Arrays, Bounded Quantification and Iteration in Logic and Constraint Logic Programming

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Abstract

We claim that programming within the logic programming paradigm suffers from lack of attention given to iteration and arrays. To convince the reader about their merits we present several examples of logic and constraint logic programs which use iteration and arrays instead of explicit recursion and lists. These programs are substantially simpler than their counterparts written in the conventional way. They are easier to write and to understand, are guaranteed to terminate and their declarative character makes it simpler to argue about their correctness. Iteration is implemented by means of bounded quantification.

1 Introduction

Any systematic course on programming in the imperative style (say using Pascal), first concentrates on iteration constructs (say **while** or **repeat**) and only later deals with recursion. Further, the data structures are explained first by dealing with the static data structures (like arrays and records) and only later with the dynamic data structures (which are constructed by means of pointers).

In the logic programming framework the distinctions between iteration and recursion, and between static and dynamic data structures are lost. One shows that recursion is powerful enough to simulate iteration and rediscovers the latter by identifying it with tail recursion. Arrays do not exist. In contrast, records can be modelled by terms, and dynamic data structures can be defined by means of clauses, in a recursive fashion (with the exception of lists for which in Prolog there is support in the form of built-in's and a more friendly notation).

One of the side effects of this approach to programming is that one often uses a sledgehammer to cut the top of an egg. Even worse, simple problems have unnecessarily complex and clumsy solutions in which recursion is used when a much easier solution using iteration exists, is simpler to write and understand, and — perhaps even more important — is closer to the original specification.

In this paper we would like to propose an alternative approach to programming in logic and in constraint logic programming — an approach in which adequate stress is put on the use of arrays and iteration. Because iteration can be expressed by means of bounded quantification, a purely logical construct, the logic programming paradigm is not "violated". On the contrary, it is enriched, clarified and better tailored to the programming needs.

Arrays are especially natural when dealing with vectors and matrices. The use of dynamic data structures to write programs dealing with such objects is unnatural. We shall try to illustrate this point by presenting particularly simple solutions to problems such as the 8 queens problem, the knight's tour, and the map colouring problem.

Further, by adding to the language operators which allow us to express optimization, i.e. minimization and maximization, we can easily write programs for various optimization problems, like the cutting stock problem.

For pedagogical reasons we limit here our attention to programs that involve arrays, iteration and optimization constructs. Of course, recursive data types and explicit recursion have their place both in logic programming and in constraint logic programming. One of the main purposes of this paper is to illustrate how much can be achieved without them.

In the programs considered in this paper recursion is hidden in the implementation of the bounded quantifiers and this use of recursion is guaranteed to terminate. Consequently, these programs always terminate. As termination is one of the major concerns in the case of logic programming, from the correctness point of view it is better to use iteration instead of recursion, when a choice arises. Also, iteration can be implemented more efficiently than recursion (see Barklund and Bevemyr [BB93] for an explanation how to extend WAM to implement iteration in Prolog).

This work can be seen as an attempt to identify the right linguistic concepts which simplify programming in the logic programming paradigm. When presenting this view of programming within the logic programming paradigm we were very much influenced by the publications of Barklund and Millroth [BM94], Voronkov [Vor92] and Kluźniak [Klu93]. In fact, the constructs whose use we advocate, i.e. bounded quantification and arrays, were already proposed in those papers. Apart from providing further evidence for elegance of these constructs in logic programming, the only, possibly new, contribution of this paper is a proposal to integrate these constructs into constraint logic programming.

2 Bounded Quantifiers

Bounded quantifiers in logic programming were introduced in Kluźniak [Klu91] and are thoroughly discussed in Voronkov [Vor92] (where also earlier references in Russian are given). They are also used in Kluźniak [Klu93] (see also Kluźniak and Miłkowska [KM94]) in a specification language SPILL-2 in which executable specifications can be written in the logic programming style.

Following Voronkov [Vor92] we write them as ∃X ∈ L Q (the bounded existential quantifier) and ∀X ∈ L Q (the bounded universal quantifier), where L is a list and Q a query, and define them as follows:

```
∃X ∈ [Y | Ys] Q ← Q{X/Y}.
∃X ∈ [Y | Ys] Q ← ∃X ∈ Ys Q.
∀X ∈ [Y | Ys] Q ← Q{X/Y}, ∀X ∈ Ys Q.
∀X ∈ [] Q.
```

To put these definitions into syntactically acceptable format, we could introduce two relations, `exists` and `forall`, and write `exists(X, L, Q)` for ∃X ∈ L Q and `forall(X, L, Q)` for ∀X ∈ L Q. For clarity, we shall use the original syntax.

The bounded quantifies can be easily expressed using the usual quantifiers, so the above language extension is subsumed by the proposal of Lloyd and Topor [LT84] (see also Lloyd [Llo87]) in which the queries and bodies of clauses can be arbitrary first-order formulas. Unfortunately, this modelling of bounded quantifiers yields unnecessarily complex programs, among others due to the use of negation and the introduction of new relation symbols.

Moreover, as pointed out by Barklund and Hill [BH95], this translation process introduces the possibility of incorrect use of negation which in some circumstances limits the use of the program to ground queries. This difficulty was originally pointed out by Bundy [Bun88] in the context of another form of bounded universal quantification.

Voronkov [Vor92] also discusses two other bounded quantifiers, written as ∃X ⊏ L Q and ∀X ⊏ L Q, where X ⊏ L is to be read "X is a suffix of L", which we do not consider here.

To some extent the use of bounded quantifiers allows us to introduce in some compact form the "and" and the "or" branching within the program computations. This reveals some connections with the approach of Harel [Har80], though we believe that the expressiveness and ease of programming within the logic programming paradigm makes Harel's programming proposal obsolete.

Even without the use of arrays the gain in expressiveness achieved by means of bounded quantifiers is quite spectacular. Consider for example the following problem which shows the power of the $\forall$X $\in$ L $\exists$Y $\in$ M combination.

Problem 1 Write a program which tests whether one list contains all the elements of another one.

Solution

```
subset(Xs, Ys) ← ∀X ∈ Xs ∃Y ∈ Ys X = Y.
```

Several other examples can be found in Voronkov [Vor92]. Here we content ourselves with just one more, in which we use delay declarations very much like in modern versions of Prolog (for example ECL^{i}PS^e [Agg95]) or the programming language Gödel of Hill and Lloyd [HL94]).

Problem 2 Write a program checking the satisfiability of a Boolean formula.

Solution We assume here that the input Boolean formula is written using Prolog notation, so for example (¬ X, Y) ; Z stands for (¬ X $\wedge$ Y) $\vee$ Z.

```
sat(X) ← X, generate(X).
generate(X) ← vars(X, Ls), ∀Y ∈ Ls ∃Z ∈ [true, fail] Y = Z.
DELAY X UNTIL nonvar(X).
```

This remarkably short program uses meta-variables and a mild extension of the delay declarations to meta-variables. The delay declaration used here delays any call to a meta-variable until it becomes instantiated; `vars(t, Ls)` for a term `t` computes in `Ls` the list of the variables occurring in `t`. Its definition is omitted. `vars(X, Ls)` can be easily implemented using the `var(X)` and `univ` built-in's of Prolog. `true` and `fail` are Prolog's built-in's.

In Gödel the calls to negative literals are automatically delayed until they become ground. In the case of the above program such an automatic delay is not advisable as it would reduce checking for satisfiability of subformulas which begin with the negation sign to a naive generate and test method.

Even though this program shows the power of Prolog, we prefer to take another course and use types instead of exploring extensions of Prolog, which is an untyped language.

3 Arrays and Bounded Quantifiers in Logic Programming

Arrays in logic programming were introduced in Eriksson and Rayner [ER84]. Barklund and Bevemyr [BB93] proposed to extend Prolog with arrays and studied their use in conjunction with the bounded quantification. In our opinion the resulting extension (unavoidably) suffers from the fact that Prolog is an untyped language. In Kluźniak [Klu93] arrays are present, as well, where they are called indexable sequences.

More recently, Barklund and Hill [BH95] proposed to add arrays and restricted quantification, a generalization of the bounded quantification, to Gödel, the programming language which does use types. Also Greco, Palopoli and Spadafora [GPS95] suggested to extend Datalog, a simple logic programming based database language, with arrays.

These developments should be contrasted with the early proposal of Kowalski [Kow83] to encode arrays by means of unit clauses.

In the programs below we use bounded quantification, arrays and type declarations. The use of bounded quantifiers and arrays makes them simpler, more readable and closer to specifications. We declare constants, types, variables and relations in a style borrowed from the programming language Pascal.

We begin with two introductory examples which involve search through a sequence. The first one uses the universal quantifier while the second one employs the existential quantifier.

Problem 3 Check whether a given sequence of 100 integers is ordered.

Solution

```
const n = 100.
rel ordered: array [1..n] of integer.
ordered(A) ← ∀I ∈ [1..n-1] A[I] ≤ A[I+1].
```

This example shows that the terms denoting the array subscripts should be evaluated (so that we can identify `1+1` with `2` etc.), very much like the right-hand side of the `is` built-in of Prolog. In a more general set up we could view here "+" as an external procedure in the sense of Małuszyński et al. [MBB+93]. This simple program appears originally in Kowalski [Kow83] though its procedural interpretation is not explained there.

Note that the bounded universal quantifier `∀I ∈ [1..n]` does *not* correspond to the imperative **for** `i:=1` **to** `n` loop. The former is executed as long as a failure does not arise, i.e. up to n times, whereas the latter is executed precisely n

times. The programming construct `∀I ∈ [1..n] Q` actually corresponds to the construct

```
for i:=1 to n do if ¬ Q then
                 begin
                   failure := true; exit
                 end
```

which is clumsy and unnatural within the imperative programming paradigm.

(Feliks Kluźniak suggested to us the following, slightly more natural interpretation of `∀I ∈ [1..n] Q`:

```
i:=1;
while i ≤ n cand Q do i:=i+1;
failure := i ≤ n,
```

where **cand** is the "conditional **and**" connective (see Gries [Gri81, pages 68-70].))

Problem 4 *Linear Search.* Check if an element is present in a given sequence of 100 integers. If yes, return its position, otherwise terminate with a failure.

```
const n = 100.
type seq: array [1..n] of integer.
rel find: (integer, seq, [1..n]).
find(E, A, J) ← ∃I ∈ [1..n] (E = A[I], J = I).
```

Here "=" is Prolog's built-in, defined by the single clause

```
X = X.
```

and called "is unifiable with". Now the query `find(e, a, J)` checks the presence of an element `e` in an array `a`. If the answer is positive, `J` is instantiated to the position of `e` in `a`. Otherwise failure results. During the execution of this query "=" is used first to compare two ground terms and then to assign a value to a variable, `J`.

It is instructive to note that the development of the corresponding solution to the linear search problem in the imperative programming style, together with the formal correctness proof, takes Sethi [Set89] three pages.

Note that in contrast to the imperative programming case, the above solution can also be used to generate all elements of `a` with their corresponding positions, by means of the query `find(E, a, J)`. In this case both uses of "=" result in assigning a value to a variable, first to `E` and then to `J`.

The bounded existential quantifier $\exists$I $\in$ [1..n] implements backtracking and has no counterpart within the imperative programming paradigm. Here the backtracking is very "shallow" and boils down to the execution of the test E = A[I] for specific values of E and A[I].

Of course, it would be preferable to use the above solution to the linear search problem for arrays of any type, not only of the type **integer**. This motivates introduction of polymorphic types in presence of arrays. Then the appropriate generalization of the above solution to an arbitrary type is obtained by using the following generalized declarations, where "*" stands for a variable denoting an unknown type:

```
type seq: array [1..n] of *.
rel find: (*, seq, [1..n]).
rel =: (*, *).
```

Of course, in general, more than one unknown type can be used in a program.

The next example shows the power of the $\forall$X $\in$ L $\exists$Y $\in$ M combination in presence of arrays and a non-trivial instance of the backtracking process.

Problem 5 Arrange three 1's, three 2's, ..., three 9's in sequence so that for all $i \in [1,9]$ there are exactly i numbers between successive occurrences of i (see Coelho and Cotta [CC88, page 193]).

Solution

```
rel sequence: array [1..27] of [1..9].
sequence(A) ← ∀I ∈ [1..9] ∃J ∈ [1..25-2I]
    (A[J] = I, A[J+I+1] = I, A[J+2I+2] = I)).
```

The range J $\in$ [1..25-2I] comes from the requirement that the indices J, J+I+1, J+2I+2 should lie within [1..27]. Thus J+2I+2 $\leq$ 27, that is J $\leq$ 25-2I.

It is useful to note here that the corresponding solution to this problem in Prolog is 15 lines long.

Next, we show the usefulness of local definitions.

Problem 6 Generate all permutations of a given sequence of 100 elements.

First we provide a solution for the case when there are no repeated elements in the sequence.

Solution 1

```
const n = 100.
rel permutation: (array [1..n] of *, array [1..n] of *).
permutation(X, Y)  ←  ∀I ∈ [1..n] ∃J ∈ [1..n] Y[J] = X[I].
```

Here, X is the given sequence.

Note the similarity in the structure between this program and the one that solves problem 1. This program is incorrect when the sequence contains repeated elements. For example for n = 3 and X:= [0,0,1], the array Y:= [0,1,1] is a possible answer.

To deal with the general case we use local array declarations and refine the above program.

Solution 2

```
const n = 100.
rel permutation: (array [1..n] of *, array [1..n] of *).
permutation(X, Y)  ←
    var A: array [1..n] of [1..n].
    ∀I ∈ [1..n] ∃J ∈ [1..n] A[J] = I,
    ∀I ∈ [1..n] Y[I] = X[A[I]].
```

This solution states that A is an onto function from [1..n] to [1..n] and that a permution of a sequence of n elements is obtained by applying the function A to its indices.

Next, consider two well-known chess puzzles.

Problem 7 Place 8 queens on the chess board so that they do not check each other.

First, we provide a naive generate and test solution. It will be of use in the next section.

Solution 1

```
const n = 8.
type board: array [1..n] of [1..n].
rel queens, generate, safe: board.

queens(X)  ←  generate(X), safe(X).

generate(X)  ←  ∀I ∈ [1..n] ∃J ∈ [1..n] X[I] = J.

safe(X)  ←  ∀I ∈ [1..n] ∀J ∈ [I+1..n]
    (X[I] ≠ X[J], X[I] ≠ X[J] + (J-I), X[I] ≠ X[J] + (I-J)).
```

To improve readability board is explicitly declared here as a type. Declaratively, this program states the conditions which should be satisfied by the values chosen for the queens. "≠" is a built-in declared as

```
rel ≠: (*, *).
```

and defined by the single clause

```
X ≠ Y  ←  ¬ (X = Y).
```

In this section we use it only to compare ground terms. A more general usage of "≠" will be explained in the next section.

Next, we give a solution which involves backtracking.

Solution 2

```
const n = 8.
type board: array [1..n] of [1..n].
rel queens: board.

queens(X)  ←  ∀J ∈ [1..n] ∃K ∈ [1..n]
    (X[J] = K,
     ∀I ∈ [1..J-1]
        (X[I] ≠ X[J], X[I] ≠ X[J] + (J-I), X[I] ≠ X[J] + (I-J))).
```

Declaratively, this program states the conditions each possible value K for a queen placed in column J should satisfy. In its last line X[J] could be replaced by K.

Problem 8 *Knight's tour.* Find a cyclic route of a knight on the chess board so that each field is visited exactly once.

Solution We assign to each field a value between 1 and 64 and formalize the statement "from every field there is a "knight-reachable" field with the value one bigger". By symmetry we can assume that the value assigned to the field X[1, 1] is 1. Taking into account that the route is to be cyclic we actually get the following solution.

```
const n = 8.
type board: array [1..n, 1..n] of [1..n²].
rel knight: board.
   go_on: (board, [1..n], [1..n]).

knight(X)  ←  ∀I ∈ [1..n] ∀J ∈ [1..n] go_on(X, I, J), X[1, 1] = 1.

go_on(X, I, J)  ←  ∃I1 ∈ [1..n] ∃J1 ∈ [1..n]
    (abs((I-I1)·(J-J1)) = 2, X[I1, J1] = (X[I, J] mod n²) + 1).
```

```
DELAY go_on(X, I, J) UNTIL ground(X[I,J]).
```

Note that the equation abs(X · Y) = 2 used in the definition of go_on has exactly 8 solutions, which determine the possible directions for a knight move. Observe that each time this call to "=" is selected, both arguments of it are ground. The efficiency of go_on could be of course improved by explicitly enumerating the choices for the offsets of the new coordinates w.r.t. the old ones.

The behaviour of the above program is quite subtle. First, thanks to the delay declaration, 64 constraints of the form go_on(X, I, J) are generated. Then, thanks to the statement X[1, 1] = 1, the first of them is "triggered" which one by one activates the remaining constraints. The backtracking is carried out by choosing different values for the variables I1 and J1. The delay declaration is not needed, but without it this program would be hopelessly inefficient.

It is interesting to note that in Wirth [Wir76], a classical book on programming in Pascal, the solutions to the last two problems are given as prototypical examples of recursive programs. These solutions are based on the same principle, namely backtracking. Here recursion is implicit in the implementation of bounded quantifiers.

We conclude this section by one more program which shows the use of another type of quantifier.

Problem 9 Let m = 50 and n = 100. Determine the number of different elements in an array X:**array** [1..m, 1..n] **of integer**.

Solution

```
const m = 50.
      n = 100.
type board: array [1..m,1..n] of integer.
rel count: (board, natural).

count(X, Number) ←
    Number = m · n -
     #(I, J: I ∈ [1..m], J ∈ [1..n]:
         (∃K ∈ [1..I-1] ∃L ∈ [1..n] X[I,J] = X[K,L])
           % X[I,J] occurs in an earlier row
         ∨ (∃L ∈ [1..J-1] X[I,J] = X[I,L]).
           % X[I,J] occurs earlier in the same row
      ).
```

In this program we used the counting quantifier introduced in Gries [Gri81, page 74] and adopted in Kluźniak [Klu93] in the specification language SPILL-

2. In general, given lists L1, L2, the term #(I, J: I ∈ L1, J ∈ L2: Q) stands for the number of pairs (i,j) such that i ∈ L1, j ∈ L2 and the query Q{I/i,J/j} succeeds. It is possible to avoid the use of the counting quantifier at the expense of introducing a local array of type board. This alternative program is more laborious to write.

This concludes our presentation of selected logic programs written using arrays, bounded quantifiers and some other features. Other examples, involving among others numerical computation, can be found in Barklund and Millroth [BM94].

4 Arrays and Bounded Quantifiers in Constraint Logic Programming

In this section we illustrate the use of arrays and bounded quantifiers in constraint logic programs. We assume from the reader some familiarity with the basic principles of constraint logic programming (see e.g. the survey article of Jaffar and Maher [JM94]).

The programs presented here are constraint programs with finite domains in the style of van Hentenryck [vH89], where we refer the reader for a number of unexplained notions. Each of these programs has a similar pattern: first constraints are generated, and then resolved after the possible values for variables are successively generated. We explain here briefly how individual constraints are processed, but do not discuss the strategies for constraint solving and constraint propagation. This calls for a generalization of the constraint solvers proposed in the literature to a more general situation in which subscripted variables are used.

We begin by providing here alternative solutions to two problems discussed in the previous section.

Problem 10 Solve problem 7 by means of constraints.

Solution

```
const n = 8.
type board: array [1..n] of [1..n].
rel queens, safe, generate: board.

queens(X) ← safe(X), generate(X).

safe(X) ← ∀I ∈ [1..n] ∀J ∈ [I+1..n]
    (X[I] ≠ X[J], X[I] ≠ X[J] + (J-I), X[I] ≠ X[J] + (I-J)).
```

```
generate(X) ← ∀I ∈ [1..n] ∃J ∈ dom(X[I]) X[I] = J.
```

Here `dom(X)`, for a (possibly subscripted) variable `X`, is a built-in which denotes the list of current values in the domain of `X`, say in ascending order. The value of `dom(X)` can change only by decreasing. This can happen only by executing a constraint, so in the above program an atom of the form `X ≠ t`.

The relation "≠" was used in the previous section only in the case when both arguments of it were ground, so known. Here we generalizes its usage, as we now allow that one or both sides of it are not known. In fact, "≠" is a built-in defined as in van Hentenryck [vH89, pages 83-84], though generalized to arbitrary non-compound types.

We require that one of the following holds:

- Both sides of "≠" are known. This case is explained in the previous section.
- At most one of the sides of "≠" is known and one of the sides of "≠", denoted below by `X`, is either a simple variable or a subscripted variable with a known subscript.

In the second case `X ≠ t` is defined as follows, where for a term `s`, `Val(s)` stands for the set of its currently possible values:

```
if Val(X) ∩ Val(t) = ∅ then succeed
elseif Val(t) is a singleton then
% t is known, so X is not known, i.e. dom(X) has at least 2 elements
    begin dom(X):= dom(X) - Val(t);
    % remove the value of t from dom(X)
      if dom(X) = [f] then X:= f
    end .
```

If neither `Val(X) ∩ Val(t) = ∅` nor `Val(t)` is a singleton, then the execution of `X ≠ t` is delayed. We treat `t ≠ X` as `X ≠ t`.

So for example in the program fragment

```
...
type bool: [false, true].
rel p: (bool,bool,bool).
p(A,B,C) ← A ≠ B, B ≠ C, C = true, ...
```

during the call of `p(A,B,C)` the constraints `A ≠ B` and `B ≠ C` are first delayed and then upon the execution of the atom `C = true` the variable `B` becomes `false` and subsequently `A` becomes `true`.

In turn, in the case of the solution to the 8 queens problem given above, during the call of `safe(X)` the execution of an atom of the form `X[I] = K` for some `I,K` $\in$ `[1..n]` can affect the domains of the variables `X[J]` for `J` $\in$ `[I+1..n]` via the execution of a constraint of the form `X[I]` $\neq$ `X[J] + t`.

This solution to the 8 queens problem is a forward checking program (see van Hentenryck [vH89, pages 122-127]). Note the textual similarity between this program and the one given in solution 1 to problem 7. Essentially, the calls to the `safe` and `generate` relations are now reversed. The `generate` relation corresponds to the `labeling` procedure in van Hentenryck [vH89]. In the subsequent programs the definition of the `generate` relation has always the same format and is omitted.

Problem 11 Solve problem 6 by means of constraints.

Solution

```
const n = 100.
rel permutation: (array [1..n] of *, array [1..n] of *).
permutation(X, Y) ←
    type board: array [1..n] of [1..n].
    rel one_one, generate: board.
    one_one(Z) ← ∀I ∈ [1..n] ∀J ∈ [I+1..n] Z[I] ≠ Z[J].
    var A: board.
    one_one(A), generate(A),
    ∀I ∈ [1..n] Y[I] = X[A[I]].
```

In this solution, apart of the local declaration of the variable A, we also use local type and relation declarations. The efficiency w.r.t. to the logic programming solution is increased by stating, by means of the call to the `one_one` relation, that `A` is a 1-1 function. This replaces the previously used statement that `A` is an onto function. The call to `one_one` generates $n \cdot (n-1)/2 = 4950$ constraints.

We conclude this section by dealing with another classical problem — that of a map colouring. It shows the use of implication.

Problem 12 Given is a binary relation `neighbour` between countries. Colour a map in such a way that no two neighbours have the same color.

Solution

```
type color: [blue, green, red, yellow].
     countries: [austria, belgium, france, italy, ...].
rel map_color, constrain, generate: array countries of color.
    neighbour: (countries, countries).
```

```
map_color(X)  ←  constrain(X), generate(X).

constrain(X)  ←  ∀I ∈ countries ∀J ∈ countries
    neighbour(I,J)  →  X[I] ≠ X[J].
```

The declarative interpretion of P → Q is as follows:

```
(P → Q)  ←  P, Q.
(P → Q)  ←  ¬P.
```

So P → Q corresponds to the IF P THEN Q statement of Gödel. Obviously, an efficient implementation of P → Q should avoid the reevaluation of P. Note that in the above program, at the moment of selection of the P → Q statement, P is ground.

Thus the constrain relation generates here the constraints of the form X[I] ≠ X[J] for all I,J such that neighbour(I,J).

5 Adding Minimization and Maximization

Next, we introduce constructs allowing us to express in a compact way the requirement that we are looking for an optimal solution. To this end we introduce the *minimization operator* X = μQ which declaratively is defined as follows:

```
X = μQ  ←  Q, ¬(∃Y ( Y < X, Q{X/Y})).
```

We assume here that X and Y are of the type **integer**. The existential quantifier ∃X Q is defined by the clause

```
∃X Q  ←  Q.
```

The efficient implementation of the minimization operator should employ some specialized methods, like the branch and bound technique, in order to limit the search process during the successive attempts of finding a minimal solution to the query Q.

A dual operator, the *maximization operator* X = νQ, is defined declaratively by:

```
X = νQ  ←  Q, ¬(∃Y ( Y > X, Q{X/Y})).
```

To put these definitions into syntactically acceptable format, we could write min(X,Q) for X = μQ and max(X,Q) for X = νQ.

In Barklund and Hill [BH95] the minimization and the maximization operators are introduced as a form of arithmetic quantifiers, in the style of the counting quantifier introduced earlier.

We now show the use of the minimization operator.

Problem 13 *The cutting stock problem* (see van Hentenryck [vH89, pages 181-187]). There are 72 configurations, 6 kinds of shelves and 4 identical boards to be cut. Given are 3 arrays:

```
Shelves:array [1..72, 1..6] of natural,
Req:array [1..6] of natural,
Waste:array [1..72] of natural.
```

`Shelves[K,J]` denotes the number of shelves of kind `J` cut in configuration `K`, `Waste[I]` denotes the waste per board in configuration `I` and `Req[J]` the required number of shelves of kind `J`. The problem is to cut the required number of shelves of each kind in such a way that the total waste is minimized.

Solution We represent the chosen configurations by the array
    `Conf: array [1..4] of [1..72]`
where `Conf[I]` denotes the configuration used to cut the board `I`.

```
rel solve: (array [1..4] of [1..72], natural).
  generate: array [1..4] of [1..72].

solve(Conf, Sol) ←
    Sol = μ
        % Sol is the minimal TCost such that:
    ∀I ∈ [1..3] Conf[I] ≤ Conf[I+1],
        % symmetry between the boards
    ∀J ∈ [1..6] $\Sigma_{I=1}^{4}$ Shelves[Conf[I],J] ≥ Req[J],
        % enough shelves are cut
    Sol = $\Sigma_{I=1}^{4}$ Waste[Conf[I]],
        % Sol is the total waste
    generate(Conf).
```

The constraints `Conf[I] ≤ Conf[I+1]`, for `I ∈ [1..3]`, limit the number of generated solutions and (like in van Hentenryck [vH89]) are added here only for the efficiency purposes.

In this program we used as a shorthand the sum notation "Σ ...". In general, it is advisable to use the sum quantifier (see Gries [Gri81, page 72]), which allows us to use $\Sigma_{I=k}^{l}$ `t` as a term. The sum quantifier is adopted in SPILL-2 language of Kluźniak [Klu93]. Kluźniak's notation for this expression is:

`(S I: k` $\leq$ `I` $\leq$ `l: t)`. The interpretation of the constraints of the form `X` $\leq$ `t`, `X` $\geq$ `t` or `X = t` is similar to that of `X` $\neq$ `t` and is omitted.

6 Conclusions

We have presented here several logic and constraint logic programs that use bounded quantification and arrays. We hope that these examples convinced the readers about the usefulness of these constructs. We think that this approach to programming is especially attractive when dealing with various optimization problems, as their specifications often involve arrays, bounded quantification, summation, and minimization and maximization. Constraint programming solutions to these problems can be easily written using arrays, bounded quantifiers, the sum and cardinality quantifiers, and the minimization and maximization operators. As examples let us mention the stable marriage problem, the knapsack problem and various scheduling problems.

Of course, it is not obvious whether the solutions so obtained are efficient. We expect, however, that after an addition of a small number of built-in's, like `deleteff` and `deleteffc` of van Hentenryck [vH89, pages 89-90] and specialized versions of the bounded quantifiers that allow us to alter the search order through the range, it will be possible to write simple constraint programs which will be comparable in efficiency with those written in other languages for constraint logic programming.

When introducing arrays we were quite conservative and only allowed static arrays, i.e. arrays whose bounds are determined at compile time. Of course, in a more realistic language proposal also open arrays, i.e. arrays whose bounds are determined at run-time, should be allowed. One might also envisage the use of flexible arrays, i.e. arrays whose bounds can change at run-time.

In order to make this programming proposal more realistic one should provide a smooth integration of arrays with recursive types, like lists and trees. In the language SPILL-2 of Kluźniak [Klu93] types are present but only as sets of ground terms, and polymorphism is not allowed. Barklund and Hill [BH95] proposed to add arrays to Gödel (which does support polymorphism) as a system module. We would prefer to treat arrays on equal footing with other types.

We noticed already that within the logic programming paradigm the demarkation line between iteration and recursion differs from the one in the imperative programming paradigm. In order to better understand the proposed programming style one should first clarify when to use iteration instead of recursion. In this respect it is useful to quote the opening sentence of Barklund and

Millroth [BM94]: "Programs operating on inductively defined data structures, such as lists, are naturally defined by recursive programs, while programs operating on "indexable" data structures, such as arrays, are naturally defined by iterative programs".

We do not entirely agree with this remark. For example, the "suffix" quantifiers mentioned in Section 2 allow us to write many list processing programs without explicit use of recursion (see Voronkov [Vor92]) and the `quicksort` program written in the logic programming style is more natural when written using recursion than iteration.

The single assignment property of logic programming makes certain programs that involve arrays (like Warshall's algorithm) obviously less space efficient than their imperative programming counterparts. This naturally motivates research on efficient implementation techniques of arrays within the logic programming paradigm.

Finally, a comment about the presentation. We were quite informal when explaining the meaning of the proposed language constructs. Note that the usual definition of SLD-resolution has to be appropriately modified in presence of arrays and bounded quantification. For example, the query `X[1] = 0, I = 1, X[I] = 1` fails but this fact can be deduced only when the formation of resolvents is formally explained. To this end substitution for subscripted variables needs to be properly defined. One possibility is to adopt one of the definitions used in the context of verification of imperative programs (see Apt [Apt81, pages 460-462]). We leave the task of defining a formal semantics of the constructs proposed here to another paper.

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