The Wouthuysen Equation

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Dedication. I dedicate this paper to Prof. P.C. Baayen, at the occasion of his retirement on 20 December 1994. The beautiful equation which forms the subject matter of this paper was invented by Wouthuysen after he retired.

Abstract.
The four complex variable Wouthuysen equation arises from an original space-time lattice approach to spinor waves and elementary particles. Here the complete space of solutions is described. It consists of one isolated point and one branched S_4-covering-space over the circle with 8 branching points of order 6, 24 branching points of order 4 and 12 "turning points". The 24 branching points of order 4 are also turning points for two of the four branches.

1. The equations

The equations are for four complex variables of unit norm

$$\begin{aligned} &2 - (z_1^2 + z_2^2 + z_3^2 + z_4^2) - (z_1 + z_2 + z_3 + z_4) \\ &+ (z_1z_2 + z_1z_3 + z_1z_4 + z_2z_3 + z_2z_4 + z_3z_4) = 0 \end{aligned} \tag{1.1}$$

$$||z_1|| = ||z_2|| = ||z_3|| = ||z_4|| = 1 \tag{1.2}$$

with in addition a stationary phase condition

$$z_1z_2z_3z_4 = 1 \tag{1.3}$$

In terms of real parameters. There are 8 parameters and (1.1), (1.2) together give 6 conditions (2 from (1.1) and 1 each from $||z_i|| = 1,\ i = 1, \dots 4$). Given (1.2), (1.3) only gives one extra condition. So by equation counting one could expect 1-dimensional families of solutions. This does indeed turn out to be the case.

Note that the equations (1.1) - (1.3) are symmetric in z_1, z_2, z_3, z_4. So there is a natural action of the symmetric group on 4 letters, S_4, and the solutions fall into S_4- orbits.

2. A NUMBER OF SPECIAL SOLUTIONS

2.1 Solutions with at least one z_i equal to 1. These are

$$(1,1,1,1), \text{ a single solution invariant under } S_4 \tag{2.2}$$

$$(1,\zeta_3,\zeta_3,\zeta_3),\ (\zeta_3,1,\zeta_3,\zeta_3),\ (\zeta_3,\zeta_3,1,\zeta_3),\ (\zeta_3,\zeta_3,\zeta_3,1) \tag{2.3}$$

$$(1,\zeta_3^2,\zeta_3^2,\zeta_3^2),\ (\zeta_3^2,1,\zeta_3^2,\zeta_3^2),\ (\zeta_3^2,\zeta_3^2,1,\zeta_3^2)\ (\zeta_3^2,\zeta_3^2,\zeta_3^2,1) \tag{2.4}$$

Here $\zeta_3 = -\frac{1}{2}+\frac{1}{2}i\sqrt{3}$ is a primitive 3-rd root of unity. These form two S_4-orbits of size 4 each.

$$\begin{gathered}(1,1,\zeta_3,\zeta_3^2),\ (1,1,\zeta_3^2,\zeta_3),\ (1,\zeta_3,1,\zeta_3^2),\ (1,\zeta_3^2,1,\zeta_3)\\ (1,\zeta_3,\zeta_3^2,1),\ (1,\zeta_3^2,\zeta_3,1),\ (\zeta_3,1,1,\zeta_3^2),\ (\zeta_3^2,1,1,\zeta_3)\\ (\zeta_3,1,\zeta_3^2,1),\ (\zeta_3^2,1,\zeta_3,1),\ (\zeta_3,\zeta_3^2,1,1),\ (\zeta_3^2,\zeta_3,1,1)\end{gathered} \tag{2.5}$$

This set of solutions forms a single S_4 orbit of size 12. As it turns out (2.2) - (2.5) are the only solutions with at least one $z_i = 1$; see section 3 below for details.

2.6 Solutions with additional symmetry (besides those in 2.1)

$$z_i = \pm\frac{1}{3}\sqrt{3} \pm \frac{1}{3}j\sqrt{6},\ \ j = \sqrt{-1} \tag{2.7}$$

This solution satisfies (up to permutations), $z_2 = -z_1,\ z_4 = -z_3$ and is in fact the only solution with the property. It also satisfies (up to permutations), $z_2 = \bar{z}_1, z_4 = \bar{z}_3$.

$$\begin{aligned} z_{1,2} &= (-\frac{1}{8}+\frac{3}{8}\sqrt{5}) \pm j(\frac{1}{8}\sqrt{15}+\frac{1}{8}\sqrt{3}),\\ z_{3,4} &= (-\frac{1}{8}-\frac{3}{8}\sqrt{5}) \pm j(\frac{1}{8}\sqrt{15}-\frac{1}{8}\sqrt{3}) \end{aligned} \tag{2.8}$$

(up to permutation). This solution also has $\bar{z}_2 = z_1,\ \bar{z}_4 = z_3$ and there is in fact, besides (2.2), (up to permutation) one one-dimensional family of such solutions on which both (2.7) and (2.8) are located.

2.9 Solutions with $z_1+z_2+z_3+z_4 = 0$

Under this additional condition $(z_1+\ldots+z_4)^2 = 0$, so $z_1^2+\ldots+z_4^2 = -2(z_1z_2+\ldots+z_3z_4)$, so

$$z_1z_2 + \ldots + z_3z_4 = -\frac{2}{3} \tag{2.10}$$

Also, using $z_1z_2z_3z_4 = 1$, $z_1z_2z_3 + z_1z_3z_4 + z_1z_2z_4 + z_2z_3z_4 = z_4^{-1} + z_2^{-1} + z_3^{-1} + z_1^{-1} = \bar{z}_4 + \bar{z}_2 + \bar{z}_3 + \bar{z}_1 = 0$ because $||z_i|| = z_i\bar{z}_i = 1$. Hence $z_1z_2z_3 + z_1z_2z_4 + z_1z_3z_4 + z_2z_3z_4 = 0$. Thus the $z_1, \ldots z_4$ are solutions of the equation

$$z^4 - \frac{2}{3}z^2 + 1 = 0 \tag{2.11}$$

The solutions of this are

$$\frac{1}{3} \pm \frac{2}{3}\sqrt{2} \tag{2.12}$$

and so the z_1, z_2, z_3, z_4 are equal to

$$\pm\sqrt{\frac{1}{3} \pm \frac{2}{3}\sqrt{2}} = \pm\frac{1}{3}\sqrt{3} \pm \frac{1}{3}i\sqrt{6} \tag{2.13}$$

which is again the special solution (2.7).

2.14 Solutions with at least one z_i equal to -1.
There are (up to permutations) three solutions with at least one $z_i = -1$. These are

$$z_1 = z_2 = -1,\ z_{3,4} = (7 + \sqrt{33})^{-1}(3 + \sqrt{33} \pm 2j\sqrt{10 + 2\sqrt{33}}) \tag{2.15}$$

making up one S_4-orbit of size 12, and

$$-1,\ \zeta_6^5 = \frac{1}{2} - \frac{1}{2}j\sqrt{3},\ \frac{1}{2}\sqrt{3} + \frac{1}{2}j,\ -\frac{1}{2}\sqrt{3} - \frac{1}{2}j \tag{2.16}$$

$$-1,\ \zeta_6 = \frac{1}{2} + \frac{1}{2}j\sqrt{3},\ \frac{1}{2}\sqrt{3} - \frac{1}{2}j,\ -\frac{1}{2}\sqrt{3} + \frac{1}{2}j \tag{2.17}$$

and all permutations (making up two complex conjugate S_4 orbits of size 24 each).

3. SOLUTIONS WITH AT LEAST ONE z_i EQUAL TO 1.
Permuting the z_i if necessary, assume $z_1 = 1$. Then (1.1) reduces to

$$z_2^2 + z_3^2 + z_4^2 = z_2z_3 + z_2z_4 + z_3z_4 \tag{3.1}$$

This scales. So take $z = z_2$ and consider

$$1 + z_3^2 + z_4^2 = z_3 + z_4 + z_3z_4 \tag{3.2}$$

Let $w_3 = z_3 - 1,\ w_4 = z_4 - 1$. Then (3.2) turns into

$$w_3^2 + w_4^2 = w_3 w_4 \tag{3.3}$$

This scales again. So consider

$$1 + w_4^2 = w_4 \tag{3.4}$$

which has the solutions

$$w_4 = \frac{1}{2} \pm \frac{1}{2} j\sqrt{3} = \zeta_6, \zeta_6^5 \tag{3.5}$$

where $\zeta_6 = \frac{1}{2} + \frac{1}{2} j\sqrt{3}$ is a 6-th root of unity. Thus the solutions of (3.2) are of the form

$$z_3 = 1 + w,\ z_4 = 1 + w\zeta_6,\quad w \in \mathbb{C} \tag{3.6}$$

(including $w = 0$). And those of (3.3) are

$$w_3 = w,\quad w_4 = w\zeta_6 \tag{3.7}$$

From (3.7) it follows that w_4 and w_3 make an angle of 60° with one another, and that they are of equal length. For $z_3 = 1 + w_3,\ z_4 = 1 + w_4$ to be on the unit circle, w_3 and w_4 must be on the circle of radius 1 with centre at -1. Hence they must be conjugate and if readily (see Figure 1) follows that the

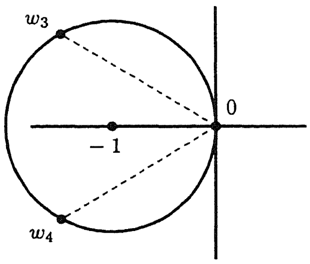


FIGURE 1.

only possibilities are

$$w_3, w_4 = -\frac{3}{2} \pm j\sqrt{3} \quad or \quad w_3, w_4 = 0$$

(e.g. because the triangle formed by $0, w_3, w_4$ must have all sides equal) and hence there are only the three possibilities

$$\zeta_3 = \zeta_3,\ z_4 = \zeta_3^2; \quad z_3 = \zeta_3^2,\ z_4 = \zeta_3; \quad z_3 = z_4 = 1$$

Thus the possible solutions of (3.1) are

$$(z_2, z_3, z_4) = (z, z, z),\ (z, \zeta_3 z, \zeta_3^2 z),\ (z, \zeta_3^2 z, \zeta_3 z)$$

and the solutions of (1.1) - (1.2) with at least one z_i equal to 1 are (up to permutations):

$$(1, z, z, z),\ z \in \mathbb{C};\ (1, z, \zeta_3 z, \zeta_3^2 z),\ z \in \mathbb{C};\ (1, z, \zeta_3^2 z, \zeta_3 z),\ z \in \mathbb{C}$$

The requirement (1.3), $z_1 z_2 z_3 z_4 = 1$, translates in all these cases to $z = 1, \zeta_3, \zeta_3^2$ and putting this in gives the 4 solution orbits (2.2) - (2.5) listed above.

4. Solutions with no z_i equal to 1.

To study the solutions of (1.1) - (1.3) for which no z_i is equal to 1, first use the transformation

$$w_i = z_i - 1,\ i = 1, 2, 3, 4 \tag{4.1}$$

(which has already proved to be useful above). This turns equation (1.1) into

$$w_1^2 + w_2^2 + w_3^2 + w_4^2 = w_1 w_2 + w_1 w_3 + w_1 w_4 + w_2 w_3 + w_2 w_4 + w_3 w_4 \tag{4.2}$$

The second tool is the Cayley transform $\phi : \mathbb{R} \to S^1 = \{z \in \mathbb{C} : ||z|| = 1\}$ given by (see Figure 2)

$$\phi(r) = \frac{r - j}{r + j},\ j = \sqrt{-1} \tag{4.3}$$

This mapping is 1-1 and onto $S^1 \backslash \{1\}$.
Let

$$z_i = \frac{r_i - j}{r_i + j}, \quad i = 1, \ldots 4 \tag{4.4}$$

Then

$$w_i = \frac{-2j}{r_i + j}, \quad i = 1, \ldots, 4 \tag{4.5}$$

Set

$$w_i = r_i + j, \quad i = 1, \ldots, 4 \tag{4.6}$$

Then the equation (4.2) becomes

$$v_1^{-2} + v_2^{-2} + v_3^{-2} + v_4^{-2} = v_1^{-1} v_2^{-1} + v_1^{-1} v_3^{-1} + \ldots + v_3^{-1} v_4^{-1} \tag{4.7}$$

Multiply this with $v_1^2 v_2^2 v_3^2 v_4^2$, to obtain

$$v_2^2 v_3^2 v_4^2 + v_1^2 v_3^2 v_4^2 + v_1^2 v_2^2 v_4^2 + v_1^2 v_2^2 v_3^2 = v_1 v_2 v_3 v_4\ (v_3 v_4 + \ldots + v_1 v_2) \tag{4.8}$$

Let e_1, e_2, e_3, e_4 be the elementary symmetric functions in the $v_1, \ldots, v_4$; i.e.

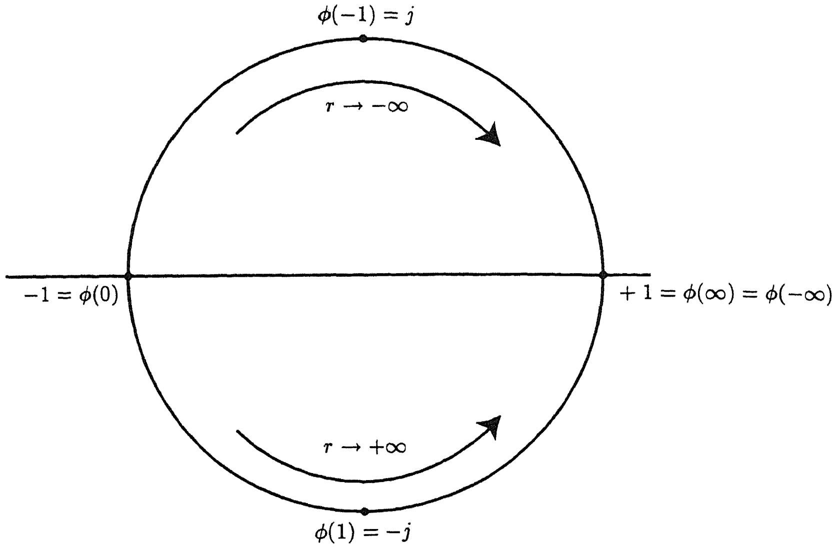


FIGURE 2.

$$e_1 = v_1 + v_2 + v_3 + v_4,\ e_2 = v_1v_2 + \ldots + v_3v_4,$$
$$e_3 = v_1v_2v_3 + \ldots + v_2v_3v_4,\ e_4 = v_1v_2v_3v_4 \tag{4.9}$$

Then (4.8) becomes

$$e_3^2 = 3e_2e_4 \tag{4.10}$$

Now let f_1, f_2, f_3, f_4 be the elementary symmetric functions in the r_1, r_2, r_3, r_4, i.e. $f_1 = r_1 + r_2 + r_3 + r_4$, etc. Then

$$e_1 = f_1 + 4j,\ e_2 = f_2 + 3jf_1 - 6 \tag{4.11}$$
$$e_3 = f_3 + 2jf_2 - 3f_1 - 4j,\ e_4 = f_4 + jf_3 - f_2 - jf_1 + 1.$$

Putting this into (4.10) gives the following equations for the f_1, f_2, f_3, f_4

$$f_3^2 + 3f_1f_3 - f_2^2 - 5f_2 + 2 - 3f_2f_4 + 18f_4 = 0$$
$$f_2f_3 + 10f_3 - 3f_1 - 9f_1f_4 = 0 \tag{4.12}$$

Now

$$z_1z_2z_3z_4 = \frac{(r_1 - j)(r_2 - j)(r_3 - j)(r_4 - j)}{(r_1 + j)(r_2 + j)(r_3 + j)(r_4 + j)} \tag{4.13}$$

Let

$$w = (r_1 - j)(r_2 - j)(r_3 - j)(r_4 - j) = f_4 - jf_3 - f_2 + jf_1 + 1 \tag{4.14}$$

By (4.13), equation (1.3) means $\bar{w} = w$ and by (4.14) this means

$$f_1 = f_3 \tag{4.15}$$

Putting this in (4.12) we see that there are two possibilities

$$f_1 = f_3 = 0 \tag{4.16A}$$

$$f_2 = 9f_4 - 7 \tag{4.16B}$$

In case A, the first equation of (4.12) becomes

$$f_2^2 + (5 + 3f_4)f_2 - (18f_4 + 2) = 0 \tag{4.17A}$$

and in case B, the first equation of (4.12) becomes

$$f_1^2 = 27f_4^2 - 30f_4 + 3 = 3(9f_4 - 1)(f_4 - 1) \tag{4.17B}$$

So, to find all solutions of (1.1) - (1.3) for which no z_i is equal to 1 it is necessary and sufficient to consider the equation

$$r^4 + f_1r^3 + f_2r^2 + f_3r + f_4 = 0 \tag{4.18}$$

under the conditions

Family A : $f_1 = f_3 = 0$ and $f_2^2 + (5 + 3f_4)f_2 - (18f_4 + 2) = 0$

Family B : $f_1 = f_3,\ f_2 = 9f_4 - 7,\ f_1^2 = 27f_4^2 - 30f_4 + 3 = 3(9f_4 - 1)(f_4 - 1)$

and to find out for which cases all four roots of (4.18) are real.

To conclude this section let's find out whether the families A and B can intersect. For an intersection we have $f_1 = 0 = f_3$ and, hence from (4.17B), $f_4 = 1/9, f_2 = -6$; $f_4 = 1, f_2 = 2$ Then and only then are all four of (4.16) - (4.17) satisfied.

If $f_4 = 1,\ f_2 = 2,\ f_1 = f_3 = 0$, The solutions of (4.18) are

$$j,\ j,\ -j,\ -j \tag{4.19}$$

i.e. two pairs of coinciding non real solutions. This gives no solution to the Wouthuysen equation, but will still be usefull later.

If $f_4 = 1/9,\ f_2 = -6,\ f_1 = f_3 = 0$, the solutions of (4.18) are

$$\pm(\frac{1}{3}\sqrt{15} + \frac{2}{3}\sqrt{3}),\ \pm(\frac{1}{3}\sqrt{15} - \frac{2}{3}\sqrt{3}) \tag{4.20}$$

which are all four real and which give the special solution (2.8):

$$\begin{aligned} z_{1,2} &= (-\frac{1}{8} + \frac{3}{8}\sqrt{15}) \pm j(\frac{1}{8}\sqrt{15} + \frac{1}{8}\sqrt{3}), \\ z_{3,4} &= (-\frac{1}{8} - \frac{3}{8}\sqrt{15}) \pm j(\frac{1}{8} - \frac{3}{8}\sqrt{15}) \end{aligned} \tag{4.21}$$

5. The family A

In this case the equation becomes

$$r^4 + f_2 r^2 + f_4 = 0.$$

with

$$f_2^2 + (5 + 3f_4)f_2 - (18f_4 + 2) = 0 \tag{5.2}$$

We shall use f_4 as the main parameter. This will turn out to be the right choice, even though (5.2) suggests that f_2 might be easier to work with.

For (5.1) to have four real roots, it is necessary and sufficient that $f_4 \geq 0,\ f_2 \leq 0,\ f_2^2 \geq 4f_4$ (besides f_2 real). The conditions $f_4 \geq 0$ and $f_2 \leq 0$ imply that only the solution

$$f_2 = -\frac{5}{2} - \frac{3}{2}f_4 - \frac{1}{2}\sqrt{9f_4^2 + 132f_4 + 33} \tag{5.3}$$

of (5.2) qualifies. If f_2 is given by (5.3) then

$$f_2^2 \geq \frac{1}{4}(9f_4^2 + 132f_4 + 33) > 4f_4$$

So, the family A consists of precisely one family of solutions parametrized by $f_4 \geq 0$. Because $f_2^2 > f_4$, the two solutions of

$$y^2 + f_2 y + f_4 = 0 \tag{5.4}$$

are unequal. So the only case in which the four solutions of (5.1) can have two or more equal is when $f_4 = 0$. Then

$$r_1 = r_2 = 0,\ r_{3,4} = \pm\frac{1}{2}\ \sqrt{10 + 2\sqrt{33}} \tag{5.5}$$

corresponding to the special solution (2.10) of the Wouthuysen equations.

6. The family B

In this case the equation becomes

$$r^4 + f_1 r^3 + f_2 r^2 + f_1 r + f_4 = 0 \tag{6.1}$$

subject to following conditions on the coefficients

$$f_2 = 9f_4 - 7, \; f_1^2 = 3(f_4 - 1)(9f_4 - 1) \tag{6.2}$$

and the question is when (6.1) will have all solutions real. This certainly requires f_1 to be real, which by (6.2) implies that $f_4 \leq 1/9$, or $f_4 \geq 1$. Thus there are four subfamilies to be considered

$$f_4 \geq 1, \quad f_1 = \sqrt{27f_4^2 - 30f_4 + 3} \tag{B1}$$

$$f_4 \geq 1, \quad f_1 = -\sqrt{27f_4^2 - 30f_4 + 3} \tag{B2}$$

$$f_4 \leq \frac{1}{9}, \quad f_1 = \sqrt{27f_4^2 - 30f_4 + 3} \tag{B3}$$

$$f_4 \leq \frac{1}{9}, \quad f_1 = -\sqrt{27f_4^2 - 30f_4 + 3} \tag{B4}$$

Under $(r_1, r_2, r_3, r_4) \mapsto (-r_1, -r_2, -r_3, -r_4), f_2$ and f_4 remain the same and f_1 and f_3 change sign. Hence (B1) (for a given value of f_4) gives four real solutions iff (B2) does so (for the same value of f_4). Similarly for (B3) and (B4). Thus it suffices to examine (B3) and (B1).

The discriminant of (6.1) is equal to

$$D = \prod_{i<k} (r_i - r_k)^2 \tag{6.3}$$

where r_1, r_2, r_3, r_4 are the four roots of (6.1). It turns out that under (6.2)

$$D = -2^{10}(2f_4^3 - 7f_2^2 + 8f_4 - 3) = -2^{10}(f_4 - 1)^2(2f_4 - 3). \tag{6.4}$$

This is a substantial calculation but it is less surprising than it maybe looks. First, D is of course a polynomial in the f_1, f_2, f_3, f_4 and it is homogeneous of degree 12 where f_i has weight i, $i = 1, \ldots, 4$. Under $r_i \mapsto -r_i$, $i = 1, \ldots, 4, D$ remains invariant. As f_1, f_3 change sign under $r_i \mapsto -r_i$ and f_2, f_4 remain invariant, f_1 and f_3 can only occur in the monomials in D in the forms $f_1^2, f_1 f_3, f_3^2$. However, the substitutions (6.2) are not homogeneous so that the degree could become as high as 12. The monomials in the discriminant of a fourth degree polynomial are of maximal degree 6 in f_1, f_3 combined. Thus a polynomial of degree 6 in f_4 could occur. A final drop in degree of 3 occurs because there are three coinciding roots at $f_4 = \infty$. Finally because there are coinciding roots of (6.1) at $f_4 = 1$ one of the roots of D must be 1.

For the subfamily (B3) (and (B4)) we have that at $f_4 = 1/9$ there are four different real solutions, see (4.20). Because $D \neq 0$ for $-\infty < f_4 < 1/9$, this must remain so for the whole family. Thus (B3) and (B4) represent two one dimensional families of solutions to the Wouthuysen equations parametrized by $f_4 \leq \frac{1}{9}$.

For $f_4 \geq 1$, i.e. the families (B1) and (B2), $D = 0$ at $f_4 = 3/2$. For this value of f_4 (6.1) becomes (for (B1))

$$r^4 + \frac{5}{2}\sqrt{3}\,r^3 + 6{\scriptstyle 1/2}\,r^2 + \frac{5}{2}\,\sqrt{3}\,r + \frac{3}{2} = 0 \tag{6.5}$$

with the solutions

$$-\sqrt{3},\ -\sqrt{3},\ -\frac{1}{4}\sqrt{3} + \frac{1}{4}j\sqrt{5},\ -\frac{1}{4}\sqrt{3} - \frac{1}{4}j\sqrt{5} \tag{6.6}$$

At $f_4 = 1$, equation (6.1) has four non real solutions, viz. $j, j, -j, -j$. So for $1 < f_4 < 3/2$, it remains the case that (6.1) has four non real solutions (because for this to change D must assume the value zero). As $D \neq 0$ for $3/2 < f_4 < \infty$, the family (B1) and (B2) have for these values of f_4 either four non real solutions or two real and two non real (complex conjugate) solutions. As it turns out the latter is the case. A numerical check shows e.g. that at $f_4 = 10$ the four solutions are approximately

$$-47.287,\ -0.606,\ -0.564 \pm 0.177j$$

In both cases (B1) and (B2) do not contribute to solutions of the Wouthuysen equation.

For later use we also need the solutions of the (B3) and (B4) families at $f_4 = 0$. The equation for the (B3) case then becomes

$$r^4 + \sqrt{3}r^3 - 7r^2 + \sqrt{3}r = 0 \tag{6.7}$$

with solutions

$$0,\ \sqrt{3},\ 2 - \sqrt{3},\ -2 - \sqrt{3} \tag{6.8}$$

7. Matching the solutions with a z_i equal to 1 to the A, B3, B4 families

Under $\phi : \mathbb{R} \to S^1$, $+\infty$ goes to 1, and so does $-\infty$. (So the true parameters space is the circle $\phi(\mathbb{R}) = \phi(\{f_4\})$). To see how the solutions with a z_i equal to 1 fit with the A, B3 and B4 families, it therefore suffices to study what happens to the corresponding solutions as $f_4 \to \infty$ (for the A-family) and as $f_4 \to -\infty$ (for the B3 and B4 families).

7.1 The A-family for $f_4 \to \infty$.
First consider an A-family of solutions

$$r^4 + f_2 r^2 + f_4 = 0 \quad f_2 = -\frac{5}{2} - \frac{3}{2} f_4 - \frac{1}{2}\sqrt{9f_4^2 + 132 f_4 + 33} \tag{7.2}$$

As $f_4 \to \infty$, $f_4^{-1} f_2$ goes to -3. Let $s = r^{-1}$. Then the equation for s is

$$s^4 + f_4^{-1} f_2 s^2 + f_4^{-1} = 0 \tag{7.3}$$

which in the limit $f_4 \to \infty$, goes to

$$s^4 - 3s^2 = 0 \tag{7.4}$$

It follows that as $f_4 \to \infty$, two solutions of (7.2) go each to $-\infty$ or $+\infty$ and the other two go to $-\frac{1}{3}\sqrt{3}$, $\frac{1}{3}\sqrt{3}$. However, the four solutions of (7.2) cannot cross as $f_4 \to \infty (f_4 > 0)$, therefore the only possibility is that one goes to $-\infty$ and the other to $+\infty$.

So up to permutations the limit solutions are

$$(-\infty, \ -\frac{1}{3}, \ \sqrt{3}, \ \frac{1}{3}\sqrt{3}, +\infty) \tag{7.5}$$

which under $\phi : \mathbb{R} \to S^1$ corresponds to the solutions (2.5)

$$(1, \zeta, \ \zeta^2, \ 1) \tag{7.6}$$

where $\zeta = \zeta_3$.

And indeed a small numerical check shows that for $f_4 = 10^3$, 10^5, respectively, the solutions of (7.2) are, respectively, approximately equal to

$$-54.890, \quad -0.576, \quad 0.576, \quad 54.890$$

$$-547.735, \quad -0.577, \quad 0.577, \quad 547.735$$

while $\frac{1}{3}\sqrt{3}$ is about 0.577.

7.7 The B3-family for $f_4 \to \infty$.
Now let's consider a B3-family of solutions

$$\begin{aligned} &r^4 + f_1 r^3 + f_2 r^2 + f_1 r + f_4 = 0, \\ &f_2 = 9f_4 - 7, \ f_1 = \sqrt{27 f_4^2 - 30 f_4 + 3} \end{aligned} \tag{7.8}$$

as $f_4 \to \infty (f_4 \leq 1/9)$. As $f_4 \to \infty$, because $f_1 \propto 3\sqrt{3}|f_4|$,

$$f_4^{-2}f_2 \to 9,\ f_4^{-1}f_1 \to -3\sqrt{3} \tag{7.9}$$

Let $s = r^{-1}$. Then the equation for s is

$$s^4 + f_4^{-1}f_1 s^3 + f_4^{-1}f_2 s^2 + f_4^{-1}f_1 s + f_4^{-1} = 0 \tag{7.10}$$

which in the limit, $f_4 \to \infty$, goes to

$$s^4 - 3\sqrt{3}s^3 + 9s^2 - 3\sqrt{3}s = s(s-\sqrt{3})^3 = 0 \tag{7.11}$$

with solutions

$$0,\ \sqrt{3},\ \sqrt{3},\ \sqrt{3} \tag{7.12}$$

It follows that as $f_4 \to \infty$ one of the solutions of (7.8) goes to ∞ or $-\infty$ and the others to $\frac{1}{3}\sqrt{3}$.

Now at $f_4 = 0$ the solutions of (7.8) are

$$-2-\sqrt{3},\ 0,\ 2-\sqrt{3},\ \sqrt{3} \tag{7.13}$$

The roots cannot cross as $f_4 \to -\infty$, and the smallest one, $-(2+\sqrt{3})$, cannot cross 0 again (because there are no zero solutions of (7.8) for $f_4 < 0$). It follows that (7.13) must go to

$$(-\infty,\ \frac{1}{3}\sqrt{3},\ \frac{1}{3}\sqrt{3},\ \frac{1}{3}\sqrt{3}) \tag{7.14}$$

which corresponds to the solution

$$(1, \zeta^2, \zeta^2, \zeta^2) \tag{7.15}$$

of the Wouthuysen equation.

A numerical check gives that for $f_4 = -10^4$, the four solutions are approximately equal to

$$-51966.143, \quad 0.572, \quad 0.577, \quad 0.583$$

while $\frac{1}{3}\sqrt{3}$ is about 0.577.

8. The topological structure of the space of solutions

Apart from the identifications at $r = \infty, -\infty$, i.e. at $z = 1$, the picture of the solution space is made up of 12 pieces as depicted in Figure 3.

Here all special (intersection) points have been made fat dots and given their r-coordinates. For the points with an $r = \pm\infty$ the corresponding z-coordinates have also been given. A crossing point of families that has not been made fat is not an existing crossing point but an artifact of the drawing. In particular there is no intersection of a B4-family with a B3-family for $-\infty \le f_4 < 1/9$.

There are twelve such pieces. The other eleven are obtained by applying appropriate elements of S_4 to the picture shown above. To get the complete

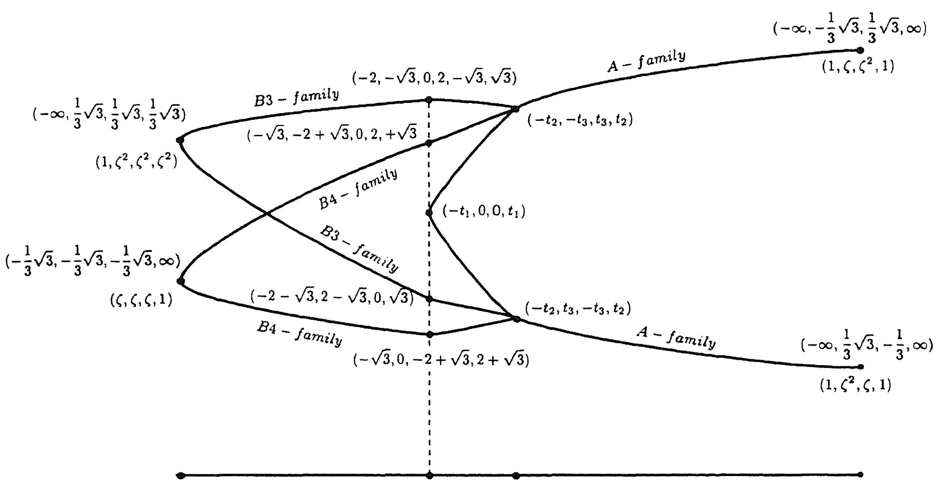


FIGURE 3.

global picture it suffices to identify the points above $f_4 = \pm\infty$ according to the coordinates attached to them. The complete topological picture is given in Figure 4.

As above let $\zeta = \zeta_3$

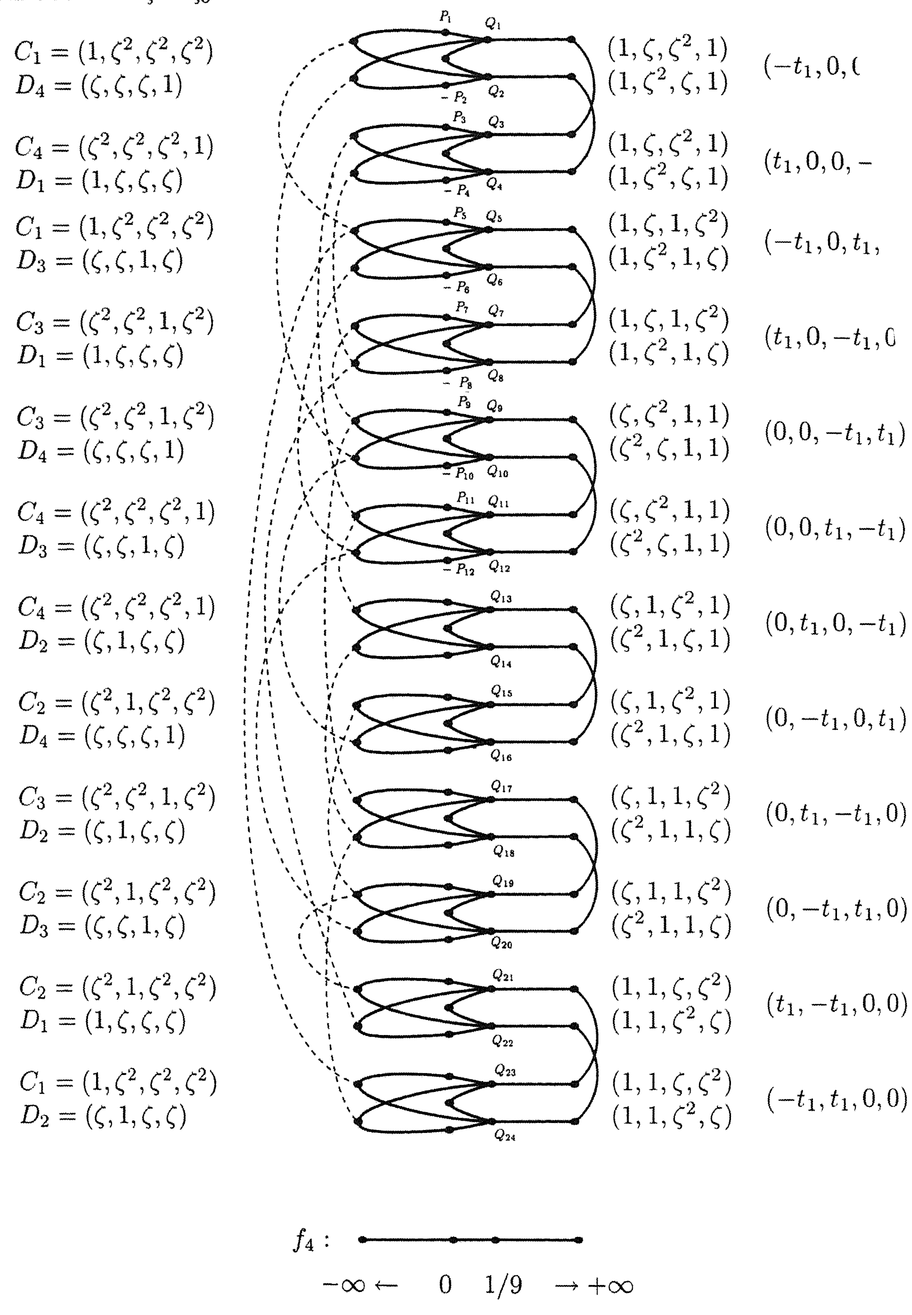


FIGURE 4.

The solution $(1,1,1,1)$ is completely isolated, and all others are connected by the scheme drawn above, where the dotted lines indicate identifications.
In the above (see also Figure 5)

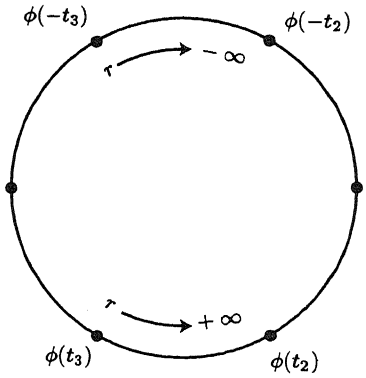


FIGURE 5.

$$Q_1 = \phi(-t_2,-t_3,t_3,t_2) = (\ -\tfrac{1}{8}+\tfrac{3}{8}\sqrt{5}+j(\tfrac{1}{8}\sqrt{3},\ -\tfrac{1}{8}-\tfrac{3}{8}\sqrt{5}+j(\tfrac{1}{8}\sqrt{15}-\tfrac{1}{8}\sqrt{3}),$$
$$-\tfrac{1}{8}-\tfrac{3}{8}\sqrt{5}-j(\tfrac{1}{8}\sqrt{15}-\tfrac{1}{8}\sqrt{3}),\ -\tfrac{1}{8}+\tfrac{3}{8}\sqrt{5}-j(\tfrac{1}{8}\sqrt{3})$$

where $t_2 = \frac{1}{3}\sqrt{15}+\frac{2}{3}\sqrt{3},\ t_3 = \frac{1}{3}\sqrt{15}-\frac{2}{3}\sqrt{3}$
And further, using the notation $\sigma(s_1,s_2,s_3,s_4) = \big(s_{\sigma(1)},s_{\sigma(2)},s_{\sigma(3)},s_{\sigma(4)}\big)$,

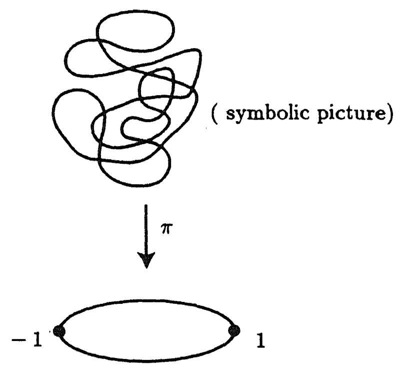
$$\begin{array}{lcrcl}
Q_2 &=& (23)Q_1 &=& \phi(-t_2,t_3,-t_3,t_2)\\
Q_3 &=& (14)Q_1 &=& \phi(t_2,-t_3,+t_3,-t_2)\\
Q_4 &=& (23)(14)Q_1 &=& \phi(t_2,+t_3,-t_3,-t_2)\\
Q_5 &=& (34)Q_1 &=& \phi(-t_2,-t_3,t_2,t_3)\\
Q_6 &=& (234)Q_1 &=& \phi(-t_2,t_3,t_2,-t_3)\\
Q_7 &=& (143)Q_1 &=& \phi(t_2,-t_3,-t_2,t_3)\\
Q_8 &=& (1423)Q_1 &=& \phi(t_2,+t_3,-t_2,-t_3)\\
Q_9 &=& (123)Q_1 &=& \phi(-t_3,t_3,-t_2,t_2)\\
Q_{10} &=& (13)Q_1 &=& \phi(t_3,-t_3,-t_2,t_2)\\
Q_{11} &=& (1234)Q_1 &=& \phi(-t_3,t_3,t_2,-t_2)\\
Q_{12} &=& (134)Q_1 &=& \phi(t_3,-t_3,t_2,-t_2)\\
Q_{13} &=& (124)Q_1 &=& \phi(-t_3,t_2,t_3,-t_2)\\
Q_{14} &=& (1324Q_1 &=& \phi(t_3,t_2,-t_3,-t_2)
\end{array}$$

$$
\begin{array}{rcrcl}
Q_{15} & = & (12)Q_1 & = & \phi(-t_3,-t_2,t_3,t_2) \\
Q_{16} & = & (132)Q_1 & = & \phi(t_3,-t_2,-t_3,t_2) \\
Q_{17} & = & (1243)Q_1 & = & \phi(-t_3,t_2,-t_2,t_3) \\
Q_{18} & = & (13)(24)Q_1 & = & \phi(t_3,t_2,-t_2,-t_3) \\
Q_{19} & = & (12)(34)Q_1 & = & \phi(-t_3,-t_2,t_2,t_3) \\
Q_{20} & = & (1342)Q_1 & = & \phi(t_3,-t_2,t_2,-t_3) \\
Q_{21} & = & (1432)Q_1 & = & \phi(t_2,-t_2,-t_3,t_3) \\
Q_{22} & = & (142)Q_1 & = & \phi(t_2,-t_2,t_3,-t_3) \\
Q_{23} & = & (243)Q_1 & = & \phi(-t_2,t_2,-t_3,t_3) \\
Q_{24} & = & (24)Q_2 & = & \phi(-t_2,t_2,t_3,-t_3)
\end{array}
$$

In words, the space of solutions of the Wouthuysen equation consists of one isolated point, $(1,1,1,1)$, and a branched curve. This branched curve, and the isolated point, come with a natural projection to the circle. The group S_4 acts on the space of solutions, leaving the isolated point invariant. The projection to the circle is invariant under this action. Let S denote the solution space and $\pi : S \to S^1 = \mathbb{R} \cup \{\infty\}$ this invariant projection.

In terms of the r-coordinates, $\mathbb{R} \cup \{\infty\}$, $-\infty = +\infty$, the picture is as follows

(i) Above all $1/9 < y < \infty$, there are 24 points which form one S_4-orbit. The inverse under π of a small enough neighborhood of such a points consists of 24 disjoint intervals.



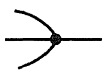
(ii) Above $y = \infty$ (corresponding to 1 under the Cayley transform) there are 21 points: the isolated solution point $(1,1,1,1)$, which is an invariant point of the S^4-action, an S^4-orbit of size 12, and two complex conjugate S_4-orbits of size 4. These are branching points of order 6. Locally around one of the points of the orbit of size 12, the branched solution curve looks like an interval turning back. Locally around a point of the two S_4-orbits of size 4 the picture is a six branched star as depicted below. Thus the inverse image of a small interval around the point ∞ of the circle $\mathbb{R} \cup \{\infty\}$ looks like the disjoint union of 12 intervals, 8 six branched stars and one isolated point.

(8)

(12)

(1)

(iii) Above $y = \frac{1}{9}$ there are 24 points which form a single S_4-orbit. They are all branching points of order 4. The inverse of a small interval around $y = 1/9$ looks like the disjoint union of 24 4-branched stars like depicted below.



(24)

(iv) Above all $0 < y < 1/9$ there are 72 points, which from three S_4-orbits; two of these are complex conjugate, the third is invariant under complex conjugation. The inverse image of a small enough interval around these y looks like 72 disjoint copies of that interval

(72)

(v) Above $y = 0$ there are 60 points. They form one orbit of size 12 and two complex conjugate orbits of size 24. The points from the orbit of size 12 are "turning points" in a sense which should be clear from the picture below. The inverse image of a small interval around such a point consists of 60 disjoint intervals of which 12 are "turn back" intervals.

(48)

(12)

(vi) Above $-\infty < y < 0$ there are 48 points which form two complex conjugate orbits of size 24 each. The inverse image of a small interval around these y looks like the disjoint union of 48 copies of that interval

(48)

So, in fact, the group $\mathbb{Z}/(2) \times S_4$ acts on the space of solutions and $\pi : S \to S^1$ is invariant under this action (of a group of order 48).

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