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Symmetric systems with semi-simple structure algebra: The quaternionic case *

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The basic theory of linear systems over the quaternions is developed.

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1. Introduction

In [5] and [2] the concept of symmetric linear systems was introduced and the fundamental role of the structure algebra was established. We recall those concepts here for the benefit of the reader. Let $\mathfrak{C}$ be a class of linear systems where the dimensions of the state and input spaces are fixed and the (A, B) satisfy some set of relations. The archetypical example is

$$\mathfrak{C}_1 = \left\{ \left(\begin{pmatrix} A & H \\ -H & A \end{pmatrix}, \begin{pmatrix} B & 0 \\ 0 & B \end{pmatrix} \right): A, B, H \text{ real matrices} \right\}$$

which arises in a variety of contexts and specifically in the modeling of twin-lift helicopters [6].

The structure algebra of the class $\mathfrak{C}$ is defined to be the algebra

$$R(\mathfrak{C}) = \{(S, T): \text{for all } (A, B) \in \mathfrak{C} \quad SA = AS, SB = BT\}.$$

The importance of the structure algebra stems from the observation that because of commutatitively relations the state space and input space become $R(\mathfrak{C})$-modules and the maps A and B are module homomorphisms. Thus if the structure of $R(\mathfrak{C})$-modules is known it is possible to reduce the structure of the maps A and B.

In the case that the algebra $R(\mathfrak{C})$ is semisimple, a great deal is known [2]. However in [2] it was shown that the symmetric systems with real semisimple structure algebra $R(\mathfrak{C})$ could always be written as the direct sum of ordinary real systems, ordinary complex systems and 'ordinary' quaternionic systems. Unfortunately there is not a well established theory of linear systems over the quaternions. The goal of this paper is to establish a minimal amount of material so that the theory of real, semisimple symmetric systems is somewhat complete.

2. Linear algebra over the quaternions

2.1. In the following we let $\mathbb{H}$ denote the real division algebra of quaternions. Recall that a typical quaternion h has the form

$$h = a + b\mathrm{i} + c\mathrm{j} + d\mathrm{k}$$

where $1, \mathrm{i}, \mathrm{j}, \mathrm{k}$ form a basis for $\mathbb{H}$ as an $\mathbb{R}$-vector space. Multiplication in $\mathbb{H}$ is determined by the formulas

$$\mathrm{i}^2 = \mathrm{j}^2 = \mathrm{k}^2 = -1, \quad \mathrm{ij} = \mathrm{k}, \quad \mathrm{jk} = \mathrm{i}, \quad \mathrm{ki} = \mathrm{j}.$$

Also recall that $\mathbb{H}$ can be represented as the set of matrices of the form

$$\begin{pmatrix} a & b & c & d \\ -b & a & d & -c \\ -c & -d & a & b \\ -d & c & -b & a \end{pmatrix}.$$

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Typically the matrix above represents the quaternion $a + bi + cj + dk$.

Thinking of i, j, k as imaginaries we define a bar conjugation on $\mathbf{H}$ by

$$h = \mathrm{Re}(h) + \mathrm{Im}(h) \to \bar{h} + \mathrm{Re}(h) - \mathrm{Im}(h):$$

in terms of the matrix representation above the bar conjugate is just the matrix transpose. So we immediately find that

$$\overline{h_1 h_2} = \bar{h}_2 \bar{h}_1.$$

We also note that, if $\mathbf{H}$ is thought of as $\mathbf{R}^4$ with the 1, i, j, k basis, the standard norm is just

$$\|h\| = (h\bar{h})^{1/2} = (\bar{h}h)^{1/2}.$$

We extend this to obtain a standard norm on $\mathbf{H}^n$, defined by

$$\|h\| = (h^* h)^{1/2},$$

where $h \in \mathbf{H}^n$ is thought of as a column vector and the asterisk denotes taking bar conjugate transpose (the multiplication is just matrix multiplication).

There is another notion of conjugation on $\mathbf{H}$ defined by

$$h^{\alpha} = \alpha^{-1} h \alpha$$

where $\alpha \in \mathbf{H}$ is nonzero. Note that there are commutation formulas

$$\begin{aligned} h_1 h_2 &= h_2 h_1^{h_2} \\ &= h_2^{h_1^{-1}} h_1 \\ &= h_2^{\bar{h}_1} h_1. \end{aligned} \tag{2.1.1}$$

We also define a conjugation on $\mathbf{H}^n$ in the same spirit by

$$h^{\alpha} = \alpha^{-1} h \alpha$$

where $\alpha \in \mathbf{H}$ is nonzero, $h \in \mathbf{H}^n$ and the multiplications are performed component by component.

2.2. By an $\mathbf{H}$-vector space we just mean a *right* module over $\mathbf{H}$.

Example 2.2.1. $\mathbf{H}$ is itself an $\mathbf{H}$-vector space in a natural way. Scalar multiplication is just multiplication on the right. Moreover, there is an isomorphism $\mathbf{H} \sim \mathrm{Hom}(\mathbf{H}, \mathbf{H})$ determined by sending h to multiplication by h on the left.

Example 2.2.2. The map $\mathbf{H} \otimes \mathbf{H} \to \mathbf{H}$ defined by

$$x \otimes a \to \bar{a} x$$

is a right $\mathbf{H}$-module structure on $\mathbf{H}$. It is isomorphic to the module structure on $\mathbf{H}$ in Example 2.2.1 via the bar conjugation map.

Example 2.2.3. $\mathbf{H}^n$ is an $\mathbf{H}$-vector space in virtue of being a direct sum of $\mathbf{H}$-vector spaces. We continue to think of vectors in $\mathbf{H}^n$ as column vectors and find that a linear map $A : \mathbf{H}^m \to \mathbf{H}^n$ is just an $m \times n$ matrix of quaternions acting on the left of vectors according to the usual matrix multiplication rules.

Now, since $\mathbf{H}$ is a division algebra, any right $\mathbf{H}$-module is a direct sum of copies of $\mathbf{H}$ in an essentially unique way. So Example 2.2.3 gives a good picture of a linear algebra over $\mathbf{H}$.

Let us also note that $\mathbf{H}^n$ has a natural choice of $\mathbf{H}$-bimodule structure extending the $\mathbf{H}$-vector space structure of 2.2.3. Consequently, given an isomorphism of $\mathbf{H}$-vector spaces

$$A : M \to \mathbf{H}^n$$

we can extend the $\mathbf{H}$-vector space structure on M to an $\mathbf{H}$-bimodule structure making A an isomorphism of $\mathbf{H}$-bimodules. Unfortunately this structure is in general dependent in a nontrivial way upon the map A. This observation essentially explains our choosing to focus attention on module rather than bimodule structures: There are not enough linear maps of $\mathbf{H}$-modules. In particular we have:

Proposition 2.2.4. *A matrix $A : \mathbf{H}^n \to \mathbf{H}^n$ preserves the canonical bimodule structures if and only if it has real entries.*

A somewhat stronger result can be proved as an easy corollary of 2.2.4. It will be useful later.

Corollary 2.2.5. *Let $A : \mathbf{H}^n \to \mathbf{H}^n$. The canonical $\mathbf{H}$-vector space structure on $\mathbf{H}^n$ extends to an $\mathbf{H}$-bimodule structure which is preserved by A if and only if A is conjugate to a real matrix.*

3. Quaternionic linear systems

3.1. In what follows, a system will be a time-invariant linear ordinary differential equation on

H-vector spaces

$$\dot{x} = Ax + Bu, \quad x \in \mathfrak{N}, u \in \mathfrak{M}. \tag{3.1.1}$$

Typically we take for granted isomorphisms

$$\mathfrak{N} \sim \mathbf{H}^n, \quad \mathfrak{M} \sim \mathbf{H}^m,$$

allowing us to think of A and B as matrices of quaternions. We also use the notation (A, B) as a shorthand designation for (3.1.1).

Now the elementary theory of linear O.D.E.'s over **H** is essentially the same as that of real or complex O.D.E.'s. In particular, we find that there is a well defined matrix exponential

$$e^{tA} = \sum_{n=0}^{\infty} \frac{t^n}{n!} A^n.$$

Moreover, the system (3.1.1) has a unique solution, denoted $x(x_0, u, t)$, which satisfies the initial condition $x(x_0, u, t) = x_0$ and which is given by

$$x(x_0, u, t) = e^{tA} x_0 + \int_0^t e^{(t-s)A} Bu(s)\, ds.$$

Now, the formula just above enables one to prove the following very important theorem.

Theorem 1. *Let $\mathfrak{R}_{A,B}$ denote the set of states which can be reached by the system (A, B) from the origin in finite time. Then*

$$\mathfrak{R}_{A,B} = \langle A \mid \mathfrak{B} \rangle = \mathfrak{B} + A\mathfrak{B} + \cdots + A^{n-1}\mathfrak{B}$$

where $\mathfrak{B}$ denotes the image of B.

Then, with the usual definition of controllability in mind one can quickly prove:

Theorem 2. *The space of matrix pairs*

$$(A, B) \in \mathrm{Mat}_{n \times n}(\mathbf{H}) \times \mathrm{Mat}_{m \times n}(\mathbf{H}),$$

with (A, B) controllable, is open and dense.

Just as easily, one can use Theorem 1 to prove that controllability is preserved by the action of the feedback groups, that is:

Theorem 3. *Change of basis in either state or input space preserves controllability.*

Theorem 4. *Controllability is preserved by state space feedback.*

3.2. The classical eigenvalue criterion for the stability of a linear system over the real or complex numbers has no clear analogue in the quaternionic case. The reason for this is, in effect, that the classical theory takes liberal advantage of the fact that linear endomorphisms of classical vector spaces preserve the canonical bimodule structures available.

One way to circumvent this difficulty is to appeal to some real or complex representation of a system whose stability is in question. This approach is considered in [2]. Of course, the control theorist is less interested in testing for stability than in stabilizing controllable systems using feedback. So we ought to prove:

Theorem 5. *The orbit of any controllable system, under the action of the feedback group, contains a stable system.*

In fact, we prove Theorem 5 as a corollary of somewhat stronger results in the next section.

4. Stabilization and invariants

4.1. The key to stabilizing quaternionic systems is a quaternionic Heymann lemma.

Lemma 4.1.1. *Let (A, B) be a controllable system over **H**. Then, for any b in the image of B, there is a feedback matrix F such that the single-input system $(A + BF, b)$ is controllable.*

The proof of 4.1.1 is trivially adapted from the proof of the real Heymann lemma. So we refer the interested reader to [7], Lemma 3.2.

Given stabilizing feedback for a system $(A + BF, b)$ with $b \in \mathfrak{B}$ we can obtain stabilizing feedback for (A, B) by an obvious lifting. So to prove Theorem 5 we need only concern ourselves with the feedback stabilization of single-input systems. We begin by exhibiting a canonical form for controllable single-input systems.

Lemma 4.1.2. *Given a controllable single-input system (A, b) over **H**, there is a unique matrix A' of the form*

$$\begin{pmatrix} 0 & 1 & 0 & \cdots & 0 \\ 0 & 0 & 1 & & \vdots \\ \vdots & & \ddots & \ddots & 0 \\ 0 & \cdots & \cdots & 0 & 1 \\ \alpha_1 & \alpha_2 & \cdots & \cdots & \alpha_n \end{pmatrix} = A'$$

such that for some P we have

$$P^{-1}AP = A' \quad \text{and} \quad P^{-1}b = e_n.$$

Proof. Let (A, b) be controllable. By Theorem 1 there is a unique n-tuple $\alpha_1, \ldots, \alpha_n$ such that

$$0 = A^n b - A^{n-1}b\alpha_1 - \cdots - b\alpha_n.$$

Now define a matrix $P = [P_1 \cdots P_n]$ by the formulas

$$P_1 = A^{n-1}b - A^{n-2}b\alpha_2 \cdots - b\alpha_n,$$

$$P_2 = A^{n-2}b - A^{n-3}b\alpha_3 \cdots - b\alpha_n,$$

$$P_{n-1} = Ab - b\alpha_n, \qquad P_n = b.$$

Then one easily checks that $AP = PA'$ and that $Pe_n = b$. But by Theorem 1 the P_i are independent, so P is invertible; and, the proof is complete.

Let us refer to the α_i in the above as the characteristic indices of the system (A, b) and observe that we have immediately an index assignability theorem.

Theorem 6. *The characteristic indices of a controllable single-input system can be altered in an arbitrary fashion by use of state-space feedback.*

In particular Theorem 6 allows us to obtain real indices. So Theorem 5 follows from its real version.

4.2. Recall that in the real Heymann lemma the characteristic indices $\alpha_1, \ldots, \alpha_n$ of the system (A, b) are such that

$$\lambda^n - \alpha_1\lambda^{n-1} - \cdots - \alpha_n$$

is the characteristic polynomial of A. One consequence of this is that the α_i are independent of the choice of the vector b, so long as the system remains controllable.

As it turns out, no such thing occurs in the quaternionic case. It is easy to see why:

Suppose the single-input system (A, b) is controllable. Suppose also that at least one characteristic index of (A, b) is nonreal, say α_i. Then, there are nonzero $\beta \in \mathbf{H}$ such that

$$\alpha_i \neq \alpha_i^{\beta}. \tag{4.2.1}$$

Now, of course, the system $(A, b\beta)$ is controllable. But, its characteristic indices are $\alpha_1^{\beta}, \ldots, \alpha_n^{\beta}$. To see this just note that by (2.1.1)

$$0 = A^n b\beta - A^{n-1}b\beta\alpha_1^{\beta} - \cdots - b\beta\alpha_n^{\beta}.$$

So by (4.2.1), (A, b) and $(A, b\beta)$ have distinct characteristic indices.

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