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Cofree coalgebras and multivariable recursiveness

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Abstract

For coalgebras over fields, there is a well-known construction which gives the cofree coalgebra over a vector space as a certain completion of the tensor coalgebra. In the case of a one-dimensional vector space this is the coalgebra of recursive sequences. In this paper, it is shown that similar ideas work in the multivariable case over rings (instead of fields). In particular, this paper contains a notion of recursiveness that exactly fits. For the case of a finite number of noncommuting variables over a field, it is the same as Schützenberger recognizability. There are applications to the question of the main theorem of coalgebras for coalgebras over rings. As should be the case, the cofree coalgebra over a finitely generated free module over a ring is the 'zero dual' of the free algebra over that module. A final application is a faithful representation theorem for coalgebras, that is representing a coalgebra as a subcoalgebra of a matrix-like coalgebra.

MSC: 16W30

1. Introduction

Let K be a field and V a vector space over K. Let $TV = K \oplus V \oplus V^{\oplus 2} \oplus V^{\oplus 3} \oplus \dots$ be the tensor coalgebra over V. It is relatively well known that a certain completion of TV which could be called the representative completion, TV_{repr}, is the cofree coalgebra over V in the categories of K-modules and coassociative K-coalgebras with counit. That means that TV_{repr} comes with a natural projection $TV_{\text{repr}} \xrightarrow{\pi} V$ which satisfies the following universal property. For any K-coalgebra C with counit and map of k-vector

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spaces $C\xrightarrow{\varphi}V$, there is a unique morphism of K-coalgebras with counit $C\xrightarrow{\tilde{\varphi}}TV_{\mathrm{repr}}$ such that $\pi\circ\tilde{\varphi}=\varphi$. This construction of the cofree coalgebra over a vector space is from [3], and has been around since a preprint version of that paper circulated in 1974. It should be called, I think, the Block–Leroux theorem.

It is the purpose of this paper to show that these constructions still work in the case of free modules over an arbitrary Noetherian integral domain and their duals, to explain in this more general setting the role of recursiveness (which plays such a nice role in the case of $\dim_k(V)=1$, see [16]) and to point out an error in [10]. There are also some results and open problems for the case of the coassociative cofree coalgebra over not necessarily free modules.

As applications, there are a new proof of the (so-called) main theorem of coalgebras over a field, a proof of the main theorem property for coalgebras over a principal ideal domain, a proof of the main theorem property for various kinds of cofree coalgebras over Noetherian integral domains, the theorem that, in appropriate cases, the cofree coalgebra over a ring is the 'zero dual' of the free algebra over the dual module, and finally, results on (faithful) representations of coalgebras, i.e. on (injective) coalgebra homomorphisms into matrix-like coalgebras. This is not the dual of the notion of a representation of an algebra; that dual notion is a corepresentation of a coalgebra, also called a comodule. As far as I know, these are among the first results on representations of coalgebras.

Probably the main contribution of this paper is the definition/recognition of an appropriate notion of recursiveness in the multivariable case and the proof that recursiveness in this sense is the same as representativeness. For the case of a finitely generated vector space over a field, this notion of recursiveness is the same as rationality or recognizability in the sense of Schützenberger, which in turn is the same as realizability in the sense of system and control theory or automata theory. This notion of recursiveness is more general than the obvious one even in the case of commutative power series in more than one variable.

There also result two different Kleene–Schützenberger theorems for noncommuting power series in infinitely many variables over a Noetherian integral domain generalizing the original result for finitely many indeterminates.

2. The tensor coalgebra over a module, the tensor algebra, and tensor power series

Let A be a commutative ring with unit element and M a module over A. The tensor powers of M (over A) are denoted

$$T^0M=A,\qquad T^1A=M,\qquad T^2M=M\otimes M=M^{\otimes 2},$$
$$\dots,T^nM=M^{\otimes n},\dots \tag{2.1}$$

Consider the direct sum, the module of tensor polynomials,

$$TM=\bigoplus_{n=0}^{\infty}T^nM. \tag{2.2}$$

The elements of $T^sM = M^{\otimes s}$ are called (homogeneous) tensors of degree s. There are natural isomorphisms for all $n \in \{0,1,2,3,\dots\} = \mathbf{N} \cup \{0\}$

$$\psi_{k,l} : T^nM \to T^kM \otimes T^lM, \quad k,l \in \mathbf{N} \cup \{0\},\ k+l=n \tag{2.3}$$

and using these, define a comultiplication on TM by assigning to a (homogeneous) tensor $t \in T^nM$, the element

$$\mu(t) = \sum_{k=0}^{n} \psi_{k,n-k}(t) \in \bigoplus_{k=0}^{n} T^kM \otimes T^{n-k}M \subset TM \otimes TM \tag{2.4}$$

and extending linearly. Together with the natural projection on the zeroth factor of (2.2), the tensors of degree zero,

$$\varepsilon : TM \to A \tag{2.5}$$

as a counit, this defines a coalgebra structure (over A) on TM, the tensor coalgebra.

There is, of course, also an algebra structure on TM

$$m : TM \otimes TM \to TM, \qquad e : A \to TM \tag{2.6}$$

determined by assigning to $t \in T^kM$ and $s \in T^lM$ the element $m(t,s) = \psi_{k,l}^{-1}(t \otimes s) \in T^{k+l}M$ and taking for e the natural inclusion $A \subset TM$ of A as the zeroth summand of TM. This is the tensor algebra of M over A. (However, (2.4), (2.5), (2.6) do not combine to define a bialgebra structure.)

We also consider the completion

$$\hat{T}M = \prod_{n=0}^{\infty} T^nM \tag{2.7}$$

of TM with product and unit element determined by (2.6). This is the algebra of tensor power series of M over A, or the module of tensor power series when the multiplication is not being considered.

Let

$$M^* = \mathrm{Mod}_A(M,A) \quad \text{and} \quad \langle\ ,\ \rangle : M \times M^* \to A \tag{2.8}$$

be the linear dual of M together with the canonical pairing $\langle x, \varphi \rangle = \varphi(x)$, $x \in M$, $\varphi \in M^*$. The elements of $\hat{T}M$ define functionals on $T(M^*)$

$$\langle f, \varphi \rangle = \sum \langle f^i, \varphi^i \rangle, \tag{2.9}$$

where the $f^i \in T^iM$ and $\varphi^j \in T^j(M^*)$ are the homogeneous components of f and φ, and the pairings are defined by

$$T^jM \times T^j(M^*) \to A, \quad \langle x_1 \otimes \cdots \otimes x_j,\ y_1 \otimes \cdots \otimes y_j \rangle = \langle x_1, y_1 \rangle \cdots \langle x_j, y_j \rangle \tag{2.10}$$

(which is well defined). The sum in (2.9) is well defined because only finitely many of the homogeneous components of φ are nonzero. In case M is free (but also in other suitable cases) the functional defined by an element of $\hat{T}M$ uniquely determines that element.

If M is free and finitely generated, say with basis $X_1,\dots,X_m$, the tensor power series algebra naturally identifies with the algebra of noncommutative power series in the $X_1,\dots,X_m$ over A and $T(M^*)$ is the algebra with as underlying module the free module with as basis all words $\alpha=[a_1,\dots,a_n]$ over the alphabet $\{1,\dots,m\}$ (including the empty word) and concatenation of words as product. For a word $\alpha=[a_1,\dots,a_n]$ write $X_a=X_{a_1},\dots X_{a_n}$ and $X_{[\,]}=1$. Then, for an element

$$f\in\hat{T}M=A\langle\langle X_1,\dots,X_m\rangle\rangle,\qquad f=\sum_{\alpha}c_\alpha X_\alpha \tag{2.11}$$

and a basis element $\beta=[b_1,\dots,b_r]\in T(M^*)$

$$\langle f,\beta\rangle=c_\beta. \tag{2.12}$$

Finally, consider the completed tensor product

$$\hat{T}M\hat{\otimes}\hat{T}M=\prod_{i,j}T^iM\otimes T^jM. \tag{2.13}$$

The coassociative comultiplication μ defined by (2.4) uniquely extends to a morphism

$$\mu:\hat{T}M\to\hat{T}M\hat{\otimes}\hat{T}M, \tag{2.14}$$

but this does not define a coalgebra structure on $\hat{T}M$ because for most tensor power series f, $\mu(f)$ does not lie in $\hat{T}M\otimes\hat{T}M$ (but only in $\hat{T}M\hat{\otimes}\hat{T}M$).

The bihomogeneous components of $\hat{T}M\hat{\otimes}\hat{T}M$ are indexed by pairs of nonnegative integers, and thus an element of $\hat{T}M\hat{\otimes}\hat{T}M$ is conveniently represented as a bi-infinite matrix:

$$a=\begin{pmatrix} a_{00} & a_{01} & a_{02} & \cdots \\ a_{10} & a_{11} & a_{12} & \cdots \\ a_{20} & a_{21} & a_{22} & \cdots \\ \vdots & \vdots & \vdots & \end{pmatrix},\qquad a_{ij}\in T^iM\otimes T^jM\subset\hat{T}M\hat{\otimes}\hat{T}M. \tag{2.15}$$

With this notation, the element $\mu(f)\in\hat{T}M\hat{\otimes}\hat{T}M$, $f=(f^0,f^1,f^2,\dots)$, $f^i\in T^iM$ is equal to the Hankel-like matrix

$$\mu(f)=\begin{pmatrix} f^0 & f^1 & f^2 & \cdots \\ f^1 & f^2 & f^3 & \cdots \\ f^2 & f^3 & f^4 & \cdots \\ \vdots & \vdots & \vdots & \end{pmatrix}. \tag{2.16}$$

3. Representative tensor power series and cofree coalgebras over free modules

A tensor power series $f \in \hat{T}M$ is called representative if for some finite k there are tensor power series $g_1, \ldots, g_k, h_1, \ldots, h_k$ in $\hat{T}M$ such that

$$\mu(f) = \sum_{i=1}^{k} g_i \otimes h_i. \tag{3.1}$$

Written out in the infinite matrix notation of Section 2 this becomes

$$\begin{pmatrix} g_1^0 & g_2^0 & \cdots & g_k^0 \\ g_1^1 & g_1^2 & \cdots & g_k^1 \\ g_1^2 & g_2^2 & \cdots & g_k^2 \\ \vdots & \vdots & \cdots & \vdots \end{pmatrix} \otimes \begin{pmatrix} h_1^0 & h_1^1 & h_1^2 & \cdots \\ h_2^0 & h_2^1 & h_2^2 & \cdots \\ \vdots & \vdots & \vdots & \vdots \\ h_k^0 & h_k^1 & h_k^2 & \cdots \end{pmatrix} = \begin{pmatrix} f^0 & f^1 & f^2 & \cdots \\ f^1 & f^2 & f^3 & \cdots \\ f^2 & f^3 & f^4 & \cdots \\ \vdots & \vdots & \vdots & \end{pmatrix} \tag{3.2}$$

(where f^i is the degree i component of f, etc.).

If the module M is free, we can interpret an element $f \in \hat{T}M$ as a functional on $T(M^*)$ and then (3.1) or (3.2) is equivalent to

$$f(ab) = \sum_{i=1}^{k} g_i(a)h_i(b) \quad \text{for all } a, b \in T(M^*). \tag{3.3}$$

Here on the left ab is the tensor algebra multiplication of $a, b \in T(M^*)$.

The terminology 'representative' comes from representation theory. If S is a semigroup and $\rho : s \mapsto A(s)$ is a finite dimensional matrix representation of S, then the matrix entries, seen as functions on S, are representative. Indeed by the definition of the notion of a representation

$$a_{ij}(ss') = \sum_{k} a_{ik}(s)a_{kj}(s').$$

For all $b \in T(M^*)$ and $f \in \hat{T}M$ define the right translate $R_b f$ of f as the functional

$$R_b f(a) = f(ab). \tag{3.4}$$

It is instructive to figure out to what element of $\hat{T}M$ the functional $R_b f$ corresponds; it is also somewhat necessary to do this, because in the case that M is not of finite rank it is not a priori totally clear that $R_b f$ is a functional that comes from some element in $\hat{T}M$. Let $\{X_j \colon j \in J\}$ be a basis of M. Write the element f of $\hat{T}M$ more precisely as

$$f = \sum_{k=0}^{\infty} \sum_{\lg(\alpha)=k}^{<\infty} c_\alpha X_\alpha, \tag{3.5}$$

where $\alpha = [a_1, \ldots, a_k]$, $a_i \in J$, is a word of length k over the alphabet J. Note that the outer sum in (3.5) can be infinite but that the inner sums in (3.5) must be finite (as has been indicated by the upper limit '$<\infty$' in the notation for the inner sum). In other words, for each length k there are only finitely many α of length k for which $c_\alpha \neq 0$. An element $b \in T(M^*)$ can be written as a sum:

$$b = \sum_{k}^{<\infty} \sum_{\lg(\beta)=k} c_\beta \beta. \tag{3.6}$$

This time the outer sum is finite (as indicated) but the inner sums may well be infinite (if the module M is of infinite rank). For instance

$$\sum_{j\in J} [j]$$

is a perfectly good element of M^*. But for $k \geqslant 2$ only certain infinite sums are actually elements of $(M^*)^{\otimes k}$. For instance a sum

$$\sum_{i,j\in J} c_{ij}[i,j]$$

is in $(M^*)^{\otimes 2}$ if and only if the matrix of coefficients

$$(c_{ij})_{i,j\in J}$$

has finite rank (assuming A to be an integral domain so that the notion of rank is well defined). For higher k similar conditions can be formulated (see Theorem 6.25). As it turns out, this does not matter much and $R_b f$ is well defined and a functional that comes from an element of $\hat{T}M$ for all b of the form (3.6).

Write

$$f = \sum_{k} \sum_{\lg(\alpha)=k} a_\alpha X_\alpha, \qquad b = \sum_{k=0}^{r} \sum_{\lg(\beta)=k} b_\beta \beta, \qquad R_b f = \sum_{\gamma} c_\gamma X_\gamma. \tag{3.7}$$

Then

$$c_\gamma = R_b f(\gamma) = f(\gamma b) = \sum_{k=0}^{r} \sum_{\beta} a_{\gamma\beta} b_\beta, \tag{3.8}$$

where $\gamma\beta$ is the concatenation of the words γ and β. Now consider the set

$$\{\gamma\beta\colon\ a_{\gamma\beta} b_\beta \neq 0\}.$$

For γ varying but of a fixed length l and β varying arbitrarily this is a finite set. Indeed if $\lg(\beta) > r$, then $b_\beta = 0$. Thus all the $\gamma\beta$ have length $\leqslant l + r$ and there are only finitely many a_α of length $\leqslant l + r$ that are different from zero. Thus, first of all, the sum (3.8) is finite so that c_γ is well defined and, second, for each length l there are only finitely many γ of that length that are nonzero.

Consider, in particular, the case that b consists of a single word, $b = \beta$. Then

$$R_\beta f = \sum_{\alpha} a_{\alpha\beta} X_\alpha \tag{3.9}$$

and

$$f = (R_\beta f)X_\beta + \sum_{\alpha'} a_{\alpha'} X_{\alpha'}, \tag{3.10}$$

where the sum on the right-hand side of (3.10) is over all words α' that do not have β as a tail. (The tails of a word $\alpha = [a_1, \ldots, a_n]$ are the words $[a_i, a_{i+1}, \ldots, a_n]$, $i = 1, 2, \ldots, n+1$; the empty word occurring for $i = n+1$.) Thus, $R_\beta f$ is obtained by dividing f on the right by X_β 'as best as possible' and a possible suggestive notation could be $R_\beta f = [f X_\beta^{-1}]$ with the square brackets indicating taking the nonnegative degree part in the ring of noncommutative Laurent series over A in the indeterminates X_j.

(It is partly for this reason that in some parts of control theory recursive sequences are often written as power series in an indeterminate t^{-1}.)

3.11. **Definition.** A torsion free A-module M over an integral domain is of finite rank if the vector space $M \otimes_A Q(A)$ over the quotient field $Q(A)$ of A is finite dimensional.

3.12. **Theorem.** *Let M be a free module over a Noetherian integral domain A with basis $\{X_j\colon j \in J\}$, and let $f \in \hat{T}M = A\langle\langle X_j\colon j \in J\rangle\rangle$ be a noncommutative power series in the X_j. The following conditions on f are equivalent*:

(i) *f is representative.*
(ii) *The A-module $Rf = \{R_b f\colon b \in TM^*\}$ is of finite rank.*
(iii) *The A-module $R_B f$ spanned by the $R_\beta f$, $\beta \in \mathrm{Word}(J)$ is of finite rank.*

Moreover, if f is representative $\mu(f)$ can be written in the form

$$\mu(f) = \sum_{i=1}^{r} g_i \otimes h_i \tag{3.13}$$

with all the g_i, h_i representative.

For a free module M over a ring A, let $TM_{\mathrm{repr}} \subset \hat{T}M$ be the module of all representative tensor power series over A. Then, TM_{repr} is a coalgebra by 3.13 (with the comultiplication induced by (2.14). Unless $M = 0$, TM_{repr} is always strictly larger than TM (and strictly smaller than $\hat{T}M$). For instance, for M free of rank 1, $\sum_{i=0}^{\infty} X^i$ is in $TM_{\mathrm{repr}} \setminus TM$.

3.14. **Theorem** (Generalized Block–Leroux theorem). *For a free module M over a Noetherian integral domain A, the coalgebra of representative tensor power series TM_{repr} is the free coalgebra over M.*

Proof of (Theorem 3.12). If f is representative, then as functionals (see (3.1), (3.3))

$$R_b f(a) = f(ab) = \sum_{i=1}^{r} g_i(a) h_i(b) \tag{3.15}$$

so that all the $R_b f$'s are linear combinations of the $g_1, \dots, g_r$. The module $R_B f$ is a submodule of Rf. This takes care of the implications (i) $\Rightarrow$ (ii) $\Rightarrow$ (iii). The final implication (iii) $\Rightarrow$ (i) needs a lemma.

3.16. **Lemma.** *Let $R_B f$ be of finite rank. Then for each k there are only finitely many words β of length k for which $R_\beta f \neq 0$.*

Of course, the lemma says something nontrivial only in the case that the free module M is of infinite rank. This lemma is due to Block and Leroux [3]. The proof given below is different.

Proof (of Lemma 3.16). Take some total ordering on the infinite index set J and order the words X_α by length first and lexicographic ordering thereafter. For each β for which $R_\beta f \neq 0$, let $\gamma(\beta)$ be the first term of $R_\beta f$ with nonzero coefficient. Take a $k \in \mathbf{N}$. Suppose there are infinitely many β of length k for which $R_\beta f$ is nonzero. There are two possibilities:

(i) There is a natural number m such that there are infinitely many β's of length k for which $\lg(\gamma(\beta)) = m$.
(ii) There is a sequence of $\beta_1, \beta_2, \beta_3, \dots, \lg(\beta_i) = k$ for all i, such that $\lg(\gamma(\beta)) \to \infty$ as $i \to \infty$.

In the second case the rank of $R_\beta f$ is obviously infinite. In the first case the coefficients in f of the $X_{\gamma(\beta)} X_\beta$ for these β's are all nonzero and that gives infinitely many different monomials of length $m + k$ in f with nonzero coefficient contradicting that $f \in \hat{T}M$. This proves the lemma.

3.17. **Lemma.** *If M is a submodule of a module of the form $A^I = \prod_{i \in I} A$ over a Noetherian integral domain A, and if M has finite rank, i.e.* $\dim_K(M \otimes K) < \infty$, *where K is the quotient field of A, then M is finitely generated.*

This will be used below. A different proof is in [20]. To see this result consider the diagram

$$\begin{array}{ccccc} M & \longrightarrow & A^I & \longrightarrow & A^n \\ \downarrow & & \downarrow & & \downarrow \\ M \otimes K & \longrightarrow & K^I & \longrightarrow & K^n \end{array},$$

where the right-most horizontal arrows are natural projections onto a suitable finite subset of the coordinates. Because $\dim_K(M \otimes K) < \infty$, there is a finite n such that the composed lower morphism is injective. As M is torsion free, the composed morphism from M along the upper edge and then down on the right is injective and, hence, the composed morphism of the upper edge is injective. Thus M is isomorphic to a submodule of a finitely generated A-module and hence is finitely generated because A is Noetherian.

Proof of Theorem 3.12 (continued). Now let $R_B f$ be of finite rank and let $g_1, \dots, g_r$ be a finite generating set. Such a set exists by Lemma 3.17. For each word β, choose coefficients $a_{i,\beta} \in A$ such that

$$R_\beta f = \sum_{i=1}^{r} a_{i,\beta} g_i \tag{3.18}$$

taking care to take $a_{i,\beta} = 0$ for all β's for which $R_\beta f = 0$. Thus, for each length k, there are only finitely many $a_{i,\beta} \neq 0$ with β of length k. Now define

$$h_i = \sum_\beta a_{i,\beta} X_\beta. \tag{3.19}$$

By the remark just made $h_i \in \hat{T}M$. Further, for all words $\alpha, \beta \in \mathrm{Word}(J)$

$$f(\alpha\beta) = R_\beta f(\alpha) = \sum_{i=1}^{r} g_i(\alpha) a_{i,\beta} = \sum_{i=1}^{r} g_i(\alpha) h_i(\beta) \tag{3.20}$$

proving that f is representative. This proves the implication (iii) $\Rightarrow$ (i) and the equivalence of the three conditions (i), (ii), (iii).

As to the last statement of the theorem, so far it has been shown that if f is representative, then there are tensor power series $g_i \in R_B f$, $i = 1, \dots, r$, and tensor power series h_i, $i = 1, \dots, r$, given by (3.19) such that (3.20) holds. Consider the matrix of coefficients $(a_{i,\beta})$. By (3.19), or rather the remarks just before that, the rows of this matrix can be considered as the $R_\beta f$ and the columns as the elements h_i. Let K be the quotient field of A, and consider the matrix $(a_{i,\beta})$ over K. Because row rank is column rank, it follows that

$$\dim\left(\sum_i K g_i\right) = \dim\left(\sum_\beta K R_\beta f\right) = \dim\left(\sum_i K h_i\right) = s. \tag{3.21}$$

Now consider left translates of f defined by

$$L_a f(b) = f(ab), \quad L_\alpha f(\beta) = f(\alpha\beta)$$

and let $L_B f$ be the vector space spanned by the $L_B f$. Reasoning as before, one finds that $L_B f$ has finite rank and that there are $h'_i \in L_B f$, $i = 1, \dots, r'$ and g'_i, $i = 1, \dots, r'$, such that

$$f(\alpha\beta) = \sum_{i=1}^{r'} g'_i(\alpha) h'_i(\beta) \tag{3.22}$$

and that

$$\dim\left(\sum_i K g'_i\right) = \dim\left(\sum_a K L_\alpha f\right) = \dim\left(\sum_i K h'_i\right) = s'.$$

Suppose that $s' \leqslant s$. Then (3.22) says that each $R_\beta f$ and, in particular, the g_i are linear combinations of the g'_j. Hence

$$\sum_i Kg_i \subset \sum_j Kg'_j. \tag{3.23}$$

Hence, $s = s'$ and the two spaces in (3.23) are equal. Thus, each g'_j is a K-linear combination of the g_i. Now, $R_\gamma(R_\beta f) = R_{\beta\gamma} f$ and thus, for some $a \in A$, $R_B(ag'_j)$ is of finite rank and thus, $R_B(g'_j)$ is of finite rank making the g'_j representative so that (3.22) holds with all the g'_j, h'_j representative. If $s' \geqslant s$, reason symmetrically using the h'_j, h_i instead of the g_i, g'_j. This finishes the proof of the theorem.

3.24. **Remarks.** In the case that A is a field or a principal ideal domain, one can thus show that if f is representative then

$$\mu(f) \in R_B f \otimes L_B f \tag{3.25}$$

because in this case the $R_B f$ (resp. $L_B f$) are free modules.

In general, the proof above does not quite give this. But introduce the pure closures of these modules (see e.g. [11, p. 372] for the notion of a pure subgroup; this is a natural analogue for modules),

$$\bar{L}_B f = \{f \in \hat{T}M\colon\ af \in L_B f \text{ for some } a \in A\},$$
$$\bar{R}_B f = \{f \in \hat{T}M\colon\ af \in R_B f \text{ for some } a \in A\}. \tag{3.26}$$

Then the Proof of Theorem 3.12 shows that

$$\mu(f) \in \bar{R}_B f \otimes \bar{L}_B f \tag{3.27}$$

and, of course, the elements of $\bar{R}_B f$ and $\bar{L}_B f$ are recursive and the ranks of these two modules are the same as those of $R_B f$ and $L_B f$.

Proof of Theorem 3.14. First of all, the theorem implies that TM_{repr} is a coalgebra under the 'comultiplication' induced from $\hat{T}M$. This is not entirely obvious because, although Theorem 3.12 says that $\mu(f)$ lands in the image of $TM_{\text{repr}} \otimes TM_{\text{repr}}$ in $\hat{T}M \otimes \hat{T}M$, if the tensor square of the inclusion $TM_{\text{repr}} \subset \hat{T}M$ were not injective, one would not know what representatives to choose and the comultiplication would not be well defined. And, of course, the tensor square of an inclusion need not be injective even in very nice looking cases, such as me well-known example mentioned just below (9.3). In the present case, there is no problem so to speak, because, the comultiplication is finitely defined, i.e. the homogeneous components $\mu(f)^n$ are completely determined by the restriction of f to T^nM.

More generally there is the following (trivial) lemma.

3.28. **Lemma.** *Let N be a submodule of $\hat{T}M$ that contains TM. Then the tensor square of the inclusion is injective.*

Proof. Let f_i, g_i be elements of N with homogeneous components f_i^m, g_i^m of degree m (in $\hat{T}M$ and hence in TM and in N). Suppose that $\sum_i f_i \otimes g_i \neq 0$ in $N \otimes N$. Then there is a finite k such that $\sum_{s+t=k,i} f_i^s \otimes g_i^t \neq 0$, these being the homogeneous components of $\sum_i f_i \otimes g_i$. But this homogeneous component of degree k is in $TM \otimes TM$ and the tensor square of $TM \subset \hat{T}M$ is certainly injective.

Proof of Theorem 3.14 (continued). The main statement of the theorem means that for every coalgebra with counit (C, μ, ε) over A and every A-module morphism $C \xrightarrow{\varphi} M$, there is a unique counit preserving morphism of coalgebras $C \xrightarrow{\tilde{\varphi}} TM_{\mathrm{repr}}$ such that $\pi\tilde{\varphi} = \varphi$.

Define inductively

$$\mu_0 = \varepsilon : M \to A, \qquad \mu_1 = id : M \to M, \qquad \mu_2 = \mu : M \to M^{\otimes 2}, \ldots,$$
$$\mu_{n+1} = (\mu \otimes id^{\otimes(n-1)})\mu_n : M \to M^{\otimes(n+1)}, \ldots \,. \tag{3.28}$$

Then the fact that $\tilde{\varphi}$ must be a counit preserving morphism of coalgebras immediately gives that it must be given by the formula

$$\tilde{\varphi}(c) = (\mu_0(c) = \varepsilon(c), \varphi(\mu_1(c)) = \varphi(c), \varphi^{\otimes 2}(\mu_2(c)), \ldots, \varphi^{\otimes n}(\mu_2(c)), \ldots). \tag{3.29}$$

It remains to show that the tensor power series (3.29) are representative. To see this, let

$$\mu(c) = \sum_j c_{1;j} \otimes c_{2;j}.$$

Then, by the coassociativity of μ and the counit property of $\mu_0 = \varepsilon$

$$\sum_j \mu_k(c_{1;j}) \otimes \mu_l(c_{2;j}) = \mu_{k+l}(c)$$

and it follows that

$$\mu(\tilde{\varphi}(c)) = \sum_j \tilde{\varphi}(c_{1;j}) \otimes \tilde{\varphi}(c_{2;j})$$

(where the left-hand side μ is the one of TM_{repr}) so that $\mu(\tilde{\varphi}(c))$ is a finite sum of tensor power series of the same type, thus proving that $\tilde{\varphi}(c)$ is representative (and reconfirming that $\tilde{\varphi}$ is a morphism of coalgebras). This finishes the proof of the theorem.

3.30. **Remarks.** Very little of the above makes sense in the case there is torsion present in the module M. The definition of ‘representative’ still makes sense. However, there seems to be no way to define anything like the $R_\beta f$ which play such a crucial role in the arguments above. To illustrate the point, consider the case that $M = \mathbf{Z} \otimes \mathbf{Z}/(n)$ over the integers. Taking the generators $X_1 = (1, 0)$, $X_2 = (0, 1)$, it is still possible to view the elements of $\hat{T}M$ as noncommutative power series in X_1, X_2 with the proviso that if an X_2 is present the coefficient is only defined as modulo n. Now take a monomial of the form $aX_1X_1\cdots X_1X_2$ and try to apply $R_{[2]}$ in the way of formula (3.10). This gives the monomial $aX_1\cdots X_1$, with, however, the coefficient only determined modulo n, while for these monomials we need coefficients in $\mathbf{Z}$.

Returning to the setting of Theorem 3.12, things should still work for more general modules than free modules. In particular, Theorem 3.12 should remain true and its proof work for, for instance, reflexive A-modules and the linear duals of free A-modules. This latter case is dealt with in Section 6. See Section 8 for the notion of reflexive A-modules and examples of those. It also should work in all cases where the functionals on the dual module of M suffice to distinguish the points in M; i.e. when the canonical morphism $M\to M^{**}$ is injective.

4. Recursive tensor power series

Let $f\in\hat{T}M$ with its homogeneous component weight n denoted f^n. A natural possible definition of left (simply polynomial) recursiveness could be as follows: There is a finite set of monomials $c_iX_{\lambda_i},\ i=1,\dots,l,\lg(\lambda_i)\geqslant 1$ such that for large enough n

$$f^n=\sum_{i=1}^{l}c_iX_{\lambda_i}f^{n-\lg(\lambda_i)}\tag{4.1}$$

and it is right (simply polynomial) recursive if there is a finite set of monomials $d_iX_{\rho_i},\ i=1,\dots,r$, such that for large enough n

$$f^n=\sum_{i=1}^{r}f^{n-\lg(\rho_i)}d_iX_{\rho_i}.\tag{4.2}$$

And f is (simply polynomial) recursive if it is both left and right (simply polynomial) recursive in the sense of formulas (4.1) and (4.2). This is probably the first guess one would make at a definition of recursiveness in the multivariable (noncommutative) case. As it turns out, this is not a general enough notion of recursiveness for the present purposes. To distinguish this possible notion from the more general one below, I shall call this simply polynomial recursiveness (as already indicated). It will be discussed a little more in Section 7.

The right notion of recursiveness that fits with cofree algebras over free modules is as follows.

4.3. **Definition.** A tensor power series f over a free module M with basis $\{X_j\colon j\in J\}$ over A is left recursive if there is a finite set of monomials $X_{\lambda_i},\ i=1,\dots,l$, and for some fixed $s>\max\{\lg(\lambda_i),\ i=1,\dots,l\}$, there are coefficients $c_{\gamma,i}\in A$, for each $i\in\{1,\dots,l\}$ and word $\gamma\in\mathrm{Word}(J)$ of length s, such that for $n\geqslant s$ for each $\alpha\in\mathrm{Word}(J)$ of length n

$$f(\alpha)=\sum_{i=1}^{l}c_{\alpha_{\mathrm{pre}(s)},i}f(\lambda_i\alpha_{\mathrm{suf}}),\tag{4.4}$$

where if β, γ are two words over J, $\beta\gamma$ is the concatenation of them and where for a word α of length $\geqslant s$, $\alpha_{\mathrm{pre}(s)}$ is the prefix of α of length s and α_{suf} is the corresponding suffix (or tail), so that $\alpha = \alpha_{\mathrm{pre}(s)}\alpha_{\mathrm{suf}}$.

The tensor power series f is right recursive if there is a finite set of monomials X_{ρ_i}, $i = 1, \ldots, r$, and for some fixed $t > \max\{\lg(\rho_i),\ i = 1, \ldots, r\}$, there are coefficients $d_{\gamma,i} \in A$, for each $i \in \{1, \ldots, r\}$ and word $\gamma \in \mathrm{Word}(J)$ of length t, such that for $n \geqslant t$ for each $\alpha \in \mathrm{Word}(J)$ of length n

$$f(\alpha) = \sum_{i=1}^{l} d_{\alpha_{\mathrm{suf}(t)}, i} f(\alpha_{\mathrm{pre}}\rho_i), \tag{4.5}$$

where this time $\alpha_{\mathrm{suf}(t)}$ is the suffix (tail) of α of length t and α_{pre} is the corresponding prefix (so that $\alpha = \alpha_{\mathrm{pre}}\alpha_{\mathrm{suf}(t)}$).

The tensor power series f is left (resp. right) recursive with finiteness condition if is left (resp. right) recursive and moreover the recursion coefficient matrix $(c_{\alpha,i})_{\lg(\alpha)=s,i}$ (resp. $(d_{\alpha,i})_{\lg(\alpha)=s,i}$) has only finitely many entries unequal to zero.

The tensor power series f is recursive if it is both left and right recursive in the sense of formulas (4.4) and (4.5); it is recursive with finiteness condition if it is left recursive with finiteness condition and right recursive with finiteness condition.

Of course, ‘with finiteness condition’ only gives something extra if there are an infinite number of indeterminates.

Note that these two formulas (4.4) and (4.5) exactly capture the idea of recursiveness in the sense that a coefficient $f(\alpha)$ for large enough $\lg(\alpha)$ is (both from the left and the right) a linear combination of coefficients for words of lesser length in a uniform manner (same coefficients).

In the case of power series in one variable, this notion of recursiveness is the same as simple polynomial recursiveness and the same as the usual notion of recursiveness for sequences.

But even in the commutative case for more than one indeterminate, this notion of recursiveness is more general than (simply) polynomial recursiveness.

As it turns out, these notions of recursiveness are closely related to a notion of recursiveness defined by Schützenberger, see (5.6). Indeed, for a finite number of noncommuting indeterminates, the notions turn out to be equivalent; for an infinite number of indeterminates there are important differences.

4.6. **Theorem.** *A tensor power series over a free module is recursive with finiteness condition if and only it is representative.*

Proof. First assume that f is representative. Then by Theorem 3.12, more precisely its proof, see (3.18), there are a finite set of monomials X_{ρ_i} and tensor power series h_i such that for all $\alpha, \beta \in \mathrm{Word}(J)$

$$f(\beta\gamma) = \sum_{i=1}^{r} R_{\rho_i} f(\beta) h_i(\gamma). \tag{4.7}$$

Now take $t = 1 + \max\{\lg(\rho_i)$ and for each α of length $\geqslant t$ take

$$\gamma = \alpha_{\mathrm{suf}(t)}, \quad \beta = \alpha_{\mathrm{pre}}, \quad \alpha = \alpha_{\mathrm{pre}}\alpha_{\mathrm{suf}(t)}$$

in formula (4.7) and observe that (4.5) holds with the monomials X_{ρ_i} for the ρ_i that occur in (4.7) and $d_{i,\gamma}=h_i(\gamma)$ for all γ of length t (because $R_{\rho_i}f(\beta)=f(\beta\rho_i)$). Note that only finitely many of the $d_{i,\gamma}=h_i(\gamma)$ are nonzero because the h_i are tensor power series and there are only finitely many of them. This gives right recursiveness with finiteness condition of f. Similarly, left recursiveness with finiteness condition follows from the fact (see (3.22)) that there are tensor power series g'_j, $j = 1,\dots,l$, and monomials X_{λ_j} such that

$$f(\beta\gamma) = \sum_{j=1}^{l} g'_j(\beta)L_{\lambda_j}f(\gamma). \tag{4.8}$$

This proves that f is recursive with finiteness condition if it is representative.

Inversely, suppose that f is recursive, so that (4.4) and (4.5) hold. Take $n = \max\{s,t\}$. Working in the completed tensor algebra $\hat{T}M$ it is easy to find polynomial tensors g_i, h_i, $i = 1,\dots,r$, such that (for any fixed s; r depending on s)

$$\begin{pmatrix} g_1^0 & g_2^0 & \cdots & g_r^0 \\ g_1^1 & g_2^1 & \cdots & g_2^1 \\ \vdots & \vdots & & \vdots \\ g_1^s & g_2^s & \cdots & g_r^s \end{pmatrix} \otimes \begin{pmatrix} h_1^0 & h_1^1 & \cdots & h_1^s \\ h_2^0 & h_2^1 & \cdots & h_2^s \\ \vdots & \vdots & & \vdots \\ h_r^0 & h_r^1 & \cdots & h_r^s \end{pmatrix} = \begin{pmatrix} f^0 & f^1 & \cdots & f^s \\ f^1 & f^2 & \cdots & f^{s+1} \\ \vdots & \vdots & & \vdots \\ f_s^0 & fs+1 & \cdots & f^{2s} \end{pmatrix}. \tag{4.9}$$

The idea is to use lots of zeros and ones. Here is the start. Let

$$\begin{aligned} f^2 &= \sum_{j=1}^{m} b_j^{12} \otimes c_j^{12}, \quad b_j^{12}, c_j^{12} \in M, \\ f^3 &= \sum_{j=1}^{m} b_j^{13} \otimes c_j^{23} = \sum_{j=1}^{m} b_j^{23} \otimes c_j^{13}, \quad b_j^{13}, c_j^{13} \in M, \quad b_j^{23}, c_j^{23} \in M^{\otimes 2}, \\ f^4 &= \sum_{j=1}^{m} b_j^{14} \otimes c_j^{34} = \sum_{j=1}^{m} b_j^{24} \otimes c_j^{24} = \sum_{j=1}^{m} b_j^{34} \otimes c_j^{14}, \quad b_j^{14}, c_j^{14} \in M^{\otimes i}, \\ &\vdots \end{aligned} \tag{4.10}$$

(There is no loss of generality in taking the same m everywhere as long as only finitely many f^i are considered.) Then for $t=3$ in (4.9)

$$\begin{pmatrix} f^0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ f^1 & 0 & b^{12} & b^{13} & b^{14} & 0 & 0 & 0 & 0 & 0 & 0 \\ f^2 & 0 & 0 & 0 & 0 & b^{23} & b^{24} & b^{25} & 0 & 0 & 0 \\ f^3 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & b^{34} & b^{35} & b^{36} \end{pmatrix}$$

$$\otimes \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & f^1 & f^2 & f^3 \\ 0 & c^{12} & 0 & 0 \\ 0 & 0 & c^{23} & 0 \\ 0 & 0 & 0 & c^{34} \\ 0 & c^{13} & 0 & 0 \\ 0 & 0 & c^{24} & 0 \\ 0 & 0 & 0 & c^{35} \\ 0 & c^{14} & 0 & 0 \\ 0 & 0 & c^{25} & 0 \\ 0 & 0 & 0 & c^{36} \end{pmatrix} = \begin{pmatrix} f^0 & f^1 & f^2 & f^3 \\ f^1 & \sum_j b_j^{12}c_j^{12} & \sum_j b_j^{13}c_j^{23} & \sum_j b_j^{14}c_j^{34} \\ f^2 & \sum_j b_j^{23}c_j^{13} & \sum_j b_j^{24}c_j^{24} & \sum_j b_j^{25}c_j^{35} \\ f^3 & \sum_j b_j^{34}c_j^{14} & \sum_j b_j^{35}c_j^{25} & \sum_j b_j^{36}c_j^{36} \end{pmatrix} = \begin{pmatrix} f^0 & f^1 & f^2 & f^3 \\ f^1 & f^2 & f^3 & f^4 \\ f^2 & f^3 & f^4 & f^5 \\ f^3 & f^4 & f^5 & f^6 \end{pmatrix},$$

where b^{ij} is the row vector $(b_1^{ij},\dots,b_m^{ij})$ and c^{ij} is the column vector $(c_1^{ij},\dots,c_m^{ij})^t$. From this the general pattern is clear.

Now extend the $g_1,\dots,g_r$ in (4.9) by the left recursion recipe for f, and extend the $h_1,\dots,h_r$ in (4.9) by the right recursion recipe for f. Then

$$\begin{pmatrix} g_1^0 & g_2^0 & \cdots & g_r^0 \\ g_1^1 & g_2^1 & \cdots & g_r^1 \\ \vdots & \vdots & & \vdots \\ g_1^s & g_2^s & \cdots & g_r^s \\ \vdots & \vdots & & \vdots \end{pmatrix} \otimes \begin{pmatrix} h_1^0 & h_1^1 & \cdots & h_1^s & \cdots \\ h_2^0 & h_2^1 & \cdots & h_2^s & \cdots \\ \vdots & \vdots & & \vdots & \\ h_r^0 & h_r^1 & \cdots & h_r^s & \cdots \end{pmatrix} = \begin{pmatrix} f^0 & f^1 & \cdots & f^s & \cdots \\ f^1 & f^2 & \cdots & f^{s+1} & \cdots \\ \vdots & \vdots & & \vdots & \\ f^s & f^{s+1} & \cdots & f^{2s} & \cdots \\ \vdots & \vdots & & \vdots & \end{pmatrix}$$

so that f is representative.

5. Recognizable tensor power series over free modules of finite rank

In this section we consider noncommutative power series in a finite number of indeterminates $X_1, X_2, \dots, X_n$, over a commutative ring with unit A, i.e. expressions of the form

$$f = \sum_{\alpha \in W} c_\alpha X_\alpha, \qquad \alpha = [a_1, a_2, \dots, a_m], \quad m \in \{0, 1, 2, \dots\},$$
$$a_i \in \{1, 2, \dots\}, \; c_\alpha \in A, \tag{5.1}$$

where W is the free monoid of all words in the alphabet $\{1, 2, \dots, n\}$, and

$$X_\alpha = X_{a_1} X_{a_2} \cdots X_{a_m}. \tag{5.2}$$

The coefficient of X_α in f is variously denoted c_α (as in (5.1)) or $f(\alpha)$, or f_α. Or, equivalently, we consider elements of the tensor power series algebra $\hat{T}M$, where M is the free module over A with basis $X_1, X_2, \dots, X_n$. We also write $A\langle\langle X_1, \dots, X_n \rangle\rangle$ for $\hat{T}M$.

A noncommutative power series (5.1) is said to be recognizable if there exists a natural number r and $r \times r$ matrices $\rho(X_i)$, $i = 1, 2, \dots, n$, an $r \times 1$ matrix b (i.e. a column vector) and a $1 \times r$ matrix c (i.e. a row vector) such that for all $\alpha \in W$

$$f_\alpha = c\rho(X_\alpha)b, \tag{5.3}$$

where

$$\rho(X_\alpha) = \rho(X_{a_1}) \cdots \rho(X_{a_m}), \tag{5.4}$$

i.e. ρ is the representation of the free monoid X^* defined by the n matrices $\rho(X_i)$, $i = 1, 2, \dots, n$. In the control theory world this is usually called realizable. There are two slightly different interpretations of (5.3).

Consider a discrete time automaton with state space A^r and initial state $b \in A^r$. When the automaton is in state $x \in A^r$ at time t and is fed the input X_i, it moves to state $\rho(X_i)x$ and outputs the scalar cx; if it is fed nothing, i.e. the empty word, it stays in the same state and outputs cx. Then, if the automaton starts in the initial state b at time zero and is fed successively the inputs $X_{a_m}, X_{a_{m-1}}, \dots, X_{a_1}$, i.e. it is fed the word X_α, $\alpha = [a_1, a_2, \dots, a_m]$ (from right to left), then the output sequence is

$$cb, c\rho(X_{a_m})b, c\rho(X_{a_{m-1}} X_{a_m})b, \dots, c\rho(X_\alpha)b = f_\alpha, f_\alpha, \dots, f_\alpha, \dots, \tag{5.5}$$

i.e. feeding in a word α produces the corresponding coefficient f_α of f.

A slightly different interpretation, much closer to the setting of the original paper [18] of Schützenberger is that of a 'transition system' with r nodes $q_1, \dots, q_r$ each of which can hold an element of $A\langle\langle X_1, \dots, X_n \rangle\rangle$ in its memory. There is an arrow form node q_i to node q_j for each $k \in \{1, \dots, n\}$ for which $\rho(X_k)_{i,j} \neq 0$ and that node is labelled $\rho(X_k)_{i,j} X_k$. The transition system works as follows. It starts at time zero with memory state b_i for node q_i and outputs cb. If at time t the memory states are y_i, then it outputs

$$c \begin{pmatrix} y_1 \\ \vdots \\ y_n \end{pmatrix}$$

and the memory states at time $t+1$ are

$$y_j(t+1) = \sum \rho(X_k)_{i,j} X_k y_i,$$

i.e. the sums of the labels of all the incoming arrows multiplied with the memory states from which they come. Then, (5.3) means that this transition system successively produces the components f^i of f at time i.

A noncommutative power series in a finite or infinite number of indeterminates (5.1) is left Schützenberger recursive if and only if there is a finite nonempty set of words S closed under taking prefixes (so that in any case the empty word is in S) such that for all words $\beta \in T = SX \setminus S$ there are coefficients $\alpha_{\beta,\alpha}$, $\alpha \in S$, $\beta \in T$ such that for all words γ

$$f_{\beta\gamma} = \sum_{\alpha \in S} a_{\beta,\alpha} f_{\alpha\gamma}. \tag{5.6}$$

At first sight this does not look all that recursive. For one thing, the words $\alpha\gamma$ occurring on the right-hand side of (5.6) may very well have longer lengths than the word $\beta\gamma$. However, for each word ω let α be the longest prefix of ω that is in S, and write $\omega = \alpha\omega'$. Then (5.6) is recursive with respect to the length of ω'.

It is left Schützenberger recursive with finiteness condition if there are only finitely many nonzero coefficients in the recursion matrix $(a_{\beta,\alpha})_{\beta \in T, \alpha \in S}$.

There are obvious corresponding notions of right Schützenberger recursive which work with suffixes instead of prefixes. A noncommutative power series $f \in A\langle\langle X_1, \dots, X_n \rangle\rangle$ is Schützenberger recursive if it is both right and left Schützenberger recursive.

It will turn out that for a finite number of indeterminates the notions of recursiveness and left right recursiveness are equivalent to Schützenberger recursiveness. In the case of an infinite number of indeterminates there are important differences. As will appear later, in the case of tensor power series over the dual of an infinite rank free module, it is the notion of recursiveness as in Section 4 that is the appropriate one.

The main point is that in the Schützenberger case, the power series is entirely determined by the recursion matrix and a finite number of initial conditions, viz. the coefficients of the X_α, $\alpha \in S$, while in the recursive case there are potentially infinitely many initial conditions, viz. the coefficients of the X_α, $\lg(\alpha) < s$.

As an example, the power series in an infinite number of variables

$$\sum_{j \in J} X_j^2$$

is recursive but not (left or right) Schützenberger recursive. Another example is

$$\sum_{j \in J} r_j X_j, \qquad r_j \in A.$$

This is always recursive. But it is left (or right) Schützenberger recursive if and only if the ideal in A generated by the r_j is finitely generated. This is even a little disturbing because one does not intuitively expect a notion like recursiveness to depend on properties of the underlying ring of coefficients.

For a noncommutative power series (5.1) (in a finite or infinite number of indeterminates) the associated Hankel matrix $H(f)$ is an infinity by infinity matrix whose columns and rows are indexed by the free monoid W and whose entry in row α and column β is the coefficient of $X_\alpha X_\beta = X_{\alpha\beta}$ in f. This is not to be confused with the Hankel-like matrix (2.16), though of course the two are far from unrelated. Let A be an integral domain with quotient field $Q(A)$. Then $f \in A\langle\langle W\rangle\rangle$ is said to be of finite Hankel rank if the rank of the matrix $H(f)$ over $Q(A)$ is finite.

It is simple but important to note that the entries in the column labelled β are the coefficients of the $R_\beta f$ and the entries in the row labelled α are the coefficients of $L_\alpha f$.

5.7. **Theorem.** *Let $f \in A\langle\langle X_1,\dots,X_n\rangle\rangle$ be a noncommutative power series in finitely many variables over a Noetherian integral domain A. Then the following are equivalent*:

(i) *f is left recursive*,
(ii) *f is right recursive*,
(iii) *f is recursive*,
(iv) *f is left Schützenberger recursive*,
(v) *f is right Schützenberger recursive*,
(vi) *f is Schützenberger recursive*,
(vii) *f has finite Hankel rank*,
(viii) *f is recognizable* (=*realizable*),
(ix) *f is representative*.

5.8. **Comments.** Over a field the equivalence (iv) $\leftrightarrow$ (v) is basically due to Schützenberger, [18]. However, the two basic constructions work just as well over an integral domain A, see [2].

The equivalence (v) $\leftrightarrow$ (vi) for A, a field, is due to Fliess [8].

A noncommutative power series f is invertible if and only if its constant term is invertible in A. The rational power series are the ones that are contained in the minimal submodule of $A\langle\langle X_1,\dots,X_n\rangle\rangle$ over A that contains $A\langle X_1,\dots,X_n\rangle$ and is closed under inversion (when applicable), sums and products. The celebrated theorem of Kleene–Schützenberger says that the rational noncommutative power series are precisely the recognizable ones, see [9].

Theorem 5.7, of course, subsumes Theorem 4.6 (for the case of finitely many indeterminates). However, the proof is more roundabout and does not show very directly the narrow connection between recursive and representative as in the Proof of Theorem 4.6.

5.9. **Corollary.** *For a finite rank free module M over a Noetherian integral domain*, *the cofree coalgebra over M* is $TM_{\mathrm{repr}} = TM_{\mathrm{real}}$, *the module of realizable noncommutative power series in the $X_1,\dots,X_n$*.

Proof. It follows directly from the realizability property that $TM_{\mathrm{repr}} = TM_{\mathrm{real}}$ is a coalgebra. The remainder of the proof is as in the proof of Theorem 3.14. □

Proof of Theorem 5.7. (i) $\Rightarrow$ (iv). This is almost immediate. Let f be left recursive in the sense of (4.4). Take S to be the set of all words of length $\leqslant s$. Then T is the set of words of length precisely $s+1$. Now take for $\alpha \in S$, $\beta \in T$

$$a_{\beta,\alpha} = \begin{cases} c_{\beta_{\mathrm{pre}(s)},i} & \text{if } \alpha = \lambda_i, \\ 0 & \text{otherwise,} \end{cases} \tag{5.10}$$

to see that f is left Schützenberger recursive.

(iv) $\Rightarrow$ (vii). Let f be left Schützenberger recursive. Then by induction on the length of ω' where $\omega = \alpha\omega'$ with α the longest prefix of ω that is in S, we see that the rows of the matrix $H(f)$ are linearly dependent on the rows with index in S. This proves (vii).

(vii) $\Rightarrow$ (viii). This is the heart of the proof of the theorem. The proof of this bit is a rather straightforward adaptation of the proof of Fliess in the case of a field [8], combined with the observation of Rouchaleou in the case of one variable (which means linear system theory), that Noetherianness is precisely what is needed to prove realization theorems given finite Hankel rank, see [17].

So suppose that f is of finite Hankel rank. Let the rows of $H(f)$ be denoted r_ω, $\omega \in W$ and let the entry indexed by ω' of r_ω, i.e. $f_{\omega\omega'} = f(\omega\omega')$ be denoted $r_\omega(\omega')$. Let ω_0 be the empty word. Because $\mathrm{rk}(H(f)) < \infty$, there are by Lemma 3.17 finitely many $\omega_0, \omega_1, \omega_2, \ldots, \omega_t$ such that each r_ω is an A-linear combination of the $r_{\omega_0}, r_{\omega_1}, \ldots, r_{\omega_t}$. Note that the row indexed by the empty word is included in the set of chosen generators (whether really needed or not).

Now define a representation ρ of the monoid X^* in the $(t+1)\times(t+1)$ matrices by $\rho(X_{\omega_0}) = \rho(1) = \mathrm{Id}$, and

$$r_{\omega_j[i]} = \sum_{k=0}^{t} \rho(X_i)_{j,k} r_{\omega_k}, \tag{5.11}$$

where, of course, $\omega_j[i]$ is the concatenation of the word ω_j with the length one word $[i]$. There is choice involved here of the $\rho(X_i)$, but that does not matter; any matrix such that (5.11) holds will do.

Claim.

$$r_{\omega_j\alpha} = \sum_{k=0}^{t} \rho(X_\alpha)_{j,k} r_{\omega_k} \quad \forall \alpha \in W, \ \forall j = 0, 1, 2, \ldots, t. \tag{5.12}$$

Because $\rho(X_{\omega_0}) = \mathrm{Id}$, using (5.11), one sees this holds for $\lg(\alpha) \leqslant 1$. So assume with induction that (5.12) has been proved for $\lg(\alpha) \leqslant s$. Consider an ω of length $s+1$. Then there is an i such that $\omega = [i]\omega'$, $\lg(\omega') = s$. Now note that

$$r_{\omega\omega'}(\omega'') = r_\omega(\omega'\omega''). \tag{5.13}$$

So

$$\begin{aligned} r_{\omega_j\alpha}(\omega') &= r_{\omega_j[i]\alpha'}(\omega') = r_{\omega_j[i]}(\alpha'\omega') \quad (\text{by } (5.13)) \\ &= \sum_k \rho(X_i)_{j,k} r_{\omega_k}(\alpha'\omega') \quad (\text{by } (5.11)) \end{aligned}$$

$$= \sum_k \rho(X_i)_{j,k} r_{\omega_k \alpha'}(\omega') \quad \text{(by (5.13))}$$

$$= \sum_{k,l} \rho(X_i)_{j,k} \rho(X_{\alpha'})_{k,l} r_{\omega_l} \quad \text{(induction hypothesis)}$$

$$= \sum_l \rho(X_i X_{\alpha'})_{j,l} r_{\omega_l} \quad \text{(matrix multiplication)}$$

$$= \sum_l \rho(X_\alpha)_{j,l} r_{\omega_l}$$

proving the claim.

Now in (5.12) substitute $j = 0$ and take the value at ω_0. This gives

$$r_\alpha(\omega_0) = \sum_k \rho(X_\alpha)_k r_{\omega_k}(\omega_0). \tag{5.14}$$

Now take

$$b = \begin{pmatrix} r_{\omega_0}(\omega_0) \\ r_{\omega_1}(\omega_0) \\ \vdots \\ r_{\omega_t}(\omega_0) \end{pmatrix}, \qquad c = (1, \underbrace{0, \ldots, 0}_{t}),$$

then $c\rho(X_\alpha) = (\rho(X_\alpha)_{0,0}, \ldots, \rho(X_\alpha)_{0,t})$ and hence from (5.14)

$$f_\alpha = r_\alpha(\omega_0) = c\rho(X_\alpha)b \tag{5.15}$$

proving this implication.

(viii) $\Rightarrow$ (ix). Now suppose that f is recognizable (= realizable) so that (5.15) holds. Then, by the definition of the representation ρ

$$f_{\alpha\beta} = c\rho(X_{\alpha\beta})b = c\rho(X_\alpha)\rho(X_\beta)b. \tag{5.16}$$

Now define g_i, $i = 0, 1, \ldots, t$, as the column vector indexed by W whose entry at α is $c\rho(X_\alpha)(i)X_\alpha$, where $c\rho(X_\alpha)(i)$ is the ith entry of the row vector $c\rho(X_\alpha)$, and define h_i, $i = 0, 1, 2, \ldots, t$, as the row vector indexed by W whose entry at β is $\rho(X_\beta)b(i)X_\beta$, where $\rho(X_\beta)b(i)$ is the ith entry of the row vector $\rho(X_\beta)b$. Then (5.16) precisely says that

$$\mu(f) = \sum_i g_i \otimes h_i$$

showing that f is representative.

(ix) $\Rightarrow$ (i), (ii). This is Part of Theorem 4.6.

This proves that (i), (iv), (vii), (viii), and (ix) are equivalent and that any of them implies (ii). Similarly, working with the corresponding right concepts, one shows that (ii), (v), (vii), (viii), (ix) are equivalent and that any of them implies (i). This proves the theorem.

6. Cofree coalgebras over infinite rank free modules and over the duals of free modules

Now consider the case of an infinity of indeterminates

$$X = \{X_j\colon\ j \in J\}. \tag{6.1}$$

In this case, there are three different notions of noncommutative power series in the infinite set of indeterminates X, viz.

$$\hat{T}M \subset \hat{T}N \subset A\langle\langle W\rangle\rangle. \tag{6.2}$$

Here, M is the infinite rank free module

$$M = \bigoplus_{j\in J} AX_j \tag{6.3}$$

with basis $\{X_j\colon\ j \in J\}$, N is the module

$$N = \prod_{j\in J} AX_j,$$

the linear dual of M, and $A\langle\langle W\rangle\rangle$ is the module of all formal sums of monomials in the X_j

$$A\langle\langle W\rangle\rangle = \left\{\sum_{\alpha\in W} f_\alpha X_\alpha\colon\ f_\alpha \in A\right\} = A^W, \tag{6.4}$$

which is also the A-module of all functions on W to A. Note that if

$$f \in \hat{T}M, \quad f = \sum_{\alpha\in W} f_\alpha X_\alpha, \tag{6.5}$$

then for each length k there are only finitely many α of that length for which the coefficient $f_\alpha = f(\alpha)$ is nonzero.

Note also that the (Cauchy) product is still well defined on $A\langle\langle W\rangle\rangle$. The coefficient at α of the product of two elements $g, h \in A\langle\langle W\rangle\rangle$ is equal to

$$(fg)_\alpha = \sum_{\beta\gamma=\alpha} g_\beta h_\gamma, \tag{6.6}$$

which is a finite sum. The two inclusions in (6.2) are both strict. For instance, any element of the form

$$\sum_{j\in J} a_j X_j \tag{6.7}$$

with infinitely many of the $a_j \neq 0$ is in $\hat{T}N$ but not in $\hat{T}M$, and any element of the form

$$\sum_{j \in J} a_j X_j^2 \tag{6.8}$$

with infinitely many of the $a_j \neq 0$ is in $A\langle\langle W \rangle\rangle$ but not in $\hat{T}N$.

In the case of a finite set of indeterminates all three modules in (6.2) coincide.

Note that the elements of TM are polynomials in the X_j in the most usual sense of the word; that is they are sums $\sum_{\alpha \in W} a_\alpha X_\alpha$ for which only finitely many of the a_α are nonzero. Note, however, that polynomials in an infinite set of variables are not always defined this way. In the theory of symmetric functions and quasisymmetric functions one works with an infinite set of commuting variables and polynomials in these are defined as power series of bounded degree, see, e.g. [13,15]. In the symmetric functions case these are then precisely the polynomials in the elementary symmetric functions in the sense that only finitely many coefficients are nonzero. The elements of TN are neither polynomials in the sense of finitely many coefficients nonzero, see (6.7), nor do they coincide with power series of bounded degree, see (6.8).

Corresponding to the three different algebras of (6.2), in the case of an infinite set of indeterminates, there are several different versions of the recursiveness Theorem (5.7) and there are different associated Kleene–Schützenberger type theorems.

6.9. **Theorem.** *Let $f \in \hat{T}M$. Then the following are equivalent*:

(i) *f is left recursive with finiteness condition.*
(ii) *f is right recursive with finiteness condition.*
(iii) *f is left Schützenberger recursive with finiteness condition.*
(iv) *f is right Schützenberger recursive with finiteness condition.*
(v) *f has finite Hankel rank.*
(vi) *f is realizable with a representation ρ for which $\rho(X_i) = 0$ for all but finitely many i.*
(vii) *f is representative* (*with the g_i and h_i in $\hat{T}M$*).

The first step in proving this is to realize that in these circumstances one is really only dealing with finitely many variables as recorded in the next two propositions.

6.10. **Proposition.** *Let $f \in \hat{T}M$ and let f be recursive with finiteness condition. Then there are only finitely many variables involved in f.*

The latter statement means the following. There is a finite subset J_0 of the set of indices of variables J, such that if $j \in J \setminus J_0$ occurs in a word α over the alphabet J, then the coefficient of X_α in f is zero.

Proof of Proposition 6.10. Let s and λ_i be as in Definition (4.4). Because f is in $\hat{T}M$ there are only finitely many monomials of length $\leqslant s$ that have nonzero coefficient in f and so these involve only finitely many variables. There are also only finitely many

λ_i and for each i only finitely many $c_{\omega,i} \neq 0$ and these also involve only finitely many variables. Let $J_0 \subset J$ be the finite subset of all these variables. Then by formula (4.4) for $lg(\alpha) > s$

$$f(\alpha) = \sum_{i=1}^{l} c_{\alpha_{\mathrm{pre}(s)},i} f(\lambda_i \alpha_{\mathrm{suf}}). \tag{6.11}$$

For a word α over the index set J, let $\mathrm{varsupp}(\alpha) = \{j \in J\colon j \text{ occurs in } \alpha\}$ and for an f in $A\langle\langle W \rangle\rangle$ let

$$\mathrm{varsupp}(f) = \bigcup_{f(\alpha)\neq 0} \mathrm{varsupp}(\alpha). \tag{6.12}$$

Thus

$$J_0 = \bigcup_{n \leqslant s} \mathrm{varsupp}(f^n) \cup \bigcup_{\substack{i,\mathrm{lg}(\omega)=s \\ c_{\omega,i}\neq 0}} \mathrm{varsupp}(\omega). \tag{6.13}$$

By induction we can assume that $\mathrm{varsupp}(f^i) \subset J_0$ for $i \leqslant m \geqslant s$. Let $\mathrm{lg}(\alpha) = m+1$ and suppose that $\mathrm{varsupp}(\alpha) \not\subset J_0$. Then one of the following holds (or both):

(i) $\mathrm{varsupp}(\alpha_{\mathrm{pre}(s)}) \not\subset J_0$,
(ii) $\mathrm{varsupp}(\alpha_{\mathrm{suf}}) \not\subset J_0$.

In the first case $c_{\alpha_{\mathrm{pre}(s)},i} = 0$ for all i, and in the second case $f(\lambda_i \alpha_{\mathrm{suf}}) = 0$ by the induction hypothesis.

6.14. **Proposition.** *Let $f \in \hat{T}M$ be left (or right) Schützenberger recursive with finiteness conditions. Then* $\mathrm{varsupp}(f)$ *is finite.*

Proof. This time let

$$J_0 = \bigcup_{\substack{\text{there is an } \alpha \\ \text{with } a_{\beta\alpha}\neq 0}} \mathrm{varsupp}(\beta) \cup \bigcup_{\alpha\in S} \mathrm{varsupp}(\alpha) \tag{6.15}$$

and prove (in the same way) with induction on the length of ω', starting with 0, that the coefficient of X_ω in f is zero unless $\mathrm{varsupp}(\omega) \subset J_0$. Here, ω' is determined by $\omega = \alpha\omega'$ with $\alpha \in S$ of maximal length.

Proof of Theorem 6.9. (i) $\Rightarrow$ (iii). By Proposition 6.10, there are only finitely many variables involved in f. Thus by Theorem 5.7, f is left Schützenberger recursive in these finitely many variables, proving (iii).

(iii) $\Rightarrow$ (v). By Proposition 6.14, only finitely many variables are involved. So this follows from Theorem 5.7.

(v) $\Rightarrow$ f is realizable. The proof of Theorem 5.7 works unchanged. But this does not yet imply that the extra finiteness condition: $\rho(X_k) = 0$ for all but finitely many k holds.

f is realizable $\Rightarrow$ f is representable. As in the proof of Theorem 5.7, this comes directly from the realization formula $f(\alpha)=c\rho(X_\alpha)b$ so that $f(\alpha\beta)=c\rho(X_\alpha)\rho(X_\beta)b$.

f is representable (and $f\in\hat{T}M$) $\Rightarrow$ (iii). By Theorem 3.12

$$f(\beta\alpha)=\sum_{i=1}^{n} g_i(\beta)h_i(\alpha) \tag{6.16}$$

with each g_i, equal to some $R_\gamma f$ and hence in $\hat{T}M$. Consider a word of length k. Because the g_i in (6.16) are in $\hat{T}M$, there are only finitely many words β of length k such that $f(\beta\omega)\neq 0$ for any ω. On the other hand by Theorem 3.12, the rank of $H(f)$ is finite, because the columns of $H(f)$ are the coefficients of the $R_\beta f$. Thus f is left (or right) Schützenberger recursive, so that there are coefficients $c_{\beta\alpha}$, $\beta\in T=SX\setminus S$, $\alpha\in S$, S a finite set of words, such that for all ω

$$f(\beta\omega)=\sum_{\alpha\in S} a_{\beta\alpha}f(\alpha\omega). \tag{6.17}$$

The length of the $\beta\in T$ is bounded because S is finite. Thus for all but finitely many β in (6.17) we can choose the $a_{\beta\alpha}$ to be zero. This then establishes that f is left Schützenberger recursive with the extra finiteness condition on the recursion matrix. Thus f involves only finitely many variables (by Proposition 6.14), and Theorem 5.17 establishes that f is left recursive (and right recursive). This proves the theorem. Alternatively use Theorem 4.6.

6.18. **Remark.** Another way to get from (iii) to (vi) in Theorem 6.9 is to use the standard construction as given in [2].

6.19. **Corollary.** *If $f\in\hat{T}M$ for an infinite rank free module M over a Noetherian integral domain, then, actually, f is in $\hat{T}M'$ for some finitely generated free submodule M' of M.*

6.20. **Corollary.** *The cofree coalgebra over an infinite rank free module over a Noetherian integral domain is the union (more precisely the inductive limit) of the cofree coalgebras over the finite rank free submodules.*

The proof is the same as above in Sections 4 and 5.

6.21. **Remark.** This last observation before Corollary 6.20 fits very well with the main theorem of coalgebras as discussed in Section 8.

The celebrated Kleene–Schützenberger theorem for noncommutative power series in a finite number of variables over a Noetherian integral domain says that the rational closure of the polynomials is the algebra of realizable (= recognizable) power series. Here the rational closure means the following. Let $P\subset A\langle\langle W\rangle\rangle$ be a subalgebra. An element of $A\langle\langle W\rangle\rangle$ is invertible if and only if its constant term is a unit. The rational closure, P_{rat}, of P is the smallest subalgebra containing P that is closed under inversion (when applicable).

In the case of finitely many variables the classical Kleene–Schützenberger theorem says

$$TM_{\mathrm{repr}} = TM_{\mathrm{real}} = TM_{\mathrm{rat}}, \tag{6.22}$$

where TM_{real} is the set of realizable power series. It is easy to see that this is indeed a subalgebra: use the direct sum and (tensor) product of representations (see below for details). The first equality in (6.22) then comes from Theorem 5.12 and the second is the classical theorem itself.

Here is a version for infinitely many variables.

6.23. **Theorem** (Kleene–Schützenberger theorem for free modules of infinite rank). *Let M be a free module over a Noetherian integral domain* (*of any rank*). *Then*

$$TM_{\mathrm{rat}} = TM_{\mathrm{real}} = TM_{\mathrm{repr}}. \tag{6.24}$$

Here TM_{real} is the module of realizable noncommutative power series in the X_j, $j \in J$, a basis of M, that is those power series f in $\hat{T}M$ that can be realized ($=$ recognized) by a triple (ρ, b, c) consisting of a finite dimensional representation ρ of X^*, say of dimension n, an $n \times 1$ vector b and a $1 \times n$ vector c such that $f(\alpha) = c\rho(X_\alpha)b$.

Proof. The second equality of (6.24) is part of Theorem 6.9. This theorem also says that if $f \in TM_{\mathrm{repr}}$ then it involves only a finite number of variables, and thus by the classical Kleene–Schützenberger theorem it is a rational power series in those finite number of variables and thus in TM_{rat}. That TM_{rat} is part of TM_{real} is established just like in the case of the classical theorem. The details will be given below (in a more general setting) because in the case of the dual of a free module the corresponding Kleene–Schützenberger theorem does not immediately follow from the classical one.

The next step is to examine the rational, representative, and realizable closures of TN in $\hat{T}N$ and in $A\langle\langle W\rangle\rangle$ where $N = \prod_{j\in J} AX_j$ and W is the free monoid on the infinite alphabet J.

A first step is to characterize the elements of TN and $\hat{T}N$ in $A\langle\langle W\rangle\rangle$.

6.25. **Theorem.** *An element $f \in A\langle\langle W\rangle\rangle$ is in $\hat{T}N$ if and only if each homogeneous component f^n has finite Hankel rank*; *it is in TN if and only if it is of bounded degree and has finite Hankel rank*.

Here 'bounded degree' means that f is of the form $\sum_{\lg(\omega)\leqslant n} a_\omega X_\omega$ for some n.

Proof. Let $f \in N^{\otimes m}$. Then f is of the form

$$f = \sum_i^{<\infty} g_{1,i} \otimes g_{i,2} \otimes \cdots \otimes g_{m,i}. \tag{6.26}$$

The Hankel matrix of each one of the summands $g_{1,i} \otimes g_{i,2} \otimes \cdots \otimes g_{m,i}$ consists of blocks of zeros and a finite number of blocks, viz. $m+1$, of rank 1 and thus is of finite rank. And thus the Hankel rank of the finite sum (6.26) is finite as the Hankel matrix of f is a sum of the finite number of Hankel matrices of the summands of (6.26). This proves that if $f \in TN$ then it has finite Hankel rank. Inversely, let $f \in A\langle\langle W \rangle\rangle$ be homogeneous of degree m and of finite Hankel rank:

$$f = \sum_{j_1, j_2, \ldots, j_m \in J} a_{j_1, j_2, \ldots, j_m} X_{j_1} X_{j_2} \cdots X_{j_m}. \tag{6.27}$$

The claim is that then $f \in N^{\otimes m}$. If $m = 1$, then there is nothing to prove. So let $m > 1$. Then the part of the Hankel matrix of f consisting of the rows indexed by elements of J, i.e. the words of length one, is also of finite rank. So there are a finite number of indices $j_1, \ldots, j_r$ so that each row of $H(f)$ indexed by a word of length one is dependent on these r rows. So, in particular, there are coefficients $c_{j_i,k}$ such that

$$a_{k, l_2, \ldots, l_m} = \sum_{i=1}^{r} c_{j_i,k} a_{j_i, l_2, \ldots, l_m} \quad \text{for all } l_2, \ldots, l_m. \tag{6.28}$$

Now let

$$g_i = \sum_k c_{j_i,k} X_k \tag{6.29}$$

and let

$$h_i = \sum_{l_2, l_3, \ldots, l_m} a_{j_i, l_2, l_3, \ldots, l_m} X_{l_2} \cdots X_{l_m}. \tag{6.30}$$

Then by (6.28)

$$f = \sum_{i=1}^{r} g_i \otimes h_i. \tag{6.31}$$

Now the Hankel matrices of the h_i are parts of the Hankel matrix of f, viz. the parts consisting of the rows indexed by words with length 1 prefix $[i]$. Thus these are of finite rank and with induction $h_i \in N^{\otimes(m-1)}$ and hence by (6.31) $f \in N^{\otimes m}$.

Now let $f \in A\langle\langle W \rangle\rangle$ be of bounded degree n and finite Hankel rank. Let f^s be the homogeneous part of f of degree $s \leqslant n$. Then the Hankel matrix of f^s consists of zero blocks and a finite number of blocks of the Hankel matrix of f, viz. the blocks indexed by rows indexed by words of length k and columns indexed by words of length l, $l + k = s$. Thus the Hankel matrix of each of the f^s is of finite rank and hence in $N^{\otimes s}$. This proves the theorem.

The remainder of this section is about recursiveness, etc. for the duals of free modules over Noetherian integral domains A. When A is a field K, and N is the dual of a free module M over K, i.e. a vector space, then N is again free, with, if the rank of M is infinite, usually a basis of larger cardinality than that of a basis of M. So in the case that A is a field there is nothing new. However, if A is an integral domain the dual of a free module is not necessarily free. Indeed, often it is not. For instance, if M is the free module of countable infinite rank over the integers, then its double dual is isomorphic to M, and so the dual M^* cannot be free because otherwise its dual M^{**} would have larger cardinality than M^* which has larger cardinality than M. This happens quite frequently. For instance the double dual of a free module M over the integers is isomorphic to that module (i.e. that module is reflexive) if and only if the cardinality of a basis of it is a non-ω-measurable cardinal. The countably infinite cardinal is non-ω-measurable. For details see [6,7].

6.32. **Theorem.** *Let $f \in A\langle\langle W \rangle\rangle$, where W is the free monoid on the infinite alphabet $\{X_j\colon j \in J\}$. Then the following are equivalent*:

(i) *f is left Schützenberger recursive.*
(ii) *f is right Schützenberger recursive.*
(iii) *f has finite Hankel rank.*
(iv) *f is realizable (= recognizable).*
(v) *f is in $\hat{T}N$ and representative (in $\hat{T}N$).*
(vi) *f is representative in $A\langle\langle W \rangle\rangle$.*
Moreover, if $f \in \hat{T}N$ these conditions are equivalent to both
(vii) *f is left recursive.*
(viii) *f is right recursive.*

6.33. **Corollary.** *The cofree coalgebra over the dual of a free module $N = \prod_{j\in J} AX_j$ is $TN_{\mathrm{repr}} = TN_{\mathrm{real}}$.*

The proof of the corollary is the same as in the case of free modules, see above. The important thing is that the realizability property (iv) immediately implies that TN_{real} is a coalgebra.

Proof of Theorem 6.32. The proofs of (i) ⇒ (iii) ⇒ (iv) are as before. Now suppose that f is realizable. Then there is a formula $f(\alpha) = c\rho(X_\alpha)b$ for the coefficients of f. Explicitly the homogeneous component of degree m of f is given by the formula

$$f^m = \sum_{j_1,\dots,j_m,k_1,\dots,k_m} c_{j_1}\rho(X_{k_1})_{j_1,j_2}\cdots\rho(X_{k_m})_{j_{m-1},j_m} b_{j_m} X_{j_1}\cdots X_{j_m}$$

showing that it is a finite linear combination of the entries of the finite dimensional matrix

$$\sum_{k_1,\dots,k_m} \rho(X_{k_1})\cdots\rho(X_{k_m})X_{k_1}\cdots X_{k_m} = \left(\sum_k \rho(X_k)X_k\right)^m$$

and hence an element of $N^{\otimes m}$. This proves that $f \in \hat{T}N$. Also because $f(\alpha\beta) = c\rho(\alpha)\rho(\beta)b$ it follows that f is representative in $\hat{T}N$. Trivially, (v) implies (vi). But if (vi) holds

$$f(\alpha\beta) = \sum_{i=1}^n g_i(\alpha)h_i(\beta)$$

showing that the Hankel matrix of f is of finite rank, the rows depending on the finite number of rows of the coefficients of the g_i. Thus (vi) implies (iii) which trivially implies (i) and (iii). Thus going around once more using right notions instead of left ones we have the equivalence of (i)–(vi).

Finally if (iii) holds than certainly f is left recursive. Inversely, if f is left recursive and it is in $\hat{T}N$, then by Theorem 6.25 its finite degree parts are in TN and of finite Hankel rank and hence the left recursiveness says that the Hankel rank of f is finite, proving (iii). Similarly, (viii) is equivalent to (iii). This concludes the proof of Theorem 6.32.

Corresponding to Theorem 6.32 there is a Kleene–Schützenberger type theorem as follows.

6.34. **Theorem.** *Let $N=\prod_{j\in J}AX_j$ be the dual of a free module over the Noetherian integral domain A . Then*

$$TN_{\text{rat}} = TN_{\text{real}} = TN_{\text{repr}}. \tag{6.35}$$

Proof. The rational power series in $\hat{T}N$ (or $A\langle\langle W\rangle\rangle$ for that matter) are the ones that can be obtained from TN by sums, scalar multiples, products, and inversions. They are all in TN_{real}. This is seen as follows. If $f\in TN$, then it is of finite Hankel rank (by Theorem 6.25) and hence realizable by Theorem 6.32. If f and g are both realizable by, say, (ρ,b,c) and (ρ',b',c') then the scalar product af is realized by (ρ,b,ac), the sum is realized by $(\rho\oplus\rho',\begin{pmatrix} b\\ b'\end{pmatrix},(c,c'))$ and the product is realized by $(\rho\otimes\rho',b\otimes b',c\otimes c')$ where the tensor product of an $m\times n$ matrix M and an $m'\times n'$ matrix M' is the $mm'\times nn'$ matrix $M\otimes M'$ whose entry at $((i,i'),(j,j'))$ is $m_{i,j}m'_{i',j'}$. Finally, let f be invertible with inverse g. We can as well assume that the constant term of f (and g) is 1. Then in [18,19], there is a construction that realizes g in one dimension more than a realization of f. That can be adapted to the present case. Another way to see that the inverse of a realizable element of $A\langle\langle W\rangle\rangle$ is realizable is as follows. Let $H(f)$ be the Hankel matrix of f and define the lower triangular $W\times W$ matrix $R(f)$ with 1's on the main diagonal by

$$R(f)_{w,\alpha}=\begin{cases}0 & \text{if } \alpha \text{ is not a suffix of } \omega,\\ f(\beta) & \text{if } \alpha \text{ is a suffix of } \omega, \text{with } \beta \text{ determined by } \omega=\beta\alpha.\end{cases} \tag{6.36}$$

Further, let $H(g)$ be the Hankel matrix of g, the inverse of f, and define the upper triangular matrix $Q(g)$ with ones on the main diagonal by

$$Q(g)_{\beta,\omega'}=\begin{cases}0 & \text{if } \beta \text{ is not a prefix of } \omega',\\ g(\gamma) & \text{if } \beta \text{ is a prefix of } \omega' \text{with } \gamma \text{ determined by } \beta\gamma=\omega'.\end{cases} \tag{6.37}$$

Then

$$\begin{aligned}(R(f)H(g)+H(f)Q(g))_{\omega,\omega'} &= \sum_{\alpha} R(f)_{\omega,\alpha}H(g)_{\alpha,\omega'}+\sum_{\alpha} H(f)_{\omega,\alpha}Q(g)_{\alpha,\omega'}\\ &= \sum_{\beta \text{ a prefix of } \omega} f(\beta)g(\gamma\omega') \quad \text{with } \beta\gamma=\omega\end{aligned}$$

$$+\sum_{\gamma \text{ a suffix of } \omega'} f(\omega\beta)g(\gamma) \quad \text{with } \beta\gamma=\omega'$$

$$=\sum_{\alpha \text{ a prefix of } \omega\omega'} f(\alpha)g(\alpha') \quad \text{with } \alpha\alpha'=\omega\omega'$$

$$=\begin{cases} 1 & \text{if } \omega=\omega'=[\ ] \\ 0 & \text{if } \lg(\omega\omega')\geqslant 1, \end{cases}$$

because g is the inverse of f. This even works if the coefficients of f and g do not commute. And thus, because $R(f)$ and $Q(g)$ are invertible, the rank of the Hankel matrix of f and its inverse g differ at most by one. Thus, by Theorem 6.32, if f is realizable and invertible so is it inverse g. This shows that $TN_{\mathrm{rat}} \subset TN_{\mathrm{real}}$.

To prove the reverse inclusion first consider an $n \times n$ matrix S with entries from $A\langle\langle W\rangle\rangle$ with zero constant terms. Then the infinite sum

$$I_n+S+S^2+S^3+\cdots$$

is well defined and equal to the inverse $(I_n-S)^{-1}$. If the entries of S were commuting, of course, the entries of $(I_n-S)^{-1}$ would be rational functions in the entries of S. It is a fundamental insight of Schützenberger [18] that this is still true if the entries of S do not necessarily commute. This can be seen as follows. If $n=1$ there is nothing to prove. If $n=2$ the inverse

$$U=\begin{pmatrix} u_{11} & u_{12} \\ u_{21} & u_{22} \end{pmatrix}$$

of

$$\begin{pmatrix} 1-s_{11} & -s_{12} \\ -s_{21} & 1-s_{22} \end{pmatrix}$$

is given by

$$u_{11}=(1-s_{11}-s_{12}(1-s_{22})^{-1}s_{12})^{-1},$$

$$u_{12}=(1-s_{11}-s_{12}(1-s_{22})^{-1}s_{21})^{-1}s_{12}(1-s_{22})^{-1}=u_{11}s_{12}(1-s_{22})^{-1}$$

$$u_{22}=(1-s_{22}-s_{21}(1-s_{11})^{-1}s_{12})^{-1},$$

$$u_{21}=(1-s_{22}-s_{21}(1-s_{11})^{-1}s_{12})^{-1}s_{21}(1-s_{11})^{-1}=u_{22}s_{21}(1-s_{11})^{-1}.$$

These formulas still work if s_{22} is an $(n-1)\times(n-1)$ matrix, s_{12} is a $1\times(n-1)$ matrix (= row vector), and s_{21} is an $(n-1)\times 1$ matrix (= column vector), all with constant terms equal to zero. This proves the statement (with induction on n). (NB: This treatment is a bit different from the one in [18], which uses quasi-inverses instead and anyway I could not make the formulas in [18] work out right.)

Now if f is realized by (ρ, b, c), define

$$S = \sum_k \rho(X_k)X_k. \tag{6.38}$$

Then

$$f = c(I_n - S)^{-1}b \tag{6.39}$$

proving that f is rational. This finishes the proof of Theorem 6.34.

Note that formula (6.39) is almost identical with the formula for the transfer matrix of a linear dynamical input–output system (which corresponds to the case of power series in a single variable).

There should be a third Kleene–Schützenberger-type theorem, namely one that describes the rational closure of the subalgebra $A\langle\langle W\rangle\rangle_{bd}$ of noncommutative power series in an infinite set of indeterminates of bounded degree. Quite likely this is the module of elements of $A\langle\langle W\rangle\rangle$ that are recursive in the sense of Section 4 (see Definition 4.3).

7. Polynomial recursiveness

Simply polynomial recursiveness has already been defined in a previous section (formulas (4.1) and (4.2)). A sum of two simply polynomially recursive power series need not be simply polynomially recursive, so define a power series to be polynomially recursive if it is a finite sum of simply polynomially recursive power series. Such a power series is rational. Indeed, formula (4.1) says that $f = (1 - \sum_{i=1}^{l} c_i X_{\lambda_i})^{-1}(f^0 + f^1 + \cdots + f^m)$, where m is some fixed number larger or equal to $\max\{\lg(\lambda_i)\}$. Thus f is rational and by the appropriate Kleene–Schützenberger theorem it is realizable and thus left and right recursive. (I know of no way to see this directly, just as there seems to be no simple direct way to translate recursiveness of Schützenberger type into rationality.) Thus f is representative and there is a formula (see 4.24 above)

$$f(\alpha\beta) = \sum_{i=1}^{r} g_i(\alpha)h_i(\beta), \quad g_i \in \bar{L}f, \;\; h_i \in \bar{R}f \tag{7.1}$$

such that the g_i, h_i are representative and recursive. It is easy to see that the $R_\beta f$ (resp. $L_\beta f$) are still left (resp. right) simply polynomially recursive. The matter of right polynomial recursiveness for the right 'translates' $R_\beta f$ is more complicated. Because $R_\beta(R_\gamma f) = R_{\gamma\beta} f$ it suffices to examine the matter in the case β has length one. So take the case $R_{[1]}f$. As an example, consider the right recursive power series in three variables, with starting term 1, and recursion monomials X_3, X_2X_1, so that

$$f^n = \sum_{\substack{\beta_i \in \{[2,1],[3]\} \\ \sum \lg(\beta_i) = n}} X_{\beta_1} \cdots X_{\beta_m} \tag{7.2}$$

and

$$(R_{[1]}f)^n = \sum_{\substack{\beta_i \in \{[2,1],[3]\} \\ \sum \lg(\beta_i)=n-1}} X_{\beta_1} \cdots X_{\beta_m} X_2. \tag{7.3}$$

This is certainly still right recursive in some sense; otherwise a nice simple formula like (7.3) could not be written. But the recursion now involves an infinite number of starting terms and an infinite number of recursion monomials, viz. starting terms $X_3^j X_2,\ j=0,1,2,\ldots$, and recursion monomials $X_1 X_3^j X_2,\ j=0,1,2,\ldots$. And both the starting terms and the recursion monomials, and the corresponding coefficients have themselves a recursive structure.

To illustrate things here is $R_{[1]}f$ up to and including the terms of degree 6.

$$\begin{aligned} R_{[1]}f = {} & X_2 + X_3X_2 + X_2X_1X_2 + X_3^2X_2 + X_3X_2X_1X_2 + X_2X_1X_3X_2 + X_3^3X_2 \\ & + X_2X_1X_2X_1X_2 + X_3^2X_2X_1X_2 + X_3X_2X_1X_3X_2 + X_2X_1X_3^2X_2 + X_3^4X_2 \\ & + X_3X_2X_1X_2X_1X_2 + X_2X_1X_3X_2X_1X_2 + X_3^3X_2X_1X_2 + X_2X_1X_2X_1X_3X_2 \\ & + X_3^2X_2X_1X_3X_2 + X_3X_2X_1X_3^2X_2 + X_2X_1X_3^3X_2 + X_3^4X_2 + \cdots \ . \end{aligned}$$

This type of 'infinite recursive recursiveness' seems to be getting somewhat close to the right notion of recursiveness as discussed in the previous sections. The reason for the interest in some polynomial versions is that those would make sense for general tensor algebras $\hat{T}M$ with M not necessarily torsion free. In [10, p. 196] there is an attempt to define such a notion. However, the condition as stated there is empty as is easily seen by introducing superfluous extra terms like $(f+g)\otimes h - f\otimes h - g\otimes h$ in the tensor sums (or much more complicated zero terms in the case of the presence of torsion).

Much remains to be examined as regards polynomial types of recursiveness.

Another idea to get at a suitable notion of recursiveness in the presence of torsion could be as follows. For an arbitrary A-module M let $\tilde{M} \xrightarrow{\pi} M$ be a covering with $\tilde{M}$ free. Then define $f \in \hat{T}M$ to be recursive if there is a recursive lift in $\hat{T}\tilde{M}$. This also has drawbacks (so far). A few words on this are in the final section.

8. The main theorem of coalgebras over rings

The (so-called) main (or fundamental) theorem for coalgebras over a field says the following. If C is a coalgebra with counit over a field K and $c \in C$ is an element of C, then there is a finite dimensional subcoalgebra C' of C that contains c. See e.g. [22, Section 2.2, p. 45ff]. A short and elegant proof of this (of which I do not know the provenance) is in [5, p. 25].

Using Lemma 8.5 these arguments can be extended to the case of principal ideal domains. However, these arguments are not strong enough to establish the main theorem for the cofree algebras over free modules and their duals over general Noetherian integral domains (see Theorem 8.10).

There is a natural analogous property that can be considered for coalgebras with counit over an arbitrary ring A (commutative and with unit element). Such a coalgebra C satisfies the 'main theorem property' if for every element $c \in C$ there is a finitely generated (as a module) subcoalgebra over A containing the element c.

This is not always true as the following example shows [12].

8.1. **Example.** Consider the Abelian group

$$C = \mathbf{Z}X_1 \oplus \left(\bigoplus_{n=2}^{\infty} \mathbf{Z}/(n)X_n \right) = \mathbf{Z}X_1 \oplus \mathbf{Z}/(2)X_2 \oplus \mathbf{Z}/(3)X_3 \oplus \cdots \tag{8.2}$$

and consider the comultiplication

$$\mu : X_n \mapsto 1 \otimes X_n + X_n \otimes 1 + nX_{n^2} \otimes X_{n^2}, \quad n \geqslant 2, \;\; X_1 \mapsto X_1 \otimes X_1, \;\; 1 = X_1 \tag{8.3}$$

and the counit which is projection onto the first factor. This is a coassociative comultiplication (because $\mathbf{Z}/(n^2) \otimes \mathbf{Z}/(n^2) \otimes \mathbf{Z}/(n) \simeq \mathbf{Z}/(n)$) and the counit does what it is supposed to do. It is easy to show, because of the term $nX_{n^2} \otimes X_{n^2}$ in (8.3), that there are no subgroups of C other than subgroups of the first summand that are stable under the comultiplication. Thus, this coalgebra does not have the main theorem property.

On the other hand, there is the following theorem.

8.4. **Theorem.** *Let C be a coalgebra with counit over a principal ideal domain A whose underlying module is a free A-module. Then C has the main theorem property.*

The (present) proof of this requires a lemma.

8.5. **Lemma.** *Let A be a principal ideal domain, and let $M = A^I$ be a free module over A. Let V be a finite dimensional subvector space of $W = M \otimes K$, where K is the quotient field of A. Then*

$$(M \otimes M) \cap (V \otimes V) = (M \cap V) \otimes (M \cap V). \tag{8.6}$$

Proof. Take a basis of V whose elements are in M. These basis elements, written as column vectors form an $\infty \times r$ matrix (more precisely an $I \times r$ matrix) with entries from A. A slight extension of the standard arguments concerning Smith canonical form (see e.g. [14, p. 337ff]), shows that there is another basis of $M = A^I$ such that suitable linear combinations of the chosen basis of V form a basis that takes the form (as a matrix whose columns are the basis elements)

$$\begin{pmatrix} d_1 & 0 & 0 & \dots & 0 \\ 0 & d_2 & 0 & \dots & 0 \\ & 0 & \ddots & \ddots & \vdots \\ \vdots & \vdots & \ddots & d_{r-1} & 0 \\ & & & 0 & d_r \\ 0 & 0 & \dots & 0 & 0 \\ \vdots & \vdots & & \vdots & \vdots \end{pmatrix}.$$

It follows that the columns of the matrix

$$\begin{pmatrix} 1 & 0 & 0 & \dots & 0 \\ 0 & 1 & 0 & \dots & 0 \\ & 0 & \ddots & \ddots & \vdots \\ \vdots & \vdots & \ddots & 1 & 0 \\ & & & 0 & 1 \\ 0 & 0 & \dots & 0 & 0 \\ \vdots & \vdots & & \vdots & \vdots \end{pmatrix}$$

form (in this new basis of M) a basis of $M \cap V$. The statement of the lemma now follows immediately.

8.7. **Example.** Formula (8.6) is most definitely not true in general. Consider the ring A of all integer linear combinations of 1 and $\sqrt{-5}$ inside the complex numbers, $A = \mathbf{Z} + \mathbf{Z}\sqrt{-5}$ and let M be the free module $M = A^2$. Let K be the quotient field of A and let V be the one-dimensional vector space generated by the vector $(2, 1-\sqrt{-5})^t$. Then, because there is no nonunit element u of A such that $u^{-1}(2, 1-\sqrt{-5})^t$ is in M, $M \cap V = A(2, 1-\sqrt{-5})$, and written in terms of matrices

$$(M \cap V) \otimes (M \cap V) = A \begin{pmatrix} 4 & 2-2\sqrt{-5} \\ 2-2\sqrt{-5} & -4-2\sqrt{-5} \end{pmatrix}. \tag{8.8}$$

On the other hand, the element of $K^2 \otimes K^2$ represented by the matrix

$$\begin{pmatrix} 2 & 1-\sqrt{-5} \\ 1-\sqrt{-5} & 2-\sqrt{-5} \end{pmatrix}$$

is in $(M \otimes M) \cap (V \otimes V)$ and it is not in the set (8.8).

The reason that things fail here is that the ring A is not an UFD. For V one-dimensional and M a free module over an UFD, formula (8.7) holds. Besides the case of PIDs (Lemma 8.5) this is the only general case that I know of where formula (8.7) is true. The general question of when formula (8.7) holds is clearly a matter that needs more investigation.

Proof of Theorem 8.4. Let c be an element of C. Then, by the main theorem of coalgebras over fields (of which another new proof is given below; see Corollary 8.11), there is a finite dimensional subvector space V of $C \otimes K$, K the quotient field of A, that contains c and is a subcoalgebra of $C \otimes K$. Let $C' = V \cap C$ (in $C \otimes K$). Then by Lemma 8.5 the submodule C' is stable under the comultiplication, and hence does the job.

There is a (large) other class of coalgebras for which the 'main theorem property' of coalgebras holds.

Let M be a module over a ring A. The module M is called reflexive if the canonical morphism $\varphi: M \to M^{**}$, $\varphi(x)(f) = f(x)$, is an isomorphism. This looks, at first sight, an unusual property. For instance if A is a field K and M is an infinite dimensional vector space over K, it is never true. But, for instance if $A = \mathbf{Z}$ and M is a free Abelian group with infinite but countable basis, it is true.

Quite generally, it is a fairly straightforward matter to prove the following theorem, [4].

8.9. **Theorem.** *Let C be a coalgebra over a ring A whose underlying module is reflexive. Then the coalgebra C has the main theorem property.*

The proof of this makes serious use of duality (like the original proof of the main theorem over fields in [22]).

The matter of when a module over a ring A is reflexive is a delicate one involving higher set theoretic notions. In the case of $A = \mathbf{Z}$ the answer is as follows. Let M be a free Abelian group. Then M is reflexive if and only if the cardinality of (a basis of) M is non-ω-measurable. (A set is ω-measurable if and only if it has a nonprincipal ultrafilter $\mathscr{D}$ such that for all countable sets of elements D_i, $i \in \mathbf{N}$, $D_i \in \mathscr{D}, \bigcap_i D_i \in \mathscr{D}$. It is easy to see that $\mathbf{N}$ is non-ω-measurable.) The reflexivity of free Abelian groups with countable basis was established by Specker in 1950, [21]. For results on higher cardinals see [1], and for a general survey of these matters see [7, Chapter 3] or [6].

An A-module M is slender iff for every morphism $\prod_{i \in \mathbf{N}} A \xrightarrow{\lambda} M$, $\lambda(e_n^*) = 0$ for all but finitely many n. Here the e_n^* are the dual 'basis' to the standard basis of $\bigoplus_{i \in \mathbf{N}} A$. A ring A is slender if and only if it is slender as an A-module. It turns out that a PID is slender iff A is not a field or a discrete complete valuation ring. So from the point of view of reflexivity properties of modules over a ring, fields and complete discrete valuation rings are exactly the wrong thing to look at.

It is a general (ununderstood) phenomenon that the universal objects of some kind tend to be rather nicer than one has any reason to expect. A manifestation of this is the following theorem.

8.10. **Theorem.** *The cofree coalgebras TM_{repr} and TM^*_{repr} for M a free module over a Noetherian integral domain A have the main theorem property.*

Proof. Let f be representative. Consider $L_B(R_B f)$. This is a finitely generated submodule. Also $R_B(L_B f) = L_B(R_B f)$. Let $\bar{L}\bar{R}_B f$ be the pure closure of $L_B R_B f$ (see 4.24). Then $\bar{L}\bar{R}_B f$ is finitely generated (because it is finite rank, see 4.28) and

$$\mu(\bar{L}\bar{R}_B f) \subset \bar{L}\bar{R}_B f \otimes \bar{L}\bar{R}_B f$$

proving the theorem for TM_{repr}. Alternatively, use Theorem 6.9 and the realizability property. If ρ is a representation involved in realizing f, and has dimension n, then the n^2 elements of $TM_{\mathrm{repr}} = TM_{\mathrm{real}}$ defined by the matrix entries $\rho_{i,j}$ of ρ

$$\rho_{i,j} = \sum_{k=0}^{\infty} (S^k)_{i,j}, \qquad S \sum_{j \in J} \rho(X_j) X_j \tag{8.11}$$

form a finite rank subcoalgebra that contains f. For the case TM^*_{repr} use Theorem 6.32 instead.

8.12. **Corollary** (Main theorem of coalgebras over a field). *Let C be a coalgebra over a field and c an element of C. Then there is a finite dimensional subcoalgebra of C containing c.*

Proof. Consider the free coalgebra TC_{repr} over the module C. The identity morphism $C \xrightarrow{id} C$ induces an imbedding of coalgebras $C \to TC_{\mathrm{repr}}$. There is a finite dimensional subcoalgebra C' of TC_{repr} that contains c. Then $C \cap C'$ is a finite dimensional subcoalgebra of C containing c (because over a field the intersection of two coalgebras is a coalgebra; this is not necessarily true over rings).

9. The 'zero dual' coalgebras of algebras over a ring

For algebras R over a field K the 'zero dual' coalgebra is defined as

$$R^0 = \{R \xrightarrow{f} A\colon \ \mathrm{Ker}(f) \text{ contains an ideal of finite codimension}\}. \tag{9.1}$$

This vector space has a natural coalgebra structure dual to the algebra structure of R, see e.g. [22]. It turns out to be exactly the right duality notion, a fact that has very much to do with the main theorem property of coalgebras over fields.

For algebras R over a ring A (where A is commutative with unit element) there is an obvious analogue. Define

$$R^0 = \{R \xrightarrow{f} A\colon \ \mathrm{Ker}(f) \text{ contains an ideal } \mathfrak{a} \text{ of finite corank}\}, \tag{9.2}$$

where 'finite corank' means that $R/\mathfrak{a}$ is finitely generated (as a module). Now define a comultiplication on R^0 by requiring

$$\mu(f)(a \otimes b) = f(ab). \tag{9.3}$$

Then $\mu(f)$, for $f \in R^0$ lands in $R^0 \otimes R^0 \subset R^* \otimes R^* \subset (R \otimes R)^*$ provided that the tensor square of the inclusion $R^0 \subset R^*$ is injective.

(This is a potentially tricky point because the tensor square of an inclusion is not necessarily injective, even in the torsion free case, as the following well-known example shows. Let $A = K[X, Y]$, $E = A$, $E_0 = (X, Y) \subset E$ the ideal generated by X and Y. Then the tensor square of the inclusion takes the nonzero element $X \otimes Y - Y \otimes X \in E_0 \otimes E_0$ to zero. In the case considered below, where R is the tensor algebra over a free finite rank module over A (i.e. a free finitely generated A-algebra) there is no problem. One thing that calls for investigation in this context is when (for arbitrary A-algebras R) the modules R^* and R^0 are flat).

9.4. **Theorem.** *Let M be a free finitely generated module over a Noetherian integral domain A. Let $R = TM$ be the free A-algebra over M. Then*

$$R^0 = (TM)^0 = T(M^*)_{\text{repr}} \tag{9.5}$$

the free coalgebra over M^*.

(As it should be.)

Proof. Let $f \in R^0$. Then $\mu(f)$ is a finite sum $\sum g_i \otimes h_i$ with $g_i, h_i \in R^0 \subset R^*$. And so, using the characterization (9.3),

$$\mu(f)(a \otimes b) = f(ab) = \sum_{i=1}^{r} g_i(a) h_i(b).$$

Thus f is representative. On the other hand, let f be representative. Then by the main theorem property of $T(M^*)_{\text{repr}}$ (see Theorem 8.10) there is a finitely generated subcoalgebra C of $T(M^*)_{\text{repr}}$ containing f. Define $\mathfrak{a} = \{a \in TM\colon\ g(a) = 0 \text{ for all } g \in C\}$. Then, $\mathfrak{a}$ is an ideal of finite corank (because C is finitely generated as a module), and $\mathfrak{a} \subset \operatorname{Ker}(f)$ so that $f \in R^0$.

10. Representations of coalgebras

Let A be a Noetherian integral domain. Consider a (finite dimensional) representation ρ of the monoid W, i.e. a collection of $n \times n$ matrices $\{\rho(X_j)\colon\ j \in J\}$. Let $N = \prod_{j \in J} AX_j$. Now define the subcoalgebra

$$M^{n\times n}_{\text{coalg}}(\rho) \subset TN_{\text{real}}. \tag{10.1}$$

As the one spanned by the n^2 entries of ρ as in (8.11).

Let

$$M^{n\times n}_{\text{coalg}},\ \ \mu(e_{i,j}) = \sum_{k=1}^{n} e_{i,k} \otimes e_{k,j} \tag{10.2}$$

be the standard matrix coalgebra over A. There is a natural surjective coalgebra homomorphism

$$\varphi\colon M^{n\times n}_{\text{coalg}} \to M^{n\times n}_{\text{coalg}}(\rho), \qquad e_{i,j} \mapsto \rho_{i,j}. \tag{10.3}$$

Now let C be a coalgebra over a Noetherian integral domain A that is (as a module) isomorphic to a submodule of a finite rank free module N. By the universality property of TN_{real} the inclusion $C \subset N$ gives rise to an associated coalgebra morphism $C \xrightarrow{\psi} TN_{\mathrm{real}}$ that is of course injective. Now let $c_1, \ldots, c_m$ be a finite set of generators of C (as a module over A). And let ρ_i, be a representation that gives a realization of $\psi(c_i)$. Let ρ be the direct sum of the representations ρ_i. Then the coalgebra (10.1) contains all the $\psi(c_i)$ so that C embeds as a coalgebra in one of the special coalgebras (10.1). So as a final application of the cofree algebra constructions we get a faithful representation theorem.

10.4. **Theorem.** *Let C be a coalgebra over a Noetherian integral domain that is* (*as a module*) *isomorphic to a submodule of a finite rank free module. For instance*, *C can be a finite rank projective module. Then C is isomorphic to a subcoalgebra of a matrix-like coalgebra $M^{n\times n}_{\mathrm{coalg}}(\rho)$.*

These coalgebras are called matrix like because $M^{n\times n}_{\mathrm{coalg}} \to M^{n\times n}_{\mathrm{coalg}}(\rho)$, $e_{i,j} \mapsto \rho_{i,j}$ (see (8.11)) is a surjective (but not necessarily injective) coalgebra morphism.

11. Coda: lifting coalgebras to coalgebras with free underlying module

In this last section I want to try to draw attention to the following problem that I think is of some importance.

Let C be a coalgebra over a ring A. Does there exist a coalgebra $\tilde{C}$ whose underlying module is free together with a surjective coalgebra morphism $\tilde{C} \to C$? If this were true in general, then the suggestion made at the end of Section 7 would be perfectly workable. That is, one could define a tensor power series f in $\hat{T}M$ for an arbitrary module M to be representative (resp. recursive) if and only if there is a representative (resp. recursive) tensor power series $\hat{f}$ in $\hat{T}(\tilde{M})$ that maps into f under $\hat{T}\pi$ where $\tilde{M}$ is a free module together with a surjective A-module morphism $\tilde{M} \xrightarrow{\pi} M$. Eventually, one may also want to lift the ring A to an integral domain.

Unfortunately, the answer to this lifting question is not an unequivocal yes. Consider the Example 8.1 of a coalgebra for which the 'main theorem property' does not hold. If this one were liftable to a coalgebra (over $\mathbf{Z}$) with free underlying Abelian group, then that lift would satisfy the 'main theorem property' (by Theorem 8.4) and hence so would the coalgebra of Example 8.1. Thus that coalgebra is not liftable.

I am inclined to think that for coalgebras over A whose underlying module is finitely generated the answer to the lifting question is yes.

The results of Section 10 seem to indicate that this may be true.

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that I take a good look at the work of the French school on noncommutative power series.

Appendix A. The non-Noetherian case

In the case the ring A is not anymore Noetherian and not even necessarily a domain, a number of the main results of the paper survive in some form. The formulations are perhaps a little less elegant and some equivalences do not survive; the main victim is the idea of finite Hankel rank; another is that representative and realizable need no longer coincide. Here is a short account.

A.1. **Theorem.** *Consider a (finite or infinite) set of indeterminates $\{X_j\colon j \in J\}$ and let M be one of the modules $M = \bigoplus_{j\in J} AX_j$ or $M = \prod_{j\in J} AX_j$. Let $f \in \hat{T}M$ be a corresponding noncommutative power series. Consider the following properties*:

(i) *The module $R_B f$ is finitely generated.*
(ii) *The module $L_B f$ is finitely generated.*
(iii) *f is left Schützenberger recursive.*
(iv) *f is right Schützenberger recursive.*
(v) *f is left recursive.*
(vi) *f is right recursive.*
(vii) *f is realizable.*
(viii) *f is representative.*

Then the following implications hold

(a) (i) $\Leftrightarrow$ (iii) $\Rightarrow$ (v) $\Rightarrow$ (vii) $\Rightarrow$ (viii).
(b) (ii) $\Leftrightarrow$ (iv) $\Rightarrow$ (vi) $\Rightarrow$ (vii) $\Rightarrow$ (viii).

If the module M is free then also

(c) (v) $\Rightarrow$ (iii).
(d) (vi) $\Rightarrow$ (iv).

If the set of indeterminates is finite (so that in particular M is free), and A is an integral domain, then also

(e) (vii) $\Rightarrow$ (i), (ii)

so that in this case (i)–(vii) *are all equivalent.*

Most of the proofs are rather similar to the ones in the main body of the paper. The exception is (e). So suppose that the set of indeterminates is finite and that f is realizable, say, by (ρ, b, c). Because there are only finitely many indeterminates involved, the

$\rho(X_j), b, c$ involve only a finite number of entries from A. Thus, f is actually defined over a subring of A generated by a finite number of elements (as a ring), and that subring is hence Noetherian.

Below are a number of examples that show that in the non-Noetherian case nothing more than Theorem A.1 can be expected.

For instance, the argument that representative implies realizable, or, e.g. that $R_B f$ is finitely generated, breaks down. Representative just gives that there are a finite number of g_i in $\hat{T}M$ such that each element of $R_B f$ is a linear combination of these elements. It does not say that there are such g in $R_B f$ itself. This can actually happen. Consider the ring over a field K of polynomials in infinitely many commuting variables $A = K[y_j\colon\ j \in J]$. Let

$$f = \sum_{j \in J} y_j X_j. \tag{A.2}$$

Then

$$R_\beta f = \begin{cases} y_\beta & \text{if } \lg(\beta) = 1, \\ 0 & \text{if } \lg(\beta) > 1, \\ f & \text{if } \beta = [\ ] \end{cases}$$

and thus all the $R_\beta f$ are A-linear combinations of two elements from $\hat{T}M$ but $R_B f$ is not finitely generated. The power series (A.2) is representative (because $\mu(f) = 1 \otimes f + f \otimes 1$ for this f) and also realizable, but it does not satisfy any of (i)–(iv). But it does satisfy (v) and (vi).

Here is an example that shows that realizability does not necessarily imply either left or right recursiveness and hence certainly not left or right Schützenberger recursiveness.

Let A be as in example (A.2), and define f by

$$f = \sum_{\alpha \in W} y_\alpha X_\alpha, \tag{A.3}$$

where, as usual, W is the free monoid on the index set J, and $y_{[\ ]} = X_{[\ ]} = 1$. This f is realizable by $c = b = 1$, $\rho(X_j) = y_j$ and hence representative; indeed $\mu(f) = f \otimes f$. Suppose it were left recursive, then, there is a finite number of words $\alpha_1, \alpha_2, \ldots, \alpha_m$ such that all rows of the Hankel matrix are linear combinations of the rows indexed by these words. Take an index j that does not occur in any of the α's; i.e.

$$j \notin J_0 = \bigcup_{i=1}^{m} \operatorname{varsupp}(\alpha_i). \tag{A.4}$$

The first entry in the row indexed by $[j]$ is y_j; and so y_j would be in the ideal generated by the y_i, $i \in J_0$, which is not the case by (A.4).

Finally, here is an example of a representative tensor power series in one variable that is not realizable. Take three sets of commuting indeterminates x_i, y_i, z_i, $i = 0, 1, 2, \ldots$. Let A be the ring of polynomials over a field K in these indeterminates subject to the relations $y_iz_j = x_{i+j}$, $\forall i,j$ (so that the x_m are really superfluous). A is an integral domain. [1]

Let f, g, h be the power series in one variable

$$f = \sum_{m=0}^{\infty} x_m X^m, \qquad g = \sum_{m=0}^{\infty} y_m X^m, \qquad h = \sum_{m=0}^{\infty} z_m X^m. \tag{A.5}$$

Then f is representative. Indeed $\mu(f) = g \otimes h$. The rows of the Hankel matrix of f indexed by the monomial X^n (i.e. the word $[\underbrace{1,1,\ldots,1}_{n}]$) is $(x_n, x_{n+1}, x_{n+2}, \ldots)$. So if $R_B f$ were to be finitely generated, there is an n such that row $(n+1)$ is linearly dependent on rows $0,1,2,\ldots,n$. Taking a look at the first entries means that there must be $a_0, a_1, \ldots, a_n \in A$ such that

$$a_0x_0 + a_1x_1 + \cdots + a_nx_n = x_{n+1}. \tag{A.6}$$

Eliminating the x_m the ring A is the ring of polynomials $K[y_i, z_j\colon i,j \in \mathbf{N} \cup \{0\}]$ modulo the ideal generated by the $y_iz_j - y_kz_l$ for $i+j = k+l$. These elements are homogeneous of degree 2, so there is a well-defined notion of degree on A. So (A.6) can only hold if there are constants $a_i^0 \in K$ such that

$$a_0^0x_0 + a_1^0x_1 + \cdots + a_n^0x_n = x_{n+1}. \tag{A.7}$$

Now consider the ring of polynomials $B = K[t]$. There are a good many ring homomorphisms from A into B. For instance, the homomorphism $\varphi : y_k \mapsto t^k,\ z_l \mapsto t^k$. Applying this to the relation (A.7) would give that the polynomial

$$a_0^0 + a_1^0 t + \cdots + a_n^0 t^n - t^{n+1}$$

is zero (which it is not). Thus, $R_B f$ is not finitely generated. Now suppose that f were realizable. Then there is a single matrix $\rho(X)$ with entries from A such that

$$f = c(I - X\rho(X))^{-1} b$$

which is of the form $\det(I - X\rho(X))^{-1}$ (some polynomial in the single variable X), and which is therefore a recursive power series in a single variable and would have $R_B f$ finitely generated. Thus (A.7) is not realizable.

[1] There are probably better ways to see this, but here is one. First show that y_0 is not a zero divisor. Then in the localized ring $z_i = z_0 y_0^{-1} y_i$, and there are the relations $z_0 y_i y_j = z_0 y_0 y_{i+j}$. Using these, every monomial in the localized ring can be written in the form $z_0^r y_0^s y_t$, $r \in \mathbf{N} \cup \{0\}$, $s \in \mathbf{Z}$, $t \in \mathbf{N}$ or $z_0^r y_0^s$. The product of two monomials of the first type is $z_0^{r_1+r_2} y_0^{s_1+s_2+1} y_{t_1+t_2}$ and the other products are obvious. It is now easy to find an ordering on these monomials (e.g. lexicographic ordering) such that, the top term of product of two sums of such monomials is the product of the two top terms of these sums.

Appendix B. The strongly representative completion of the tensor coalgebra and the cofree coalgebra over a module

Let M be module over A. Consider submodules $T \subset \hat{T}M$ such that

$$TM \subset T \subset \hat{T}M \quad \text{and} \quad \mu(T) \subset T \otimes T. \tag{B.1}$$

Then the tensor square of the injection $T \subset \hat{T}M$ is injective (Lemma 3.37), and thus T is a coalgebra (with the comultiplication μ of (2.14) restricted to T and with the canonical projection onto the zeroth component $\hat{T}M \to A$ restricted to T as counit. An example of such a submodule is $TM \subset \hat{T}M$. If T and T' are two submodules that satisfy (B.1), then so does the sum $T + T'$. Thus there is a largest such submodule that contains all the others. This largest such submodule is denoted TM_{srepr} and is called the strongly representative completion of TM. From what has been said just now it is evident that it has a natural coalgebra structure. Its elements are called strongly representative tensor power series.

Obviously, $TM \subset TM_{\text{srepr}}$, but TM_{srepr} is always larger (unless $M = 0$). For instance, if $0 \neq x \in M$, the non-terminating tensor power series

$$(1, x, x \otimes x, x \otimes x \otimes x, \ldots) \tag{B.2}$$

is in TM_{srepr}. Indeed, using the notations of (2.15), (2.16)

$$\begin{pmatrix} 1 \\ x \\ x^{\otimes 2} \\ \vdots \end{pmatrix} \otimes (1 \;\; x \;\; x^{\otimes 2} \;\; \cdots) = \begin{pmatrix} 1 & x & x^{\otimes 2} & \cdots \\ x & x^{\otimes 2} & x^{\otimes 3} & \cdots \\ x^{\otimes 2} & x^{\otimes 3} & x^{\otimes 4} & \cdots \\ \vdots & \vdots & \vdots & \end{pmatrix}. \tag{B.3}$$

(The element (B.2) is group like.)

To check for a given element $f \in \hat{T}M$ whether it is in TM_{srepr}, potentially involves an infinity of conditions. Specifically, it means that for every finite sequence $i_1, i_2, \ldots, i_k$ of 1's and 2's there are a finite number of tensor power series $g_{i_1,i_2,\ldots,i_k;j_1,j_2,\ldots,j_k}$ such that

$$\mu(f) = \sum g_{1;j} \otimes g_{2;j},$$
$$\mu(g_{i_1,i_2,\ldots,i_k;j_1,j_2,\ldots,j_k}) = \sum_j g_{i_1,i_2,\ldots,i_k,1;j_1,j_2,\ldots,j_k,j} \otimes g_{i_1,i_2,\ldots,i_k,2;j_1,j_2,\ldots,j_k,j} \tag{B.4}$$

B.5. **Theorem.** *Let $\pi : TM_{\text{srepr}} \to M$ be the module morphism of projection onto the first factor. Then $\pi : TM_{\text{srepr}} \to M$ is the cofree coalgebra with counit over M.*

Proof. Exactly as in the proof of Theorem 3.14 it follows that the sought for morphism $\tilde{\varphi}$ associated to $\varphi : C \to M$ must be given by the formula

$$\tilde{\varphi}(c) = (\mu_0(c) = \varepsilon(c), \varphi(\mu_1(c)) = \varphi(c), \varphi^{\otimes 2}(\mu_2(c)), \ldots, \varphi^{\otimes n}(\mu_n(c)), \ldots). \tag{B.6}$$

It remains to show that the tensor power series (B.6) is strongly representative. To see this, let

$$\mu(c)=\sum_j c_{1,j}\otimes c_{2,j}.$$

Then, by the coassociativity of μ and the counit property of $\mu_0=\varepsilon$

$$\sum_j \mu_k(c_{1,j})\otimes\mu_l(c_{2,j})=\mu_{k+l}(c)$$

and it follows that

$$\mu(\tilde{\rho}(c))=\sum_j \tilde{\rho}(c_{1;j})\otimes\rho(c_{2,j})$$

(where the left-hand side μ is the one of TM_{srepr}) so that $\mu(\tilde{\rho}(c))$ is a finite sum of tensor products of power series of the same type, thus proving that $\tilde{\rho}(c)$ is strongly representative (and reconfirming that $\tilde{\rho}$ is a morphism of coalgebras). This finishes the proof of the theorem.

B.7. **Corollary.** *If A is a Noetherian integral domain and M is a free module or the linear dual of a free module, then $TM_{\mathrm{srepr}}=TM_{\mathrm{repr}}$.*

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