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# Linear vs Nonlinear Mathematics
## with special emphasis on
# the KdV Equation and other Liouville Integrable Dynamical Systems

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**Abstract.** After some words on non-linear versus linear mathematics, arguments are presented to show that Liouville integrable systems are just one step away from linear ones. In particular attention is drawn to the fact that many Liouville integrable systems are fractional linear quotients of linear ones. Conjecturally this is always the case.

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## 1. Introduction.

The original title that the coordinators and organizers of this KdV course inflicted on me was "The importance of nonlinear mathematics". And in an unguarded moment, possibly distracted by other concerns, I was unwise enough to raise no objections.

Also I thought that surely somewhere in the past, at least one of those giants, would have given some glowing lecture, praising the nonlinear, and raising the aim of truly understanding nonlinear phenomena to some sort of holy grail. Maybe so. But though a general consensus certainly can be detected that the nonlinear world is indeed where the real rewards are, I have found no convincing account, and very few general real hard results of the type "such and such phenomena can not happen in a linear setting".

Another argument would be that such a glowing account as I failed to find is hardly needed. Just look at the Mathematics Subject Classification Scheme. With the exception of (most of) sections 15 (Linear algebra), 46 (Functional analyis), 47 (Operator theory), and a handfull of subsections dealing specifically with linear somethings (such as linear ordinary differential equations, linear partial differential equations, linear programming, linear control systems, ...) all of mathematics is about nonlinear mathematics. From this point of view it is surprising that there are in fact books with 'non-linear' in the title.

In the past I have had occasion to write (in the Series Editor's prefaces of books in the series MIA, e.g. [1]):

"The nonlinear world is where the rewards are. Linear models are honest and also a bit sad and depressing: proportional efforts and results. It is in the nonlinear world that infinitesimal inputs may result in macroscopic outputs (or vice versa). To appreciate what I am hinting at: if electronics were linear we would have no fun with transistors and computers, we would have no TV; in fact you would not be reading these lines."

Few people would quarrel with this statement. Though the possibly implied suggestion that there is no fun to be had, and that there are no challenges left, in linear mathematics has to be taken with large numbers of grains of sand; more like an ocean full. Indeed, quantum mechanics is all about Hilbert space and self-adjoint linear operators, ... . It is also, of course, infinite

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dimensional, giving lots of room for all kinds of fascinating phenomena. But also in finite dimensional linear algebra, I can easily find hosts of problems that are important, fascinating, and, seemingly, out of reach for the moment.

I see no way of providing anything like a scientific mathematical foundation for the statement in quotes above. Much depends on what is the precise meaning of 'linear mathematics', and that, as we shall note below, is not as easy to define as one might think. Below are some few initial remarks on the general case at hand: "the linear versus the nonlinear". The matter will not be settled shortly.

I shall concentrate on aspects relevant to the KdV equation and other Liouville integrable (systems of) equations (both finite and infinite dimensional ones). More concretely, I shall try to give some evidence that while these equations are definitely non-linear, they are in fact the next class.

## 2. A few general remarks on linear and nonlinear mathematics.

Let me start with a quotation from [12] (in several parts):

"the closely guarded secret of this subject (differential equations) is that it has not yet attained the status and dignity of a science but still enjoys the freedom and freshness of pre-scientific study .... The work of classification and systematization of specimens has hardly begun.

This is true even of differential equations which belong to the genus technically described as 'ordinary, linear equations'. ... In the case of non-linear equations, Lie's theory of transformation groups has done little but suggest a scheme of classification. An inviting flora or rare equations and exotic problems lies before a botanical excursion in the nonlinear field. ....

The history of mathematical physics during the last century may be divided into two periods—the linear period and the non-linear period. In those happy far-off times of the linear period, all differential equations were linear and the principle of superposition reigned supreme. In the present distressful times most differential equations are non-linear and no effective general method of solution has yet been proposed."

I would submit that the first paragraph of the quotation (which also has been quoted several times by others, e.g. [7]), holds for all of nonlinear mathematics (in so far as that term is defined); I would also submit that at least one recognizable species has been identified and to some extent decribed—the species of Liouville integrable dynamical systems.

One difficulty is that if linearity is abandoned, in general no structure is left at all. No one is optimistic enough to think that we will have anything like a complete classification of all possible phase portraits of ordinary differential equations (even qualitatively) in the next few centuries (or even ever). Thus strong substitute structures are needed that are compatible (in some sense) with the equations involved. These can be algebraic (symmetry groups e.g.), geometric, combinatorial, .... In the case of Liouville integrable systems there are both strong algebraic and strong geometric structures present and that has certainly played a major role in their emergence as a recognized species.

## 3. What is linear, what is nonlinear.

Here are a few past, present, and future buzzwords from nonlinear mathematics: singularities, bifurcations and catastophes, pattern formation, chaos and strange attractors, fractals and selfsimilarity, crystal growth, .... Of all it is generally believed that we are dealing with (mostly) genuinely nonlinear phenomena. However, that statement could use some clarification.

3.1. *Meaning of non-linear*? To find out what the phrase nonlinear means lets turn to the encyclopaedic sources. The encyclopaedia [3] has articles on Non-linear boundary value problem, Non-linear connection, Non-linear differential equation, Non-linear equation, Non-linear functional, Non-linear functional analysis, Non-linear integral equation, Non-inear operator, Non-linear oscillations, Non-linear partial differential equation, Non-linear potential, Non-linear programming. In all cases there is an underlying linear setting—usually a vector space—and the non-linear something is defined as a something that does not respect that underlying linear structure. Without such an underlying linear setting the phrase 'non-linear' is not defined, and as such 'non-linear mathematics' is an undefined notion (though of course it carries intuitive meaning). The material in [5] agrees.

3.2. *Superposition principle.* Similarly the phrase that there is a superposition principle is largely meaningless. Consider an ordinary differential equation in $\mathbf{R}^n$, and for simplicity assume that solutions (with perscribed initial conditions) are unique and always exist for all time. Take two solutions $x(t)$, $y(t)$, and let $z(t)$ be the unique solution that takes the value $z(0)=x(0)+y(0)$ at time 0. Then $(x(t),y(t))\mapsto z(t)=x(t)*y(t)$ is a nonlinear superposition principle. Its presence means nothing. It is shocking that there is in fact a published article devoted to just this observation and nothing more.

3.3. *Transformations.* Consider the set of polynomial equations

$$\begin{aligned}
&x_2+x_3-x_2x_3-x_1x_3^2=1\\
&-x_1-2x_2x_3-x_1x_3-x_1x_3^2+x_2x_3^2-x_2x_3-x_3^2=0\\
&x_1-x_2+x_3+2x_2x_3+2x_1x_3^3+x_1x_3+x_1x_3^2=0
\end{aligned}$$

This is without question a nonlinear set of polynomial equations. The fact that they are in fact a nonlinear invertible transformation (and a rather simple one) of a set of linear equations, viz

$$\begin{aligned}
&-x_1+x_2+x_3=1\\
&-x_1\qquad\;\, -x_3=0\\
&2x_1-x_2+x_3=0
\end{aligned}$$

is besides the point. Especially because there is in all likelyhood no algorithmic test to decide whether a set of nonlinear polynomial equations is a non-linear invertible transformation of a linear set or not.

3.4. *Finite dimensional Liouville integrable dynamical systems.* The remark just made is important for the case of the KdV equation and its finite dimensional relatives. A Hamiltonian dynamical system on $\mathbf{R}^{2n}$ (or more generally on a finite dimensional symplectic manifold) given by a Hamitonian $H=H_1$ is called *Liouville integrable* (unfortunately more generally also called '*completely integrable*'), if there are $n$-1 additional independent conserved quantities $H_2,\cdots,H_n$ in involution with $H_1$ and each other, i.e. such that $\{H_i,H_j\}=0$ for all $i, j$, where $\{,\}$ denotes the Poisson bracket.

It follows that there are new coordinates $p_i, q_j$, obtainable from the other ones by means of a socalled canonical transformation, such that in these new coordinates the dynamical system is given by

$$\begin{aligned}
&\dot{p}_i=0\\
&\dot{q}_i=p_i
\end{aligned}$$

(*action-angle coordinates*). See e.g. [2] for some more details. Here, again, there is in all probability no deciding algorithm.

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3.5. *Inbeddings.* Consider a differentiable manifold $M$ and a dynamical system on it given by a vectorfield $X$. For instance one with a strange attractor (a chaotic one). Let $C(M)$ be the linear space of real valued differentiable functions on $M$. Let $L_X$ be the first order differential operator on $C(M)$ defined by the vectorfield. Then the evolution of any function $f$ on $M$ (including the coordinates on $M$) is given by the first order linear differential equation

$$\dot{f} = L_X f.$$

3.6. This last observation is one reason why it is quite difficult to formulate a precise and defendable statement of the kind: "Such and such a phenomenon belongs essentially to non-linear mathematics".

Even without the caveats already mentioned, such statements tend to be doubtful. For instance in that pre-eminently nonlinear field of singularity theory, singularities and relations between them turn out to be controlled by simple Lie algebras, [1], themselves originally invented as a linearization of transformation groups.

## 4. Covering linearizations.

The category of differentiable manifolds and dynamical systems on them is not noticeably self-dual and there is a vast difference between being inbeddable and being a quotient. Here are some observations concerning quotient properties of Liouville integrable systems.

4.1. *The KdV equation.* Let us start with the KdV equation in its integrated form

$$u_t + 6u_x^2 + u_{xxx} = 0 \tag{4.1.1}$$

The standard KdV equation arises from this one by differentiation with respect to $x$ (and replacing $u_x$ with $u$). Now consider an associated set of linear equations

$$\begin{aligned} &\Gamma_t + 4\Gamma_{xxx} = 0 \\ &\Gamma_{xx} = a^2\Gamma \end{aligned} \tag{4.1.2}$$

Let $\Gamma$ be a solution of (4.1.2). Then, as is easily verified,

$$u = \Gamma_x / \Gamma$$

is a solution of (4.1.1). It is somewhat remarkable that this still works for operator valued functions. I.e. the commutativity of the multiplication on the real line plays no role in this. This is the starting point of the monograph [6], and probably should be explored much further (and also for other Liouville integrable systems).

Thus the KdV equation (4.1.1) emerges as a fractional linear quotient of the system (4.1.2) at least for these solutions. This, however, can be generalized, and in fact, at least at the formal level, the KdV is indeed a fractional linear quotient of a linear dynamical system, [4]. This is implied by the socalled inverse scattering method in the case of the KdV. The analytic details remain to be worked out.

4.2. *Matrix Riccati equations.*

The matrix Riccati equation is the following equation for an $n \times m$ matrix $P$

$$\dot{P} = AP + PD + PBP + C \tag{4.2.1}$$

where the $A, B, C, D$ are known matrices of the right sizes so that (4.2.1) makes sense. Riccati

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equations turn up all over mathematics, for instance in atmospheric tranport problems, oscillation theory, the calculus of variations, linear Hamiltonian systems, ... . There are several thousands of articles on it and one monograph, [9], and another monograph is in preparation. One particularly important area of application of in particular matrix Riccati equations is in optimal control of linear systems and filtering problems for such systems (Kalman filtering), cf e.g. [13]. Riccati equations are definitely nonlinear and exhibit finite escape times, one characteristic phenomenon of nonlinearity (and this implies that they are not in any way non-linear transformations of a linear system).

For our purposes here assume that the matrices $A, B, C, D$ are constant (independent of time). In this case consider the following linear dynamical system

$$\frac{d}{dt}\begin{pmatrix} X \\ Y \end{pmatrix} = \begin{pmatrix} A & C \\ -B & -D \end{pmatrix}\begin{pmatrix} X \\ Y \end{pmatrix} \tag{4.2.2}$$

for the $n \times m$ matrix $X$ and the $m \times m$ matrix $Y$. Now let $X, Y$ be a solution of (4.2.2) and observe that $P = XY^{-1}$ is a solution of (4.2.1) (as long as $Y$ is invertible; non invertibility of $Y$ corresponds to finite escape time phenomena). Carrying this analysis a bit further it leads to the observation that the Grassmann manifold of $m$-planes in $n+m$ space is the proper place to understand the matrix Riccati equation, it leads to a detailed understanding of its finite escape time behaviour, and it leads to the observation that the matrix Riccati equation is a fractional linear quotient of a linear system.

4.3. *The Toda lattices.* The nonlinear Toda lattices

$$\begin{aligned} \ddot{q}_n &= -e^{q_{n+1}-q_n} + e^{q_n - q_{n-1}} \\ q_0 &= \infty,\ q_{N+1} = -\infty,\ n = 1,\cdots,N \end{aligned} \tag{4.3.1}$$

also are fractional linear qoutients of linear systems, [11], and more generally this holds for the AKSRS (Adler, Kostant, Symes, Reyman, Semenov Tian-Shansky) systems. Those are obtained by a splitting of a Lie algebra and the covering linear space is the socalled double of that Lie algebra [10].

4.4. *KP equation.* The same picture holds for the KP equation, [4]. Indeed, following the ideas of Michio Sato, the KP equation emerges as a projective limit of matrix Riccati equations. This holds for the whole KP hierarchy (and it is as a hierarchy that this picture emerges most clearly).

Now the KP hierarchy is thought to be rather universal in the world of Liouville integrable systems (in the sense that other Liouville integrable systems arise from it by 'specialization'). Thus it becomes a research problem to investigate when specialization preserves a fractional linear quotient situation. Given this and the observations in 4.1 - 4.3 above, it looks like Liouville integrability and being a fractional linear quotient of a linear system have a great deal to do with one another.

## 5. Universality of the KdV and NLS.

One remarkable fact about the KdV equation, the nonlinear Schrödinger (NLS) equation, and their many relatives is that they turn up so frequently in applications. There is a good reason for that. As soon as one goes one step better than a linear approximation, they tend to turn up. This illustrates besides the universality also their closeness to the linear world.

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5.1. *Universality of the KdV equation.* Let us start with the usual standard linear one-dimensional wave equation

$$\frac{\partial^2\psi}{\partial t^2} = v_0^2\frac{\partial^2\psi}{\partial x^2} \tag{5.1.1}$$

Weak dispersion allows us to treat separately waves travelling in the two directions. So

$$\frac{\partial\psi}{\partial t} + v_0\frac{\partial\psi}{\partial x} = 0 \tag{5.1.2}$$

Now consider the dispersion law between the wave vector $k$, the velocity $v$ and the frequency $\omega$ for a linear wave $\exp(i\omega t - kr)$

$$\omega = kv(k). \tag{5.1.3}$$

Here $v(k)$ goes to $v_0$ as $k \to 0$ and is in general an analytic function of $k$ which can be expanded in a power series. In the absence of dissipation it is a power series in $k^2$. To the next approximation (with respect to no dispersion), therefore

$$\omega = v_0 k - \beta k^3 \tag{5.1.4}$$

and this gives rise to a third order correction term in (5.1.2) giving

$$\frac{\partial\psi}{\partial t} + v_0\frac{\partial\psi}{\partial x} + \beta\frac{\partial^3\psi}{\partial x^3} = 0. \tag{5.1.5}$$

Now assume, as is reasonable for classical systems, that there is a conservation law

$$\frac{\partial\psi}{\partial t} + \frac{\partial j}{\partial x} = 0 \tag{5.1.6}$$

This gives

$$j = v_0\psi + \beta\frac{\partial^2\psi}{\partial^2 x}$$

and inserting the next correction term this gives

$$j = v_0\psi + \beta\frac{\partial^2\psi}{\partial^2 x} + \frac{\alpha}{2}\psi^2 \tag{5.1.7}$$

and hence

$$\frac{\partial\psi}{\partial t} + v_0\frac{\partial\psi}{\partial x} + \beta\frac{\partial^3\psi}{\partial x^3} + \alpha\psi\frac{\partial\psi}{\partial x} = 0 \tag{5.1.8}$$

Now make the changes of variables $\xi = x - v_0 t$ ( so go to a coordinate system that moves along with (the centre of) the travelling waves), and $\psi = (\alpha/\beta)\eta$ to find the KdV equation in the form

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$$\frac{\partial\eta}{\partial t}+\frac{\partial^3\eta}{\partial\xi^3}+\eta\frac{\partial\eta}{\partial\xi}=0. \tag{5.1.9}$$

Thus the Korteweg de Vries equation arises in quite general circumstances as the next approximation to the linear one as soon as dispersion and non-linearity are taken into account (but no dissipation).

5.2. *Universality of the non-linear Schrödinger equation.* The NLS equation (non-linear Schrödinger equation) looks as follows.

$$i(\frac{\partial\psi}{\partial t}+v_0\frac{\partial\psi}{\partial x})=-\beta\frac{\partial^2\psi}{\partial x^2}+\alpha|\psi|^2\psi \tag{5.2.1}$$

It is also Liouville integrable (a notion that is not really yet well defined for infinite dimensional systems such as PDE's). This equation also can be shown to arise naturally in a wide variety of circumstances as a result of incorporating weak nonlinearity and weak dispersion. In this case for fields differing but little from harmonic ones.

The material of 5.1 and 5.2 above was taken from the introduction of [8]. Incidentally, the interested reader is advised to consult the original Russian edition (Teoriya solitonov, Nauka, 1980); the translation is so poor that referring to the original will anyway be frequently necessary.

## 6. Coda.

In the above a number of arguments have been given indicating that Liouville integrable systems are so to speak the next one one encounters after the linear approximation. These arguments are briefly as follows: covering linearizations with a fractional linear mapping giving the Liouville integrable systems as a quotient (section 4), and the universality properties hinted at in section 5. There are many more, including superposition properties (of a rather more substantial nature than the nonsense of subsection 3.2 above). One more potential line of reasoning I would like to add very briefly. The familiar transcendental function such as exp and the trigonometric ones can be seen as arising from the simplest linear differential equations. Going a little further one finds the socalled Painlevé transcendents which in several ways seem to be fundamentally connected to Liouville integrable systems.

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