II.3 An Introduction to Stochastic Geometry

Marie-Colette N.M. van Lieshout

CWI, P.O.Box 94079, 1090 GB Amsterdam, The Netherlands
E-mail: Marie-Colette.van.Lieshout@cwi.nl

ABSTRACT
In this brief primer on stochastic geometry we introduce the fundamental concept of a random closed set, discuss characteristics such as the volume fraction and spatial covariance, and review a range of summary statistics based on morphological operators. After considering in some detail the so-called bombing model, we turn to estimation issues, and conclude with a pointer to the literature.

1. Introduction

Binary images often arise in material science and cell biology, possibly after thresholding, but also in the agricultural sciences, for instance as indicators of presence/absence in ground cover data. Each such image determines a set, namely the collection of foreground pixels. As the underlying natural phenomenon is essentially continuous, we are led to study random sets in $\mathbb{R}^2$, a formal definition of which is given in Section 2. The probability distribution of such a random set may be quite a complicated object. Hence, an exploratory data analysis often begins by listing (estimates of) a few characteristics that can be thought of as the spatial analogues of quantiles and moments for real-valued random variables. In this spirit, Section 3 is devoted to the basic notions of volume fraction and spatial covariance, whereas Section 4 presents some more geometric statistics.

The best known and most tractable random set is the bombing (or Boolean) model which can be thought of as the impact of particles dropped randomly and independently of each other over some region. As such, the Boolean model not only provides a natural probabilistic description for relatively sparse configurations of overlapping objects such as bomb craters or plants, but its amenability to explicit calculations also makes it a useful approximation to as well as a benchmark for more complicated random set models. In Section 5, we define the bombing model with spherical particles, and derive explicit expressions for most summary statistics introduced in Sections 3–4.

In Section 6, we turn to the estimation of random set characteristics from data images. A range of techniques is discussed, including stereology, minus sampling and Hanisch style edge correction. The paper is concluded with a brief summary, and pointers to the literature.

2. Random sets

It would be natural to define a random set as a map from some probability space $(\Omega, \mathcal{A}, P)$ onto the power set of $\mathbb{R}^2$. However, due to measurability problems attached to this naive approach, we have to restrict ourselves to *closed* sets. The formulation below follows the exposition by G. Matheron [17]; independently, D.G. Kendall and his collaborators [10] developed a related but somewhat different theory.

Definition 1. *A* random closed set *(rcs) is a map $X : \Omega \to \mathcal{F}$ from a probability space $(\Omega, \mathcal{A}, P)$ onto the class $\mathcal{F}$ of closed subsets of $\mathbb{R}^2$ such that*

$$\{\omega : X(\omega) \Uparrow G\} \text{ and } \{\omega : X(\omega) \Uparrow K\}$$

are measurable for all open sets $G \in \mathcal{G}$ *and all compact sets* $K \in \mathcal{K}$. *Here we write* $A \Uparrow B$ *for the event that* A *hits* B, *i.e.* $A \cap B \neq \emptyset$.

The above definition might look a bit forbidding, but readers not familiar with measure theory could just remember that the only meaningful statements about random closed sets are those concerning hitting/missing an open or compact set.

In fact, in definition 1 it is sufficient to consider only either $\mathcal{G}$ or $\mathcal{K}$. Furthermore, by Choquet's theorem [17, pp. 30–35], see also [3], any random closed set is completely characterised by its *capacity functional*, the mapping $T : \mathcal{K} \to [0,1]$ defined by

$$T(K) = P(X \Uparrow K), \quad K \in \mathcal{K}, \tag{2.1}$$

the probability that X is trapped by the probing set K.

Finally, the random set X is called ***stationary*** if its distribution is invariant under translations, or equivalently if its capacity functional does not depend on the choice of the origin.

3. Volume fraction and covariance

Although the Choquet theorem yields a valuable characterisation of the distribution of a random closed set, the family $\mathcal{K}$ of traps is huge and lower-dimensional summary statistics are called for. This section lists some of the main characteristics of a random set, the role of which can be compared to that of the mean and variance in more classical statistics. Throughout we shall assume stationarity.

The first summary statistic is based on restricting the capacity functional (2.1) to singletons $\{x\}$, $x \in \mathbb{R}^2$.

Definition 2. *The* volume fraction *or* coverage probability $p_X \in [0,1]$ *of a stationary rcs* X *is defined by*

$$p_X = P(0 \in X).$$

Since X is assumed to be stationary, $p_X = T(\{0\}) = T(\{x\}) = P(x \in X)$ for any $x \in \mathbb{R}^2$.

The spatial correlation between points may be expressed in terms of the hitting probability for a set consisting of two points, say x and $y \in \mathbb{R}^2$. Now,

$$\begin{aligned} T(\{x,y\}) &= P(X \Uparrow \{x,y\}) = P(x \in X \text{ or } y \in X) \\ &= P(x \in X) + P(y \in X) - P(x \in X;\, y \in X) \\ &= 2p_X - P(0 \in X;\, y - x \in X) \end{aligned}$$

where the last equation follows from the stationarity of X.

Definition 3. *The* covariance $C_X : \mathbb{R}^2 \to [-1,1]$ *of a stationary rcs* X *is defined by*

$$C_X(z) = P(0 \in X;\, z \in X) - p_X^2.$$

Note that the spatial covariance $C_X(z)$, $z \in \mathbb{R}^2$, is identical to the (classical) covariance of the indicator random variables $1_X(0)$ and $1_X(z)$. Indeed [17, 19, 20, 21],

$$\mathrm{Cov}(1_X(0), 1_X(z)) = E\,[1_X(0)\,1_X(z)] - E1_X(0)\,E1_X(z) = P(0 \in X;\, z \in X) - p_X^2$$

as $E1_X(0) = E1_X(z) = p_X$. The related *semivariogram* [16]

$$\gamma_X(z) = \frac{1}{2}\mathrm{var}(1_X(0) - 1_X(z)) = p_X(1 - p_X) - C_X(z)$$

is popular amongst geostatisticians [12].

As we would expect, swapping fore- and background does not affect the covariance and semivariogram. More precisely [1], if X is a stationary rcs with $P(0 \in \partial X) = 0$, then

$$p_{\overline{X^c}} = P(0 \in \overline{X^c}) = P(0 \in X^c) = 1 - p_X.$$

Indeed, $1_{\overline{X^c}}(z) = 1 - 1_X(z)$ almost surely and hence

$$\mathrm{Cov}(1_{\overline{X^c}}(0), 1_{\overline{X^c}}(z)) = \mathrm{Cov}(1 - 1_X(0), 1 - 1_X(z)) = \mathrm{Cov}(1_X(0), 1_X(z)).$$

4. MORPHOLOGICAL STATISTICS

Unfortunately, the covariance C_X has limited power to discriminate between different probability models [19], which emphasises the need for summary statistics that are more sensitive to the shape of X. A first step in this direction is to evaluate (2.1) for closed balls. Doing so, we obtain the *empty space function*

$$F_X(r) = T(B(0,r)) = P(X \Uparrow B(0,r))$$

defined for all radii $r \geq 0$. In particular, $F_X(0) = p_X$, the volume fraction of X.

In order to present a wider range of statistics, it is convenient to introduce some basic morphological operators [17, 20].

Definition 4. *Let A be a fixed subset of $\mathbb{R}^2$. Write X_a for the translation of $X \subseteq \mathbb{R}^2$ by a vector $a \in \mathbb{R}^2$. Then we define the* Minkowski addition *of X and A as*

$$X \oplus A = \bigcup_{a \in A} X_a = \{h : \check{A}_h \cap X \neq \emptyset\}$$

and the Minkowski subtraction *by*

$$X \ominus A = \bigcap_{a \in A} X_a = \{h : \check{A}_h \subseteq X\}$$

where $\check{A}$ is the reflection $\{-a : a \in A\}$ *of A.*

As an aside, the definition of Minkowski subtraction in the earlier work of Hadwiger [6] is slightly different, namely $\{h : A_h \subseteq X\} = \bigcap_{a \in A} X_{-a}$. Seen as mappings, $\delta_A : X \mapsto X \oplus \check{A}$ and $\epsilon_A : X \mapsto X \ominus \check{A}$ are called *dilation* and *erosion* respectively. The fixed set A is referred to as the *structuring element*. In image analysis applications, A is usually small, for instance a three by three square, a cross or a (discretised) disc. The mappings δ_A and ϵ_A are *dual* in the sense that eroding a set X amounts to dilating its complement, that is $\epsilon_A(X) = (\delta_A(X^c))^c$.

Since $X \Uparrow B(0,r)$ if and only if $0 \in X \oplus B(0,r)$, the empty space function $F_X(r)$ at range r can be expressed as the coverage probability of the dilation of X by a ball of radius r.

Definition 5. *The empty space function $F_X : [0,\infty) \to [0,1]$ of a stationary rcs X is defined by*

$$F_X(r) = P(0 \in X \oplus B(0,r)).$$

Some authors [21, p. 178] prefer to work with the *spherical contact distribution function* $H_X : [0,\infty) \to [0,1]$, which is well-defined whenever $p_X < 1$ and related to the empty space function as follows:

$$H_X(r) = P(0 \in X \oplus B(0,r) \mid 0 \notin X) = \frac{F_X(r) - F_X(0)}{1 - F_X(0)}.$$

Thus, we may interpret $1 - H_X(r)$ as the conditional probability that $B(0,r)$ does not hit X given that the test point 0 itself is not an element of X.

Definition 6. *Let A be a fixed subset of $\mathbb{R}^2$. Write X_a for the translation of $X \subseteq \mathbb{R}^2$ by a vector $a \in \mathbb{R}^2$. Then we define the* opening *of X by the structuring element A as*

$$X \circ A = (X \ominus \check{A}) \oplus A = \bigcup_{h: A_h \subseteq X} A_h$$

and the closing *of X by A as*

$$X \bullet A = (X \oplus \check{A}) \ominus A = \bigcap_{h: A_h \subseteq X^c} A_h^c.$$

Since erosion and dilation are dual, the same is true for opening and closing. Opening has the effect of removing sharp edges; closing tends to fill small gaps [20].

The family $\{\psi_r : r \geq 0\}$ of openings $\psi_r(X) = X \circ B(0,r)$ with discs of increasing radius is useful to quantify the 'size' of the components of X. More precisely, following [17], we define the spherical size of X at a point $x \in X$ to be the largest r for which $x \in X \circ B(0,r)$, i.e. for which x is covered by a disc of radius r fitting completely in X. By duality, the size $\eta(x, X)$ of the empty space at $x \in X^c$ is given by

$$\begin{aligned} \eta(x,X) &= \sup\{r \geq 0 : x \in X^c \circ B(0,r)\} = \sup\{r \geq 0 : x \notin X \bullet B(0,r)\} \\ &= \inf\{r \geq 0 : x \in X \bullet B(0,r)\}. \end{aligned} \tag{4.1}$$

The concept is illustrated in Fig. 3 (left), where the closing transform $\eta(\cdot,\cdot)$ is given for the set X shown in Fig. 1. As X^c is open, $P(\eta(x,X) \leq r) = P(x \in X \bullet B(0,r))$ for all $r \geq 0$. This observation motivates the following definition [14].

Definition 7. *The* size distribution function $P_X : \mathbb{R} \to [0,1]$ *of a stationary rcs X is defined by*

$$P_X(r) = \begin{cases} P(0 \in X \bullet B(0,r)) & r \geq 0 \\ P(0 \in X \circ B(0,|r|)) & r < 0 \end{cases}.$$

The function $P_X(\cdot)$ is called the *granulométrie bidimensionelle* in metallurgy [4]. As the empty space function, the size distribution function takes the value p_X at $r = 0$, and is increasing and semi-continuous from the right. Provided $p_X < 1$, conditioning on the event $\{0 \notin X\}$ yields the *size distribution law* $G_{0,X}$ of the pores [20, p. 335]

$$G_{0,X}(r) = P(0 \in X \bullet B(0,r) \mid 0 \notin X) = \frac{P_X(r) - P_X(0)}{1 - P_X(0)}$$

which can be interpreted as the conditional probability that a background point is included by closing with respect to $B(0,r)$.

To conclude this section, we would like to emphasise that although we have concentrated on discs for clarity of exposition, it is straightforward to define size distribution and empty space functions for other non-empty convex and compact structuring elements.

5. Example: the bombing model

Let Y be a stationary Poisson process on $\mathbb{R}^2$ with intensity $\lambda > 0$, and define the stationary random closed set X by

$$X = \bigcup_{y \in Y} B(y,r)$$

for some radius $r > 0$. The points $y \in Y$ are called the *germs*, the set $B(0,r)$ is the *primary grain*, and X is referred to as the *bombing model* [13, 15]. A realisation of the bombing model with $\lambda = 100$ and $r = 0.05$ is given in Fig. 1.

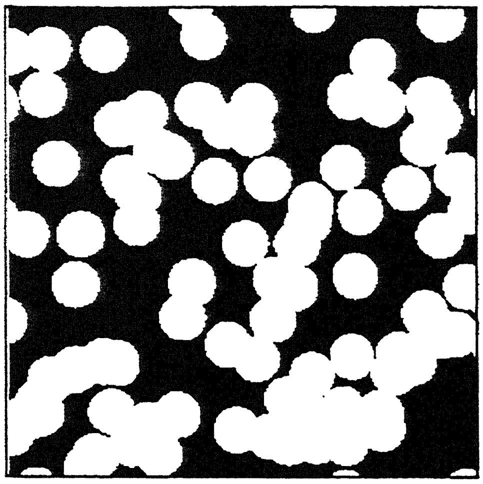

Figure 1: Sample of the bombing model in the unit square with intensity 100 and disc radius 0.05

Theorem 1. *The capacity functional of the bombing model is given by [17, 18, 20, 21]*

$$T(K) = 1 - \exp\{-\lambda|K \oplus B(0,r)|\}$$

for $K \in \mathcal{K}$, where $|K \oplus B(0,r)|$ denotes the area of the r-envelope of K.

Proof: As

$$B(y,r) \Uparrow K \Leftrightarrow y \in K \oplus B(0,r),$$

those germs whose associated grain hits K form a Poisson process Z with intensity λ on $K \oplus B(0,r)$. Therefore

$$1 - T(K) = P(Z = \emptyset) = e^{-\lambda|K \oplus B(0,r)|}$$

from which the result follows. □

If additionally K is convex, by the Steiner formula [6]

$$T(K) = 1 - \exp\left\{-\lambda\left(|K| + U(K)r + \pi r^2\right)\right\}$$

where $U(K)$ denotes the boundary length of the set K.
We are now ready to compute some of the summary statistics introduced in Sections 2–3.

Volume fraction

$$p_X = T(\{0\}) = 1 - e^{-\lambda\pi r^2}.$$

For the bombing model with $\lambda = 100$ and $r = 0.05$ (cf. Fig. 1), $p_X = 1 - e^{-\pi/4} \approx 0.54$.

Covariance

First, note that by the conditional independence property of a Poisson process, $T(\{0,z\}) = 1 - e^{-2\lambda\pi r^2}$ whenever $||z|| > 2r$. Otherwise,

$$\begin{aligned} T(\{0,z\}) &= 1 - \exp\{-\lambda|B(0,r) \cup B(z,r)|\} \\ &= 1 - \exp\left\{-\lambda\left(2\pi r^2 - 2r^2\arccos(||z||/(2r)) + \frac{1}{2}||z||(4r^2 - ||z||^2)^{1/2}\right)\right\} \end{aligned}$$

hence $T(\{0,z\})$ depends on z only through its norm $||z||$. Next, since both the covariance and the semivariogram can be expressed in terms of $T(\{0,z\})$ and p_X, we derive

$$\begin{aligned}\gamma_X(z) &= T(\{0,z\}) - p_X \\ &= (1-p_X) \\ &- (1-p_X)^2 \exp\left\{\lambda\left(2r^2 \arccos(||z||/(2r)) - \frac{1}{2}||z||(4r^2 - ||z||^2)^{1/2}\right)\right\}\end{aligned}$$

and

$$C_X(z) = p_X(1-p_X) - \gamma_X(z).$$

For the bombing model with $\lambda = 100$ and $r = 0.05$, the spatial covariance is plotted in Fig. 2 (left) as a function of $||z||$.

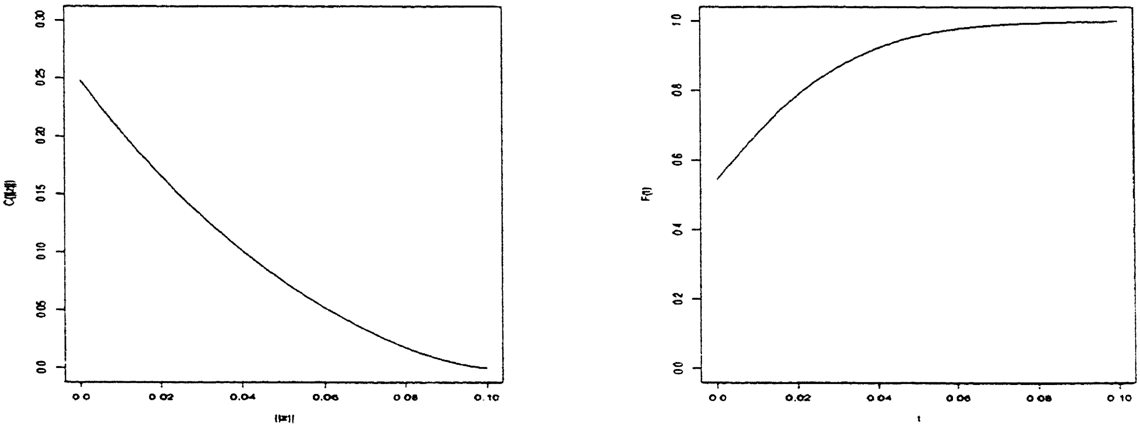

Figure 2: The covariance (left) and empty space (right) functions of the bombing model in the unit square with intensity 100 and disc radius 0.05

Empty space function

$$F_X(t) = T(B(0,t)) = 1 - e^{-\lambda\pi(r+t)^2}.$$

For $\lambda = 100$ and $r = 0.05$, the empty space function is plotted in Fig. 2 (right) against its argument t.

The size distribution function is notoriously hard to compute [7]. Thus, in the next section we shall turn to estimation of $P_X(\cdot)$ as well as of the other main random set characteristics [18].

6. Estimation

Throughout this section, assume a stationary random closed set X is observed in a compact window S.

Since the volume fraction can be interpreted as the mean proportion of space covered by the random set, a natural estimate is

$$\widehat{p_X} = \frac{|X \cap S|}{|S|}$$

where $|\cdot|$ denotes area. We have

$$E|X \cap S| = \int_{\mathcal{F}} \left(\int_S 1_X(s)\, ds \right) dP(X) = \int_S \left(\int_{\mathcal{F}} 1_X(s)\, dP(X) \right) ds = p_X |S|,$$

hence the estimator is unbiased. If one is not willing to compute $|X \cap S|$ either explicitly or by image analysis software, alternatives are provided by the geometric sampling techniques of ***stereology*** [11]. The basic *Delesse formula* reads

$$p_X := A_A = L_L = P_P$$

where p_X is by definition the mean area per unit area (A_A), which is equal to the mean length per unit length (L_L) or the mean number of points per point (P_P). Using the latter, the point count method [5] places a grid $\{g_1, \ldots, g_n\}$ over the image and counts the fraction of grid points covered by X, i.e.

$$\widehat{p_X^G} = \frac{1}{n} \sum_{i=1}^{n} 1\{g_i \in X\}.$$

Since $E\widehat{p_X^G} = \frac{1}{n} \sum_{i=1}^{n} P(g_i \in X) = p_X$, the estimator is unbiased.

In principle, the above approach applies to all summary statistics introduced in Sections 2–3. Indeed, $C_X(z)$ is related to the volume fraction of $X \ominus \{0, -z\}$; $F_X(r)$ is the coverage probability of $X \oplus B(0, r)$, and $P_X(r)$ that of either $X \bullet B(0, r)$ or $X \circ B(0, |r|)$ depending on the sign of r. However, some care is required due to edge effects. For example, $(X \bullet B(0, r)) \cap S$ cannot be determined from $X \cap S$ alone, and we have to restrict ourselves to $S \ominus B(0, 2r)$ as follows

$$\widehat{P_X^M(r)} = \frac{|((X \cap S) \bullet B(0, r)) \cap (S \ominus B(0, 2r))|}{|S \ominus B(0, 2r)|} \tag{6.1}$$

to obtain an unbiased, observable estimator. The above approach is known as ***minus sampling*** and can be traced back to Miles's contribution to [10]. When applying minus sampling ideas to the empty space function, restriction to $S \ominus B(0, r)$ suffices. As for the covariance, we may estimate the volume fraction of the random set $X \ominus \{0, -z\}$ based on the window $S \ominus \{0, -z\}$, and subtract $\widehat{p_X}^2$.

Although generally applicable, minus sampling tends to discard a lot of information (especially for large r or $||z||$) and more sophisticated edge corrections have been proposed. For instance [9] use ideas from survival analysis to construct a Kaplan–Meier estimator for the empty space function; [2] show that the ideas underlying this approach are reminiscent of the Hanisch estimator [8] of point process theory. By way of illustration, in the remainder of this paper, we derive a Hanisch style estimator [14] for the size distribution function $P_X(r)$, $r \geq 0$. To do so, recall the notion of spherical void size defined by (4.1) in Section 3. Furthermore, write

$$\zeta(s, S^c) = \inf\{r \geq 0 : B(s, 2r) \cap S^c \neq \emptyset\}$$

for the 'distance' from $s \in S$ to the window boundary. As noted in Section 3, $X \bullet B(0, r) = \{x \in \mathbb{R}^2 : \eta(x, X) \leq r\}$ for any closed set X and all $r \geq 0$. Similarly, $S \ominus B(0, 2r) = \{s \in S : \zeta(s, S^c) \geq r\}$. The closing transform $\eta(x, X)$ for the realisation X of the bombing model shown in Fig. 1 is given in Fig. 3 (left).

The minus sampling estimator (6.1) can be expressed in terms of η and ζ as

$$\widehat{P_X^M(r)} = \frac{\int_S 1\{\eta(s, X \cap S) \leq r \leq \zeta(s, S^c)\}\, ds}{\int_S 1\{\zeta(s, S^c) \geq r\}\, ds}.$$

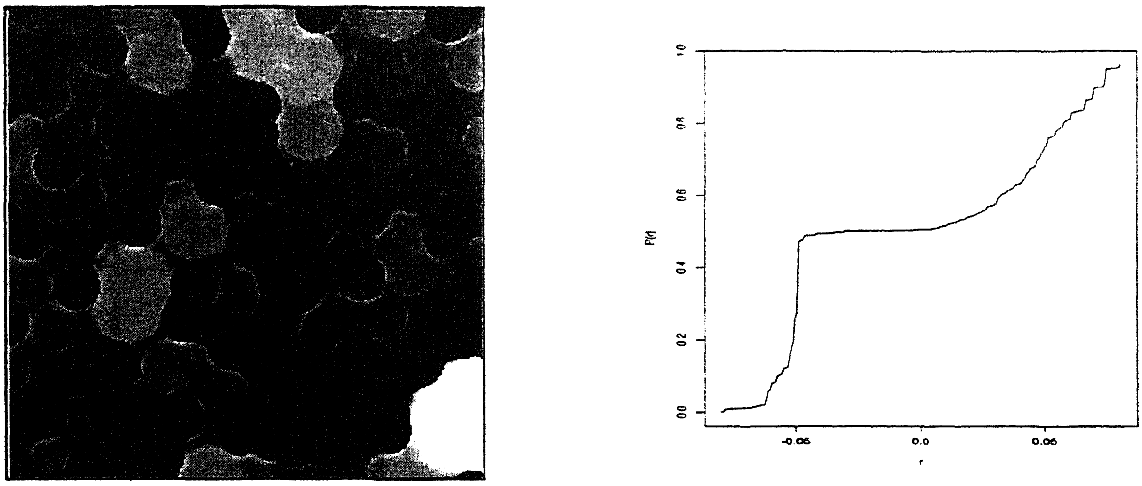

Figure 3: The closing transform η (left) and the estimated size distribution function $\widehat{P_X^H}$ (right) for the sample of the bombing model in the unit square with intensity 100 and disc radius 0.05 depicted in figure 1

Now, the key observation is that if $s \notin S \ominus B(0, 2r)$ and $\eta(s, X \cap S) \leq \zeta(s, S^c)$ the correct void size $\eta(s, X)$ at s is measured, but not taken into account by $\widehat{P_X^M(r)}$. If we do include such points, we obtain the Hanisch style estimator

$$\widehat{P_X^H(r)} = \int_S \frac{1\{\eta(s, X \cap S) \leq \min(r, \zeta(s, S^c))\}}{|S \ominus B(0, 2\eta(s, X \cap S))|} \, ds.$$

Theorem 2. *Let X be a stationary rcs observed in a compact window S. Then, the Hanisch style estimator $\widehat{P_X^H(r)}$ is pointwise unbiased for $P_X(r)$. It is increasing and semi-continuous from the right.*

Rescaling if necessary ensures values in $[0, 1]$.

Proof: (sketch) To see the unbiasedness, note that

$$\begin{aligned} E\widehat{P_X^H(r)} &= \int_S E\left[\frac{1\{\eta(s, X) \leq r\}\, 1\{s \in S \ominus B(0, 2\eta(s, X)\}}{|S \ominus B(0, 2\eta(s, X))|}\right] ds \\ &= \int_S \int_{[0,r]} \frac{1\{s \in S \ominus B(0, 2t)\}}{|S \ominus B(0, 2t|} \, dP_X(t)\, ds \\ &= \int_{[0,r]} 1\, dP_X(t) = P_X(r). \end{aligned}$$

Clearly, $\widehat{P_X^H(r)}$ is increasing. The semi-continuity follows from [17, 1-5-1,1-5-2]. □

The estimated size distribution function $\widehat{P_X^H(r)}$ for the binary image of Fig. 1 is plotted in Fig. 3 (right).

7. Summary and conclusions

In this paper, an introduction to stochastic geometry was given, with emphasis on its basic object of study: a random closed set. Clearly, due to space restrictions, we could only give a flavour. For more in depth information, the interested reader is referred to the bibliography below. Indeed, many excellent textbooks are available, on subjects ranging from the integral geometric background [6] of stochastic geomety, through the pioneering monographs on random set theory [10, 17], Boolean models [18] and coverage problems [7], to the more recent comprehensive introductions of [1, 21]. Regarding the applications of random sets, see [4, 12] for geology and geostatistics, cf. [15, 19, 20] for image analysis and spatial statistics, and for stereology, cf. [11].

REFERENCES

1. Baddeley, A.J. (1993) *Spatial statistics and image analysis.* Lecture notes, Leiden University.
2. Chiu, S.N. and Stoyan, D. (1998) Estimators of distance distributions for spatial patterns. *Statistica Neerlandica*, 52:239–246.
3. Choquet, G. (1953/1954) Theory of capacities. *Annales de l'Institut Fourier* V:131–295.
4. Fabbri, A.G. (1984) *Image processing of geological data.* Van Nostrand Reinhold, New York.
5. Glagolev, A.A. (1933) On geometrical methods of quantitative mineralogic analysis of rocks. *Transactions of the Institute of Econ. Mineralogy, Moscow*, 59:1–47.
6. Hadwiger, H. (1957) Vorlesungen über Inhalt, Oberfläche und Isoperimetrie. Springer Verlag, Berlin.
7. Hall, P. (1988) *Introduction to the theory of coverage processes.* John Wiley and Sons, New York.
8. Hanisch, K.-H. (1984) Some remarks on estimators of the distribution function of nearest neighbour distance in stationary spatial point processes. *Mathematische Operationsforschung und Statistik, Series Statistik*, 15:409–412.
9. Hansen, M.B., Gill, R.D. and Baddeley, A.J. (1996) Kaplan-Meier type estimators for linear contact distributions. *Scandinavian Journal of Statistics*, 23:129–155.
10. Harding, E.F. and Kendall, D.G. (1974) *Stochastic geometry: a tribute to the memory of Rollo Davidson.* John Wiley and Sons, London.
11. Jensen, E.B.V. (1998) *Local Stereology.* World Scientific Publishing, Singapore.
12. Journel, A. and Huibregts, C. (1978) *Mining geostatistics.* Academic Press, London.
13. Kolmogorov, A.N. (1937) On the statistical theory of metal crystallisation. *Izvestia Academy of Science, USSR, Series Mathematics*, 3:355–360.
14. Lieshout, M.N.M. van (1999) Size-biased random closed sets. *Pattern Recognition*, 32:1631–1644.
15. Matérn, B. (1960) *Spatial Variation.* Lecture Notes in Statistics (Vol. 36), Springer Verlag, Berlin.
16. Matheron, G. (1955) Application des méthodes statistiques à l'estimation des gisements. *Annales des Mines*, XII:25–50.
17. Matheron, G. (1975) *Random sets and integral geometry.* John Wiley and Sons, New York.
18. Molchanov, I. (1997) *Statistics of the Boolean model for practitioners and mathematicians.* John Wiley and Sons, Chichester.
19. Ripley, B.D. (1988) *Statistical inference for spatial processes.* Cambridge University Press.
20. Serra, J. (1982) *Image analysis and mathematical morphology.* Academic Press, London.
21. Stoyan, D., Kendall, W.S. and Mecke, J. (1987) *Stochastic geometry and its applications.* Akademie Verlag, Berlin. Second edition, 1995.