varphi_c(\mathbf{y}) \neq 1$. Then, by definition, any two points u and v in $\mathbf{y}$ are neighbours with respect to $\sim_c$, as $\varphi_c(\mathbf{y}^* \cup \{u, v\}) \neq 1$ with $\mathbf{y}^* = \mathbf{y} \backslash \{u, v\} \in \Omega_{\mathcal{X}'}$.

Appendix B. Appendix: Some results for locally scaled sequential Markov point processes

First, we give an analogue of the Hammersley–Clifford factorization (1) for locally scaled sequential Markov point processes.

Proposition B.1. *Let X be a locally stable, homogeneous point process with density f_X on $\mathcal{X} \in \mathbb{B}^k$. If X is Markov with interaction function φ, then the density $f^{(c)}_{\vec{X}_c}$ of the corresponding locally scaled sequential process can be factorized as follows:*

$$f^{(c)}_{\vec{X}_c}(x_1, \cdots, x_n) \propto \prod_{j=1}^{n} \prod_{\mathbf{y} \subseteq \mathbf{x}_{<j}} \varphi\left(\frac{\mathbf{y} \cup \{x_j\}}{c(x_j)}\right), \tag{35}$$

where $\mathbf{x}_{<j} = \{x_1, \cdots, x_{j-1}\}$.

Proof. From the Hammersley-Clifford factorization of f_X one derives

$$\lambda_X\left(\frac{x_j}{c(x_j)} \middle| \frac{\mathbf{x}_{<j}}{c(x_j)}\right) = \frac{\prod_{\mathbf{y} \subseteq \mathbf{x}_{<j+1}} \varphi(\mathbf{y}/c(x_j))}{\prod_{\mathbf{y} \subseteq \mathbf{x}_{<j}} \varphi(\mathbf{y}/c(x_j))} = \prod_{\mathbf{y} \subseteq \mathbf{x}_{<j}} \varphi\left(\frac{\mathbf{y} \cup \{x_j\}}{c(x_j)}\right),$$

from which the result follows by (31).

We may define a sequence $\vec{\mathbf{y}}$ to be an x-clique if $\vec{\mathbf{y}}$ either has length zero or all its members $y \in \vec{\mathbf{y}}$ satisfy $x \rightsquigarrow_c y$. Note that if $\vec{\mathbf{y}}$ is no x-clique, a point $y \in \vec{\mathbf{y}}$ can be found such that $x \not\rightsquigarrow_c y$, or equivalently $\frac{x}{c(x)} \not\sim \frac{y}{c(x)}$. But then $(\mathbf{y} \cup \{x\})/c(x)$ is no $\sim$–clique,which implies $\varphi((\mathbf{y} \cup \{x\})/c(x)) = 1$. It follows that (35) is a product of genuine clique interaction functions.

As already mentioned in Section 7, the density of a classical non ordered point process can be obtained by averaging the density of a sequential process over permutations. If we use the geometric average, the resulting process is identical to that obtained by φ-averaging in Section 6.1.

Proposition B.2. *Let X be a homogeneous Markov point process with density f_X on $\mathcal{X} \in \mathbb{B}^k$ and with interaction function φ, and let $f^{(c)}_{\vec{X}_c}$ be the density of the corresponding locally scaled sequential process. Then the geometric average of $f^{(c)}_{\vec{X}_c}(\vec{\mathbf{x}})$ over all permutations of $\mathbf{x}$ is equal to the density with respect to Π_c of a locally scaled point process obtained by φ-averaging, i.e.*

$$\bar{f}^{(c)}_{\vec{X}_c}(\mathbf{x}) = \left(\prod_{\vec{\mathbf{x}}} f^{(c)}_{\vec{X}_c}(\vec{\mathbf{x}})\right)^{1/n(\mathbf{x})!} \propto \prod_{\mathbf{y} \subseteq_1 \mathbf{x}} \hat{\varphi}_c(\mathbf{y}),$$

where the product $\prod_{\vec{\mathbf{x}}}$ is over all permutations of $\mathbf{x}$ and $\hat{\varphi}_c(\mathbf{y})$ is as defined in (21),

$$\hat{\varphi}_c(\mathbf{y}) = \left(\prod_{y \in \mathbf{y}} \varphi(\mathbf{y}/c(y))\right)^{1/n(\mathbf{y})}, \quad n(\mathbf{y}) \geq 1.$$

Proof. Note that

$$\prod_{j=1}^{n} \prod_{\mathbf{y} \subseteq \mathbf{x}_{<j}} g(\mathbf{y}, x_j) = \prod_{\mathbf{y} \subseteq_1 \mathbf{x}} g(\mathbf{y} \setminus \{y_{m_{\vec{\mathbf{x}}}}\}, y_{m_{\vec{\mathbf{x}}}}),$$

where $y_{m_{\vec{\mathbf{x}}}}$ is the latest member of $\mathbf{y}$ in a given permutation $\vec{\mathbf{x}} = (x_1, \cdots, x_n)$ of $\mathbf{x} = \{x_1, \cdots, x_n\}$. Furthermore, for any given non empty subset $\mathbf{y} \subseteq \mathbf{x}$,

$$\prod_{\vec{\mathbf{x}}} g(\mathbf{y} \setminus \{y_{m_{\vec{\mathbf{x}}}}\}, y_{m_{\vec{\mathbf{x}}}}) = \left(\prod_{y \in \mathbf{y}} g(\mathbf{y} \setminus \{y\}) \right)^{n(\mathbf{x})!/n(\mathbf{y})},$$

where the product $\prod_{\vec{\mathbf{x}}}$ is over all permutations of $\mathbf{x}$. To see the latter, fix $y \in \mathbf{y}$. Then $y = y_{m_{\vec{\mathbf{x}}}}$, i.e. y occurs as latest arrival out of $\mathbf{y}$ in $n(\mathbf{x})!/n(\mathbf{y})$ permutations of $\mathbf{x}$. To complete the proof, combine the two identities, let $g(\mathbf{y}, y) = \varphi(\frac{\mathbf{y} \cup \{y\}}{c(y)})$ and apply (35).