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A PARTIAL SURVEY OF THE USES OF ALGEBRAIC GEOMETRY IN SYSTEMS AND CONTROL THEORY (*)

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Preface and apology.

This is an expanded version of the talk with this title which I gave at the occasion of the F. Severi centennial conference at INDAM in Rome, April 1979.

By its very nature algebraic geometry ought to be applicable virtually everywhere, but the applied side of the subject has not been much in evidence in the last decennia it seems, until a few years ago when two new areas of applicability arose: one of these is of course more or less described by the key words: Korteweg-de Vries equations, solitons, finite gap operators, Yang-Mills fields, instantons, and a selection of references is [AHS, AHDM, DM 1, DM 2, DMN, BLS, GD, Kri, MT, Ve]; the other one concerns the uses of algebraic-geometric ideas (especially) and results (to a lesser extent) in control and system theory, which is my subject today.

The word algebraic geometry in the title must be understood in a fairly wide sense. For one thing some of the applications below rest on the underlying ring theory or commutative algebra rather then on algebraic geometry itself; for another many of the results have their topological analogues and use differential topology rather than algebraic geometry. It is true though that for most of the results below the original inspiration came from algebraic geometry, even if the final, and for the moment most important version (over the reals) bears few or no traces of that fact.

The word partial in the title also reflects that I shall deal only

(*) I risultati conseguiti in questo lavoro sono stati esposti nella conferenza tenuta il 13 aprile 1979.

with (families of) linear systems, and that I shall not touch upon various algebraic, geometric and topological ideas which already play, or are very likely to play an important role in especially nonlinear system theory like Lie algebras of vector-fields, connections, foliations and (analytic) stratifications. A selection of references dealing also with such aspects of system and control theory is [Bro 1, Bro 2, Bro 3, Bru 1, Her 4, Her 5, Ell, HH, Hir, HKr, HM 6, Kre, Lo, LoW, JS, MB, MMO, MW, SJ, So 4, Su 1, Su 2, Wi].

Finally let me mention the recent survey paper [BF], the paper [Haz 3], the recent collection [MH], and the reasonably soon to be expected proceedings of the NATO-AMS Advanced Study Inst. and Summer Sem. on algebraic and geometric methods in linear system theory (Harvard Univ., June 1979), as good sources for similar material, discussed in a variety of ways and styles, for those whose appetite was awakened by the present paper, and for those who could not get through it, but still feel they cannot afford to neglect the subject entirely.

1. Introduction.

The basic object under consideration in this lecture is a linear dynamical system Σ. This is a set of linear differential or difference equations

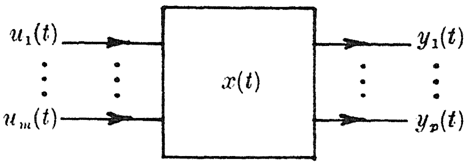
$$(1.1)\qquad \begin{cases} \dot{x}(t) = Fx(t) + Gu(t)\,, & x(t+1) = Fx(t) + Gu(t)\,, \\ y(t) = Hx(t)\,, & y(t) \quad = Hx(t)\,, \\ \text{(continuous time)} & \text{(discrete time)} \end{cases}$$

where the F, G and H are time independent matrices with coefficients in some appropriate field k, and where $x(t) \in k^n =$ state space, $u(t) \in \in k^m =$ input space or control space, and $y(t) \in k^p =$ output space. We speak of a system of dimension n with m inputs and p outputs.

Occasionally one adds a direct feedthrough term to $y(t)$, so that then $y(t) = Hx(t) + Ju(t)$ in (1.1) instead of $y(t) = Hx(t)$. For the mathematical problems discussed below the presence or absence of the term $Ju(t)$ makes little difference. Thus a system (whether discrete or continuous time) is specified by giving three matrices F, G, H, and possibly a fourth one J, of dimensions $n \times n$, $n \times m$, $p \times n$ and $p \times m$.

One common interpretation of the set of equations (1.1) is in terms of some device which accepts input functions $u(t) = (u_1(t), \ldots, u_m(t))$ and produces output functions $y(t) = (y_1(t), \ldots, y_p(t))$.

(1.2)



Assuming that we start the device at time zero in state $x(0) = 0$ the corresponding input/output map f_Σ of Σ is

$$f_\Sigma\colon\ u(t) \mapsto y(t) = \int_0^t H \exp\big(F(t-\tau)\big) G u(\tau)\,d\tau \quad \text{(continuous time)} \tag{1.3}$$

$$f_\Sigma\colon\ u(t) \mapsto y(t) = \sum_{i=1}^{t} A_i u(t-i)\,, \qquad A_i = HF^{i-1}G\,,\ i = 1, 2, \ldots \quad \text{(discrete time)} \tag{1.4}$$

In both cases f_Σ is completely determined by the matrices A_i, sometimes called the Markov parameters of the system.

Taking the Laplace transform in the continuous time case, and the z-transform in the discrete time case, one finds the input/output relations

$$\hat{y}(s) = T(s)\hat{u}(s)\,, \qquad T(s) = H(sI - F)^{-1}G \tag{1.5}$$

where $T(s)$ is called the *transfer function* (*matrix*).

Two systems $\Sigma = (F, G, H)$, $\Sigma' = (F', G', H')$ over k are said to be isomorphic if there is an invertible matrix $S \in GL_n(k)$ such that $\Sigma' = \Sigma^S = (SFS^{-1}, SG, HS^{-1})$. This notion of isomorphism corresponds to a base change $x' = Sx$ in state space. It also fits in well with the input/output point of view in that the input/output maps of Σ and Σ^S are the same for all $S \in GL_n(k)$. The converse is not always true but holds generically, cf. section 3 below.

In principle thus, a linear dynamical system seems a very simple object indeed (if taken one at a time), of which it is hard to believe that any sophisticated mathematics will be needed to deal with it. To a large extent this appears to be true. The fun starts when instead of considering single systems (1.1) one considers families of them; that is one considers e.g. real continuous time systems where now the matrices F, G, and H are allowed to depend continuously or polynomially on some extra parameters $\sigma = (\sigma_1, \ldots, \sigma_r)$.

It is when studying families of systems, and when trying to extend to families various useful known single system constructions and results, that we shall employ fairly sophisticated algebraic geometric ideas and results like fine moduli spaces, vector bundles, the Quillen-Suslin theorem, the quadratic Serre problem, Stein spaces, intersection numbers and 1-st Chern numbers.

One way to look at this study of families is to regard it as a systematic investigation to see which of the standard constructions in control and system theory are continuous in the system parameters. Viewed in this way the study of families (rather than single systems) is obviously relevant in an uncertain world full of (small) measurement errors.

As it happens there are—in this author's opinion—many more compelling reasons for studying families rather than single systems. Section 2 below is devoted to this. Section 4 discusses moduli (and some of their uses) and section 5-11 treat of various standard system theoretic notions like feedback, realizations, model matching, pole assignment, completely reachable subsystems, ... In each case I shall try to describe briefly the system/control theoretic idea, the single system solution or construction (in so far this has not already been done in the basic system theory section 3) and then discuss the family-wise versions of these (if available).

Thus our central object is a family of linear dynamical systems Σ, that is a system valued function, which we shall regard from different viewpoints proceeding along a contour around it. By the time we are finished, adapting a method of Henri Pétard [Pe] in big game hunting, we shall presumably know all about the residue in the middle.

2. Assorted reasons for studying families rather than single systems.

2.1. *Families of systems* (definition). Intuitively a family of systems is a set of equations (1.1) where the matrices F, G, H depend in some way on a set of parameters σ. For various reasons this definition is not quite general enough, notably if one wants to discuss and use universal families of systems (and this is not the only reason for considering somewhat more general families). A better definition (in the topological case) is:

A *family of* real or complex *systems* Σ *over a topological space* V consists of an n-dimensional real or complex vector bundle E over V, a vector bundle endomorphism $F\colon E\to E$ and two vector bundle homomorphisms $G\colon V\times k^m\to E$, $H\colon E\to V\times k^p$ where $k=\boldsymbol{R}$ or $\boldsymbol{C}$. Taking n independent sections of E in a small neighbourhood V' of

$v \in V$ and writing out the matrices of F, G, H with respect to the obvious bases in $\{v'\}\times k^m$, $\{v'\}\times k^p$ and the basis of $E(v')$ defined by the n sections for all $v' \in V'$, we see that locally Σ is given by a continuous map into $L_{m,n,p}(k)$, the space of all triples of matrices over k of sizes $n\times n$, $n\times m$ and $p\times n$. So locally Σ is just like the intuitive notion of a family, but globally it need not be. The family Σ is differentiable (resp. analytic) if all the ingredients which go into its definition, i.e. V, E, F, G, H are differentiable (resp. analytic).

Similarly an algebraic geometric family of systems Σ over a scheme V consists of an algebraic vector bundle $E \to V$ and morphisms of algebraic vector bundles

$$F\colon E \to E\,, \quad G\colon V\times A^m \to E\,, \quad H\colon E \to V\times A^p\,,$$

where A^i is affine i-space. Locally this corresponds to a morphism of schemes $V \to L_{m,n,p}$ where $L_{m,n,p} \simeq A^{n^2+nm+pn}$ in the obvious way. For every point of V with residue field $k(v)$ there is an associated system over $k(v)$, viz. $F(v)\colon E\otimes k(v) \to E\otimes k(v)$, $G(v)\colon k(v)^m \to E\otimes \otimes k(v)$, $H(v)\colon E\otimes k(v) \to k(v)^p$.

Two families $\Sigma = (E; F, G, H)$ and $\Sigma' = (E'; F', G', H')$ are said to be isomorphic if there is an isomorphism of vector bundles $\varphi\colon E \to E'$ such that $\varphi F = F'\varphi$, $\varphi G = G'$, $H'\varphi = H$.

2.2. *Systems over rings*. The difference discrete-time equations (1.1) also make perfect sense if the matrices F, G, H are assumed to have their coefficients in a commutative ring R and $x(t) \in R^n$, $y(t) \in R^p$, $u(t) \in R^m$. In fact the linear machine

$$(2.2.1) \qquad x(t+1) = Fx(t) + Gu(t)\,, \qquad y(t) = Hx(t)$$

still makes perfect sense in the more general setting that we have three R-modules: U = input module, X = state module, Y = output module, and three R-module homomorphisms $G\colon U \to X$, $F\colon X \to X$, $H\colon X \to Y$.

Note that the input/output operator of the linear machine, cf. (1.4), is a convolution operator so that the theory of linear discrete time systems also has things to say about e.g. convolutional codes. There are more reasons for studying systems over rings, some of which will be touched on below; cf. also [So 1], [Kam 2].

Assuming that the input module U and the output module Y are free and that the state module X is projective there is an obvious way of associating a family of systems over Spec (R) in the sense of 2.1 above to the data U, X, Y, F, G, H. Indeed let E be the vector bundle associated to the projective module X and let $\tilde{F}, \tilde{G}, \tilde{H}$ be the bundle morphisms defined by F, G, H. Then $(E; \tilde{F}, \tilde{G}, \tilde{H})$ is an algebraic geometric family in the sense of 2.1 above.

For each prime ideal $\mathfrak{p}$ of R let $k(\mathfrak{p})$ be the quotient field of $R/\mathfrak{p}$. Then the system over the point $\mathfrak{p}$ defined by this family is simply given by the triple of matrices $F(\mathfrak{p}) = F \otimes k(\mathfrak{p})$, $G(\mathfrak{p}) = G \otimes k(\mathfrak{p})$, $H(\mathfrak{p}) = H \otimes k(\mathfrak{p})$.

2.3. *Delay-differential systems.* Consider a real delay-differential system, e.g.

$$\text{(2.3.1)} \qquad \begin{cases} \dot{x}_1(t) = x_1(t-a_1) + 2x_2(t) + x_2(t-a_2) + u(t)\,, \\ \dot{x}_2(t) = x_1(t) + 2x_2(t-a_1) + u(t-a_2)\,, \\ y(t) \;= 2x_1(t-a_2) + x_2(t)\,, \end{cases}$$

where a_1 and a_2 are two incommensurable positive real numbers. Introducing the delay operators $\sigma_1\alpha(t) = \alpha(t-a_1)$, $\sigma_2\alpha(t) = \alpha(t-a_2)$ we can rewrite (2.3.1) formally as

$$\text{(2.3.2)} \qquad \dot{x}(t) = Fx(t) + Gu(t)\,, \qquad y(t) = Hx(t)$$

with the matrices F, G and H given by

$$\text{(2.3.3)} \qquad F = \begin{pmatrix} \sigma_1 & 2+\sigma_2 \\ 1 & 2\sigma_1 \end{pmatrix}, \qquad G = \begin{pmatrix} 1 \\ \sigma_2 \end{pmatrix}, \qquad H = (2\sigma_2 \quad 1)$$

and in turn this triple of matrices can be viewed as a triple of matrices with coefficients in the ring $\boldsymbol{R}[\sigma_1, \sigma_2]$ that is a system over the ring $\boldsymbol{R}[\sigma_1, \sigma_2]$, or, equivalently, as a family of systems parametrized by the parameters $\sigma = (\sigma_1, \sigma_2)$. Thus the infinite dimensional system (2.3.1) gets turned into a family of finite dimensional systems. That this is not a completely formal exercise is shown by a nice paper of Kamen [Kam 1] in which he relates the spectral theory of (2.3.1) to the (commutative) algebra which goes into the study of (2.3.2).

One thing which is suggested by this point of view is that two delay-systems Σ, Σ' like (2.3.1) be considered isomorphic if there is an invertible matrix $S \in \boldsymbol{Gl}_n(\boldsymbol{R}[\sigma_1, \sigma_2])$ which takes Σ into Σ'; i.e. they are isomorphic if one can be obtained from the other by means of an invertible transformation $x' = Sx$ where S may involve delays. This turns out to be precisely the right notion of isomorphism in connection with degeneracy phenomena for delay-differential equations, cf. [Kap]. Similarly the system-over-rings-as-family-of-systems point of view also seems to suggest useful notions of e.g. complete reachability, cf. below in section 10.

2.4. (*Singular*) *perturbation, deformation, approximation.* These reasons for studying families depending on a small parameters rather than only the objects themselves are almost as old as mathematics itself. Certainly (singular) perturbations are a familiar topic in the theory of boundary values of differential equations. And in the control world O'Malley, [OMa], for instance discusses a singularly perturbed regulator problem which consists of the following data

$$(2.4.1)\qquad \begin{cases} \dot{x}_1 = A_{11}(\varepsilon)x_1 + A_{12}(\varepsilon)x_2 + B_1(\varepsilon)u\,, & x_1(0,\varepsilon) = x_1^0(\varepsilon)\,,\\ \varepsilon\dot{x}_2 = A_{21}(\varepsilon)x_2 + A_{22}(\varepsilon)x_2 + B_2(\varepsilon)u\,, & x_2(0,\varepsilon) = x_2^0(\varepsilon)\,,\\ J(\varepsilon) = {}^t x_1(1,\varepsilon)\pi(\varepsilon)x_1(1,\varepsilon) + \\ \qquad + \int_0^1 \big({}^t x_1(\tau,\varepsilon)Q(\varepsilon)x_1(\tau,\varepsilon) + {}^t u(\tau,\varepsilon)R(\varepsilon)u(\tau,\varepsilon)\big)\,d\tau & \end{cases}$$

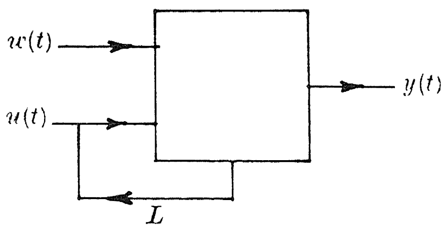
where the upper ^{t} denotes transposes.

Here the matrix $R(\varepsilon)$ is positive definite, the matrices $Q(\varepsilon)$ and $\pi(\varepsilon)$ are positive semidefinite, and it is desired to find the control which drives the initial state $(x_1^0(\varepsilon), x_2^0(\varepsilon))$ to $(0, 0)$ in time 1 and which minimizes the cost $J(\varepsilon)$. All matrices may depend on time as well. For fixed small ε there is a unique optimal solution. Here one is interested however in the asymptotic solution as $\varepsilon \to 0$, which is, still quoting [OMa] a problem of considerable practical interest, in particular, in view of an example of Hadlock et al. [HJK] where the asymptotic results are far superior to the physically unacceptable results obtained by setting $\varepsilon = 0$ directly in (2.4.1).

Another interesting perturbation type problem arises may be when we have a system

$$(2.4.2)\qquad \dot{x} = Fx + G_1u + G_2w\,, \qquad y = Hx$$

where w is some undesirable noise input, and where F, G_1, G_2, H depend on a small parameter ε. It is desired to try to remove the influence of the noise input w by means of state feedback



That is one tries to find a matrix L such that in the new system with state feedback loop L, which is given by the equations

$$(2.4.3) \qquad \dot{x} = (F + GL)x + G_1 u + G_2 w\,, \qquad y = Hx\,,$$

the disturbances do not show up any more in the output y. Suppose we can solve this for $\varepsilon = 0$. Can we then find a disturbance decoupler $L(\varepsilon)$ by perturbation methods, i.e. as a power series in ε which converges (uniformly) for ε small enough and of which the various terms can be calculated by successive approximation?

2.5. There are still more reasons for being interested in families rather then single systems. E.g. $2-d$ and $n-d$ systems which we shall meet briefly in section 6.3 below; parameter uncertainty, where one tries to perform certain constructions so as to attain certain desirable properties for systems some of whose parameters are uncertain or for systems which have parameters which may vary somewhat; cf. also 7 below; identification problems; and, not least, time varying systems which can on occasion be fruitfully viewed as triples of matrices depending on a parameter t, cf. also 11.2 below.

3. A little basic system theory.

In this section we describe briefly as background material and for later use a few of the more elementary concepts and results pertaining to a single system over a field k.

3.1. *Complete reachability and complete observability.* Let k be a field and $\Sigma = (F, G, H)$ a linear dynamical system over k. The triple (F, G, H) can be interpreted either as a continuous time system (given by differential equations) or as a discrete time system (given by difference equations), cf. (1.1). Given Σ one defines the *reachability matrix*

$$(3.1.1) \qquad R(\Sigma) = R(F, G) = (G \;\vdots\; FG \;\vdots \dots \vdots\; F^n G)$$

as the $n \times (n+1)m$ matrix consisting of the $n+1$ blocks $G, FG, \dots$ $\dots, F^n G$. Dually one defines the *observability matrix*

$$(3.1.2) \qquad Q(\Sigma) = Q(F, H) = \begin{pmatrix} H \\ HF \\ \vdots \\ HF^n \end{pmatrix}$$

as the $(n+1)p\times n$ matrix consisting of the $n+1$ blocks $H, HF, \ldots$ $\ldots, HF^n$.

The system Σ is said to be *completely reachable*, abbreviated cr, if $R(\Sigma)$ has its maximal rank n and the system is said to be *completely observable*, abbreviated co, if $Q(\Sigma)$ has its maximal rank n.

These notions have the following interpretation in terms of the sets of equations (1.1). The system is cr if for every $x\in k^n$, there is an input function $u(t)$ such that starting in $x(0)=0$ at time zero the solution of the first equation using this control $u(t)$ passes through x. The system is co if for every two states x, x' and input function $u(t)$, the two output functions $y(t)$, $y'(t)$ resulting from starting in x, x' at time zero and using this input function are equal if and only if $x=x'$.

Finally one associates to Σ its Hankel matrix $\mathcal{H}(\Sigma)$ which is defined as the infinite block Hankel matrix

$$\mathcal{H}(\Sigma)=\begin{pmatrix} A_1 & A_2 & A_3 & \cdots \\ A_2 & A_3 & A_4 & \cdots \\ A_3 & A_4 & A_5 & \cdots \\ \vdots & \vdots & \vdots & \end{pmatrix} \tag{3.1.3}$$

built from the $p\times m$ blocks $A_i=HF^{i-1}G$, $i=1,2,\ldots$ Note that $\mathcal{H}(\Sigma)$ depends only on the isomorphism class of Σ because

$$\mathcal{H}(\Sigma^S)=\mathcal{H}(SFS^{-1}, SG, S^{-1}H)=\mathcal{H}(F, G, H)=\mathcal{H}(\Sigma))\,.$$

We note that

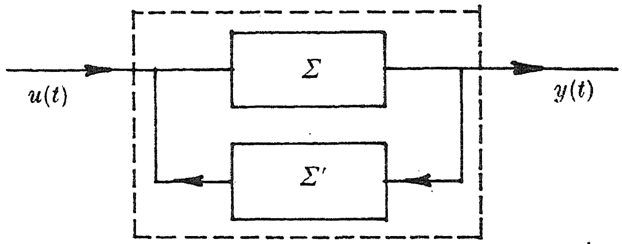
$$\mathcal{H}(\Sigma)=\begin{pmatrix} H \\ HF \\ HF^2 \\ \vdots \end{pmatrix}(G \vdots FG \vdots F^2G\ldots)$$

so that rank $\mathcal{H}(\Sigma)\leqslant n$ for a system of dimension n and rank $\mathcal{H}(\Sigma)=n$ for a system Σ of dimension n iff Σ is both cr and co (using the Cayley Hamilton theorem).

3.2. *Realization theory*. Let $\Sigma\in L_{m,n,p}(k)$ be a system over k. Then as we have seen, cf. (1.3), (1.4), Σ determines an input/output map f_Σ which is completely determined by the infinite sequence of matrices

$$\mathcal{A}(\Sigma)=\big(A_1(\Sigma), A_2(\Sigma), \ldots\big)\,,\qquad A_i(\Sigma)=HF^{i-1}G\,,\quad i=1,2,\ldots\,. \tag{3.2.1}$$

is processed through another linear system Σ' and then fed back into Σ. The block diagram is of course



If the transfers function of Σ is $T(s)$ and that of Σ' is $T'(s)$, then the transfer function of the total system is

(3.4.1) $$\frac{T(s)}{1-T(s)\,T'(s)}.$$

4. Fine moduli spaces, universal families and canonical forms.

4.1. *The quotient scheme* $M^{\mathrm{co,cr}}_{m,n,p}$. Let k be any field, then $GL_n(k)$ acts on $L^{\mathrm{cr}}_{m,n,p}(k)$ the set of all cr linear dynamical systems $\Sigma=(F, G, H)$ of dimension n with m inputs and p outputs. Let $M^{\mathrm{cr}}_{m,n,p}(k)$ be the set of orbits. We note that the stabilizer subgroup of each $\Sigma \in L^{\mathrm{cr}}_{m,n,p}(k)$ is trivial (because $R(\Sigma^S) = SR(\Sigma)$ and $R(\Sigma)$ has full rank), which (morally) goes some way towards suggesting the following theorem.

4.1.1. THEOREM: There exists a scheme $M^{\mathrm{cr}}_{m,n,p}$ over $\boldsymbol{Z}$ such that for each field k the k-rational points of $M^{\mathrm{cr}}_{m,n,p}$ are precisely the orbits of $GL_n(k)$ acting on $L^{\mathrm{cr}}_{m,n,p}(k)$. There is an open subscheme $M^{\mathrm{cr,co}}_{m,n,p}$ corresponding to the orbits of cr and co systems.

Locally $M^{\mathrm{cr}}_{m,n,p}$ is isomorphic to affine space A^{nm+pn} and the way these pieces are glued together is very reminiscent of Grassmann varieties. For details cf. [Haz 2] for the topological version, [Haz 3] and also [BH] for the case of varieties over a field, and [Haz 6] for the fact that $M^{\mathrm{cr}}_{m,n,p}$ is defined and is classifying over $\boldsymbol{Z}$.

4.2. *Universal families*. There are a number of universal families of systems. Let us start with a topological one

4.2.1. THEOREM: There exists a family $\Sigma^u = (E^u; F^u, G^u, H^u)$ of real cr systems over the smooth differentiable manifold $M^{\mathrm{cr}}_{m,n,p}(\boldsymbol{R})$ such that the following universality property holds. For each continuous family Σ of real cr systems over a topological space V there is a unique continuous map φ_Σ such that Σ is isomorphic to the pullback

$$\varphi_\Sigma^! \Sigma^u = (\varphi_\Sigma^! E^u;\ \varphi_\Sigma^! F^u,\ \varphi_\Sigma^! G^u,\ \varphi_\Sigma^! H^u)\ .$$

There are corresponding statements for differentiable and real analytic families over differentiable and real analytic varieties. ($M^{\mathrm{cr}}_{m,n,p}(\boldsymbol{R})$ is real analytic). There is also an analogous theorem for complex systems.

On the algebraic-geometric side of things we have

4.2.2. THEOREM: There exists an algebraic family Σ^u of cr systems over the scheme $M^{\mathrm{cr}}_{m,n,p}$ such that for every algebraic family Σ of cr systems over a scheme V there is a unique morphism of schemes $\varphi_\Sigma\colon V \to M^{\mathrm{cr}}_{m,n,p}$ such that Σ is isomorphic over V to the pullback family $\varphi_\Sigma^! \Sigma^u$.

Here a family $\Sigma = (E; F, G, H)$ over a scheme V is said to be cr if for every $v \in V$ the system over v, i.e. the system $(E \otimes k(v); F \otimes \otimes k(v), G \otimes k(v), H \otimes k(v))$ over the residue field $k(v)$, is cr.

4.3. *The Kronecker nice selection.* Most will agree that the Jordan canonical form is a useful gadget when dealing with matrices. What it does is select one particular element out of each orbit of $\boldsymbol{GL}_n(\boldsymbol{C})$ acting on $M_n(\boldsymbol{C})$, the space of all $n \times n$ matrices, by similarity, i.e. as $(S, A) \mapsto SAS^{-1}$. Similarly it would be nice to have a canonical form for $\boldsymbol{GL}_n(k)$ acting on $L_{m,n,p}(k)$, or at least $L^{\mathrm{co,cr}}_{m,n,p}(k)$. For one thing they can be useful when trying to identify a system from its input/output data, because the input/output data only specify an orbit, (not the system itself, so that there are a number of redundant parameters to get rid off before trying to estimate the remaining ones, cf. also [GW]). One particular canonical form proceeds via what is called the Kronecker nice selection, which we now describe. It will also be useful in 10.3 below when studying feedback.

Let $\Sigma = (F, G, H)$ be a cr system over a field k. Consider an array of $n \times (n+1)m$ dots. For each (i, j), $i = 0, \dots, n$; $j = 1, \dots, m$, in this array put a cross at this spot if and only if the column vector $F^i g_j$ where g_j is the j-th column of G, is linearly independent of the vectors $F^a g_b$, with $(a, b) < (i, j)$ where the order is the lexicographic one (i.e. $(a, b) < (i, j) \Leftrightarrow a < i$ or $(a = i$ and $b < j)$). This yields a pattern of n crosses (because rank $R(\Sigma)$ is n). For example the result for $n = 6$, $m = 4$ might be

$$\begin{array}{ccccccc}
\cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot \\
\times & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot \\
\times & \times & \times & \cdot & \cdot & \cdot & \cdot \\
\times & \times & \cdot & \cdot & \cdot & \cdot & \cdot
\end{array} \qquad (4.3.1)$$

which means e.g. that $g_1 = 0$ and that Fg_2 is linearly dependant on g_1, g_2, g_3, g_4.

Note that the pattern above has the property that whenever a $\times$ appears in a row than all positions in this row left of this $\times$ are also occupied by $\times$'s. This is no accident (and it is this property that the word nice in the title of this subsection refers to). It follows that the pattern obtained is uniquely described by the m-numbers $\tilde{\varkappa}(\Sigma) = (\tilde{\varkappa}_1(\Sigma), \ldots, \tilde{\varkappa}_m(\Sigma))$ of $\times$'s in each row. This sequence of m numbers $\tilde{\varkappa}(\Sigma)$, or more precisely the corresponding pattern of crosses, is what I call the Kronecker nice selection.

Note that $\tilde{\varkappa}(\Sigma^S) = \tilde{\varkappa}(\Sigma)$ for all $S \in \boldsymbol{GL}_n(k)$ so that these numbers are discrete invariants.

4.4. *Canonical forms.* The Kronecker selection $\tilde{\varkappa}(\Sigma)$ defined above now can be used to define a canonical form on $L^{\mathrm{cr}}_{m,n,p}(k)$. We label the columns of $R(\Sigma) = R(F, G) = (G \,\vdots\, FG \,\vdots\, \ldots \,\vdots\, F^n G)$ by the spots in the array of 4 3 above, *i.e.* by the pairs (i, j), $i = 0, \ldots, n$; $j = 1, \ldots, m$. For each subset α of this set of pairs let $R(\Sigma)_\alpha$ be the matrix obtained from $R(F, G)$ by removing all columns whose index is not in α. Note that for all $S \in \boldsymbol{GL}_n(k)$,

$$(R(\Sigma^S))_\alpha = S(R(\Sigma)_\alpha)\,. \tag{4.4.1}$$

It follows that each orbit of $\boldsymbol{GL}_n(k)$ in $L^{\mathrm{cr}}_{m,n,p}(k)$ contains precisely one element Σ such that $R(\Sigma)_{\tilde{\varkappa}(\Sigma)} = I_n$. This defines a canonical form

$$c_\varkappa\colon L^{\mathrm{cr}}_{m,n,p}(k) \to L^{\mathrm{cr}}_{m,n,p}(k)\,, \qquad \Sigma \mapsto \Sigma^S\,, \text{ where } S = (R(\Sigma)_{\tilde{\varkappa}(\Sigma)})^{-1}\,. \tag{4.4.2}$$

This is but one example of a large number of canonical forms in use in system and control theory, and one may ask whether this construction is continuous. The Jordan canonical form for matrices e.g. is discontinuous which severely limits its usefulness for instance in numerical matters, [GWi]. Similarly it would be nice to have a continuous canonical form for systems for identification and numerical purposes. However,

4.4.3. THEOREM: There exists a continuous canonical form $c\colon L^{\mathrm{cr,co}}_{m,n,p}(\boldsymbol{R}) \to L^{\mathrm{co,cr}}_{m,n,p}(\boldsymbol{R})$ if and only if $p = 1$ or $m = 1$.

There is a similar statement concerning canonical forms which are morphisms on the algebraic varieties $L^{\mathrm{cr,co}}_{m,n,p}(k)$, k an algebraically closed field. For details and more theorems like this, cf. [Haz 2, Haz 3]. The reason behind this theorem is the following. As is easily seen, a continuous canonical form exists on all of $L^{\mathrm{co,cr}}_{m,n,p}(\boldsymbol{R})$ if and only if the universal bundle E^u restricted to $L^{\mathrm{co,cr}}_{m,n,p}(\boldsymbol{R})$ is trivial. It turns out that this is the case if and only if $m = 1$ or $p = 1$.

4.5. *Pointwise-local isomorphism problems.* It is an immediate consequence of the fine moduli space theorems 4.2.1, 4.2.2 that if two families Σ and Σ' of cr systems over V are pointwise isomorphic then thay are isomorphic as families over V. A similar statement holds for families which are co everywhere; in fact the whole body of definitions and statements has a co (i.e. output) counterpart.

In general, however, such a statement is definitely false just as in the case of matrices depending holomorphically on a parameter with respect to similarity, [Wa]. In analogy with the positive results one has in that case.

4.5.1. THEOREM: Let Σ, Σ' be two families of dynamical systems over V. Suppose that $\Sigma(v)$ and $\Sigma'(v)$ are isomorphic for all $v \in V$. Suppose moreover that the stabilizer subgroup of $\Sigma(v)$ has constant dimension as a function of v in some neighbourhood of $v_0 \in V$. Then there exists an open neighbourhood U of v_0 such that Σ and Σ' are isomorphic as families over U.

The theorem holds both for continuous real families over a topological space and for algebraic families over schemes, so in particular for systems over rings. Cf. [HP] for details of the proofs and various examples.

5. Realization with parameters and variations.

5.1. *Pointwise realization theory.* As was remarked in section 3 a strictly proper rational matrix function $T(s)$ with coefficients in a field k, or, equivalently, a sequence of matrices $\mathcal{A} = (A_1, A_2, \ldots)$ with finite rank Hankel matrix can be realized by means of a finite dimensional system, *i.e.* we can find a $\Sigma = (F, G, H)$ over k such that

$$(5.1.1) \qquad T(s) = H(sI - F)^{-1}G\,, \qquad A_i = HF^{i-1}G\,, \quad i = 1, 2, \ldots$$

and it is even possible to find a realization which is co and cr. A more or less standard way of proving the first statement is as follows. The hypothesis that the rank of the Hankel matrix

$$\mathcal{H} = \begin{pmatrix} A_1 & A_2 & A_3 & \cdots \\ A_2 & A_3 & A_4 & \cdots \\ \vdots & \vdots & \vdots & \end{pmatrix}$$

is finite means that there is an r and that there are matrices $T_1, \ldots, T_r$ such that the $(r+1)$-th column of $\mathcal{H}$ is equal to T_r(1-st column) +

$+ T_{r-1}$(2-nd column) $+ \dots T_1$(r-th column), which means that

$$(5.1.2) \qquad A_{r+i} = T_r A_i + T_{r-1} A_{i+1} + \dots + T_1 A_{r+i-1}\,, \; i = 1, 2, \dots .$$

Now let

$$(5.1.3) \qquad \begin{pmatrix} 0 & \cdots & & 0 & T_r \\ I & \ddots & & \vdots & \vdots \\ 0 & \ddots & \ddots & \vdots & \vdots \\ \vdots & \ddots & \ddots & 0 & T_2 \\ 0 & \cdots & 0 & I & T_1 \end{pmatrix} = F\,, \quad \begin{pmatrix} I \\ 0 \\ \vdots \\ 0 \end{pmatrix} = G\,, \quad (A_1, \dots, A_r) = H\,.$$

Then $A_i = HF^{i-1}G$ for all $i = 1, 2, \dots$ because $(A_i, \dots, A_{i+r-1})F = (A_{i+1}, \dots, A_{i+r})$ by (5.1.2). Thus the system Σ defined by (5.1.3) realizes $\mathcal{A}$. One then proceeds to find the canonical cr subsystem Σ^{cr} of the system just constructed, cf. 11.1 below, and then constructs the canonical co quotient system of the Σ^{cr} just constructed, to find a cr and co system which (also) realizes Σ, cf. also 6.2.

5.2. *Realization with parameters* ([By 4]). It is not at all clear, however, that the realization construction of 5.1 above is continuous in the parameters of $\mathcal{A}$ (or in the parameters of $T(s)$). Also one usually prefers a realization of minimal dimension, i.e. a co and cr realization, and it is also not clear that the construction which associates to a system Σ its cr and co subquotient with the same input/output map is continuous. This question is in fact the topic of section 11 below, cf. also 6.2.

Let $\mathcal{A}(a)$ be a family of sequences of matrices depending on a parameter with uniformly bounded MacMillan degree, or, equivalently, let

$$(5.2.1) \qquad T_a(s) = \sum_{i=1}^{\infty} A_i(a)\, s^{-i}$$

be a family of rational strictly proper transfer functions (with the same boundedness property). Then an obvious necessary condition for the existence of a family $\Sigma(a)$ in the sense of 2.1, which is co and cr everywhere, such that $\Sigma(a)$ realizes $\mathcal{A}(a)$ (or, equivalently $T_a(s)$) for all a is that the MacMillan degree of $\mathcal{A}(a)$ (cf. 3.2 above) be constant as a function of a. This is also sufficient.

5.2.2. THEOREM: Let $\mathcal{A}(a)$ be an algebraic (resp. continuous) family of sequences of matrices of constant MacMillan degree. Then ere exists an algebraic (resp. continuous) family of systems $\Sigma(a)$ alizing $\mathcal{A}(a)$.

Indeed, one shows without too much difficulty (using the Zariski ain theorem as in [By 4], or by constructing local inverses [Haz 3]) at $\Sigma \mapsto \mathcal{A}(\Sigma)$ induces an isomorphism of $M^{co,cr}_{m,n,p}$ with the space of l sequences of MacMillan degree n. Thus the family $\mathcal{A}(a)$ defines morphism into $M^{co,cr}_{m,n,p}$ and the pullback of the universal family by eans of this morphism is the desired family.

This does not mean that we can always find a family of co and cr *atrix* triples $(F(a), G(a), H(a))$ realizing $\mathcal{A}(a)$. Indeed this will be ssible if and only if the pullback of the underlying bundle E^u of e universal family of systems by means of the morphism defined r the family $\mathcal{A}(a)$ is trivial. Yet precisely such a family of matrix iples is what is desired on occasion; in particular when $\mathcal{A}(a)$ is a mily of matrix sequences coming from a sequence $\mathcal{A} = (A_1, A_2, \ldots)$ matrices with coefficients in a ring R.

5.2.3. COROLLARY: Let R be a ring such that all projective mod- es of rank n are free. Let $\mathcal{A} = (A_1, A_2, \ldots)$ be a sequence of matrices th coefficients in R, such that the MacMillan degree of $\mathcal{A}(\mathfrak{p}) =$ $(A_1(\mathfrak{p}), A_2(\mathfrak{p}), \ldots)$ over the quotient field $k(\mathfrak{p})$ of $R/\mathfrak{p}$ is equal to n r all prime ideals $\mathfrak{p}$. Then there exists a triple of matrices (F, G, H) er R, i.e. a system over R, which realizes $\mathcal{A}$ (i.e. such that $A_i = HF^{i-1}G$, $i = 1, 2, \ldots$) and which is such that $(F(\mathfrak{p}), G(\mathfrak{p}), H(\mathfrak{p}))$ is co d cr for all $\mathfrak{p}$. (I.e. we have a split realization in the terminology [So 3].)

By the Quillen-Suslin theorem the condition on R is in particular lfilled for rings of polynomials over a field, which is e.g. the case interest when discussing realization by means of delay-differential stems.

5.3. *Realization by means of delay-differential systems.* Let $\Sigma(\sigma) =$ $(F(\sigma), G(\sigma), H(\sigma))$ be a delay-differential system with r incommen- rable delays. Here $\sigma = (\sigma_1, \ldots, \sigma_r)$ and σ_i stands for the delay erator $\sigma_i z(t) = z(t - a_i)$, so that we have written $\Sigma(\sigma)$ as a system er the ring of polynomials $k[\sigma_1, \ldots, \sigma_r]$. The transfer function of (σ) is

$$(s) = H\big(\exp(-a_1 s), \ldots, \exp(-a_r s)\big)\cdot$$
$$\big(sI - F(\exp(-a_1 s), \ldots, \exp(-a_r s))\big)^{-1} G\big(\exp(-a_1 s), \ldots, \exp(-a_r s)\big)$$

hich can be seen as a rational function in s whose coefficients are lynomials over k in $\exp(-a_1 s), \ldots, \exp(-a_r s)$.

Now inversely suppose that we have a transfer function $T(s)$ which can be written as a rational function in s with coefficients which are polynomials in the exponential functions $\exp(-a_1 s), \ldots, \exp(-a_r s)$, and we ask whether it can be realized by means of a delay-differential system $\Sigma(\sigma)$. Now if the a_i are incommensurable then the functions s, $\exp(-a_1 s), \ldots, \exp(-a_r s)$ are algebraically independent, and there is precisely one transfer function $T'(s) = T'(s; \sigma_1, \ldots \sigma_r)$ whose coefficients are polynomials in the $\sigma_1, \ldots, \sigma_r$ such that

$$T(s) = T'\big(s; \exp(-a_1 s), \ldots, \exp(-a_r s)\big) .$$

Thus the problem is mathematically identical with the one just discussed above in 5.2, and by Corollary 5.2.3 and the Quillen-Suslin theorem we get a positive answer in the case that the MacMillan degree of $T'(s; \sigma_1, \ldots, \sigma_r)$ is constant for all complex values of the parameters $\sigma_1, \ldots, \sigma_r$.

5.4. *Network synthesis.* An n-port is an electronic gadget with n pairs of terminals (over which voltages and currents can be measured). An n-port which is constructed on a finite graph consisting only of lumped resistors, inductors, capacitators, ideal transformers and gyrators can be described by an $n \times n$ scattering matrix $S(p)$ which essentially, after a normalization, relates the voltages and the currents across the n ports. The matrix $S(p)$ is rational and it is symmetric if no gyrators are present. When discussing the inverse problem of how to realize an $S(p)$ by means of a network (i.e. the network synthesis problem, which has been solved) one hits the following symmetric version of the system realization problem discussed above.

Given a symmetric, rational, proper $n \times n$ matrix $W(s)$ (the matrix $W(s)$ is related to the scattering matrix $S(p)$ by a simple fractional substitution), find an internally symmetric realization, where the last phrase means that we want to find a triple (F, G, H) of matrices of sizes $r \times r$, $r \times n$, $n \times r$ such that

$$(5.4.1) \qquad W(s) = H(sI - F)^{-1} G \,, \qquad I_{p,q} F = {}^t F I_{p,q} \,, \qquad I_{p,q} G = {}^t H$$

where the upper ^{t} denotes transposes, and where $I_{p,q}$, $p + q = r$, is the standard symmetric form of signature $p - q$ (consisting of p $+1$'s and q -1's on the diagonal and zero's elsewhere). Note that r and $p - q$ are given by $W(s)$ as the MacMillan degree of $W(s)$ and the signature of the Hankel matrix of $W(s)$.

In [YT] Youla and Tissi show that internally symmetric realizations of minimal degree always exist (op. cit. Lemma 8) and that any two of them are transformed into one another by an element of $O(p, q) \subset \boldsymbol{GL}_n(\boldsymbol{R})$.

The situation is now entirely analogous to the one for linear dynamical system (realization) theory discussed above, and one can ask about fine moduli spaces, etc. In particular one can ask about the existence of continuous symmetric canonical forms. It turns out that these exist only in the case where they have long been known to exist, [BD]. (The Foster and Cauer canonical forms for RL and RC networks). Again, the problem is ruled by a certain universal bundle, which, again, is nontrivial as soon as it has a decent chance to be so. (There seems to be a kind of Murphy's law also in this highly theoretical branch of electrical engineering.)

Another question which it is now natural to ask is whether there exist polynomial families of internally symmetric realizations for polynomial families of symmetric matrices $W(s)$. Especially in connection with delay networks, i.e. networks with transmission lines, [An, Ko, RMY, Yo]. Here instead of the old Serre problem, one hits the quadratic analogue which asks whether any quadratic space over $k[\sigma_1, \ldots, \sigma_r]$ is induced from one over k, [Ba]. Here the general answer is negative ([Pa], $k = \boldsymbol{R}$, $r = 2$), but the answer is yes if $r = 1$ ([Har]), if k is algebraically closed ([Ra]), and if the quadratic space is not definite ([Oj]).

6. **Realization over rings** (2).

Let $\mathcal{A} = (A_1, A_2, \ldots)$ be a sequence of $p \times m$ matrices over a ring R. Suppose we want to realize $\mathcal{A}$ over R, i.e. we want to find matrices (F, G, H) with coefficients in R such that $A_i = HF^{i-1}G$, $i = 1, 2, \ldots$. One way to tackle this was discussed above and consists of treating $\mathcal{A}$ as a family over Spec (R) and using the fine moduli space of co and cr systems and the Quillen-Suslin theorem ([Sus, Qu]). The hypotheses to make this work, however, are rather strong: viz. the MacMillan degree of $\mathcal{A}(\mathfrak{p})$ must be constant as a function of $\mathfrak{p}$, and R must be projective free in the appropriate dimensions.

Another way to get realizations of $\mathcal{A}$ goes as follows. Assume for simplicity that R is an integral domain; if R is not an integral domain but is reduced, then these ideas generalize rather easily. Let K be the quotient field of R. Then $\mathcal{A}$ is realizable over K if and only if the rank of the Hankel matrix of $\mathcal{A}$, viewed as a matrix over K, is finite. Let $d(\mathcal{A})$ denote this number. Thus we are left with the problem: which integral domains are such that if a sequence of matrices over R is realizable over K, then it is also realizable over R (possibly using higher dimensional matrices).

This method is not particularly thrifty in terms of the dimension of the realization obtained, but has the advantage of requiring far weaker hypotheses, as we shall see.

6.1. *The Fatou property.* An integral domain R is said to be *Fatou* if for every rational function $p(s)/q(s)$, where $p(s)$ and $q(s)$ are polynomials with coefficients in the quotient field K of R, such that its expansion $p(s)/q(s) = \sum a_i s^{-i}$ has all its coefficients in R, there exist polynomials $p'(s)$, $q'(s)$ over R such that $q'(s)$ has leading coefficient equal to 1 and such that $p'(s)/q'(s) = \sum a_i s^{-i}$.

Fatou proved in 1906 that the ring of integers $\mathbf{Z}$ has this property, whence the name. The Fatou property is actually equivalent to the realization property: if $\mathcal{A}$ over R is realizable over K then it is realizable over R.

For the one input/one output case this is immediate because firstly the polynomial part of $T(s) = \sum a_i s^{-i}$ causes no difficulties at all, showing that the realization property for the one input/one output case implies the Fatou property. Secondly, a power series $\sum a_i s^{-i}$ is the expansion of a rational function $p'(s)/q'(s)$ with the leading coefficient of $q'(s)$ equal to one iff $a_{i+r} = t_r a_i + \ldots t_1 a_{i+r-1}$ for all $i = 1, 2, \ldots$, (where the t_j are the coefficients of $q'(s)$), and then the realization procedure 5.1 above gives the desired realization. In the more input/more output case one simply observes that $T(s)$ consists of rational functions as entries. Realizing each of these we find in the case of three inputs and two outputs the realizations (F_{ij}, G_{ij}, H_{ij}), $i = 1, 2$; $j = 1, 2, 3$, of $\sum a_r(i, j) s^{-r}$, where $a_r(i, j)$ is the (i, j)-th coefficient of A_r and $T(s) = \sum A_r s^{-r}$. Now put all these together in the following way

$$F = \begin{pmatrix} F_{11} & 0 & 0 & 0 & 0 & 0 \\ 0 & F_{12} & 0 & 0 & 0 & 0 \\ 0 & 0 & F_{13} & 0 & 0 & 0 \\ 0 & 0 & 0 & F_{21} & 0 & 0 \\ 0 & 0 & 0 & 0 & F_{22} & 0 \\ 0 & 0 & 0 & 0 & 0 & F_{23} \end{pmatrix}, \qquad G = \begin{pmatrix} G_{11} & 0 & 0 \\ 0 & G_{12} & 0 \\ 0 & 0 & G_{13} \\ G_{21} & 0 & 0 \\ 0 & G_{22} & 0 \\ 0 & 0 & G_{23} \end{pmatrix},$$

$$H = \begin{pmatrix} H_{11} & H_{12} & H_{13} & 0 & 0 & 0 \\ 0 & 0 & 0 & H_{21} & H_{22} & H_{23} \end{pmatrix}.$$

Then $A_r = HF^{r-1}G$ for all r, and of course this trick works in general.

6.1.1. Theorem ([RWK]): Every noetherian integral domain is Fatou.

PROOF ([So 1]): Let $\mathcal{A}$ be a sequence of $p\times m$ matrices over R which is realizable over K. The first step now consists of the following elegant realization procedure by means of a not necessarily free state module ([Rou, Fl 1, Fl 2]). Write down the Hankel matrix $\mathcal{H}$ of $\mathcal{A}$, and let X be the R module generated by the columns of $\mathcal{H}$. Now define $G'\colon R^m \to X$ by $G'(a_1, \ldots, a_m) = a_1 b_1 + \ldots + a_m b_m$, where the b_i are the columns of $\mathcal{H}$; define $F'\colon X \to X$ by $F'(b_j) = b_{j+m}$, and let $H'(b_j)$ be the column vector consisting of the first p entries of b_j. (Note that F' is well defined because by the structure of the Hankel matrix any linear relation $c_1 b_{i_1} + \ldots + c_r b_{i_1} = 0$ implies $c_1 b_{i_1+m} + \ldots + c_r b_{i_r+m} = 0$.)

The second step consists in showing that the module X is finitely generated. Let $v_1, \ldots, v_n$ be n columns of $\mathcal{H}$ which form a basis for $X \otimes_R K$ over K. Then every column of $\mathcal{H}$ can be written as a sum $\sum d^{-1} d_i v_i$, where $d_i \in R$ and where $d \in R$ is the determinant of a full rank $n\times n$ submatrix of the matrix formed by the v_i. Let X' be the R module generated by the vectors $d^{-1}v_i$, $i = 1, \ldots, n$. Then X is a submodule of the finitely generated module X' and so is finitely generated because R is noetherian.

Finally let $R^n \to X$ (different n in general) be any surjective module homomorphism. Then because R^n is free there are homomorphisms F, G, H such that the following diagram is commutative,

$$\begin{tikzcd}
& R^n \arrow[r, "F"] \arrow[dd] & R^n \arrow[dr, "H"] \arrow[dd] & \\
R^m \arrow[ur, "G"] \arrow[dr, "G'"'] & & & R^p \\
& X \arrow[r, "F'"'] & X \arrow[ur, "H'"'] &
\end{tikzcd}$$

and then $HF^{i-1}G = H'F'^{i-1}G' = A_i$, $i = 1, 2, \ldots$, proving the theorem.

Not all integral domains are Fatou, cf. [Cha, CCh]. A closely related property called strong Fatou is also relevant for system theoretic considerations ([SR 2]), and it in turn implies that the ring in question is almost projective free. (For such rings it suffices to add one copy of R to a projective module to make if free.)

6.2. *Minimal realizations*, ([Eil]). Let $F\colon X \to X$, $G\colon R^m \to X$, $H\colon X \to R^p$ be a (discrete time) system over a ring R whose state module is not necessarily free. Define $\underline{G}\colon R^m[z] \to X$ by $az^i \mapsto F^i Ga$ and define $\overline{H}\colon X \to R^p[\![z]\!]$ by $\overline{H}x = \sum HF^n xz^n$. Then the appropriate (and obvious) notions of cr and co for systems over rings are: the system is (ring) cr if $\underline{G}$ is surjective, and the system is (ring) co if $\overline{H}$

is injective. (For the family over Spec (R) associated to the system the property « ring cr » is equivalent to the requirement that every member of the family be cr; but the property that every member of the family be co is stronger than the property « ring co ».) The system is said to be minimal if it is both cr and co.

Now let $X^{cr} \subset X$ be the image of $\underline{G}$. Then $G(R^m) \subset X^{cr}$ and $F(X^{cr}) \subset X^{cr}$, and the induced cr system $(X^{cr}; F, G, H)$ has the same input/output behaviour as the original system $(X; F, G, H)$. More or less dually let C be the kernel of $\overline{H}$ and let X^{co} be the R-module $X^{co} = X/C$. Now $F(C) \subset C$ and $H(C) = 0$ so that we have an induced system $(X^{co}; F, G, H)$, which is co and which has the same input/output behaviour as the original system.

Performing both constructions we find a co and cr system $((X^{cr})^{co}; F, G, H)$ with the same input/output behaviour as the original system; i.e. we find a minimal system. All minimal systems realizing a given $\mathcal{A}$ are isomorphic (so that in particular it does not matter which of the two constructions is carried out first).

Of course the minimal realization of a given $\mathcal{A}$ need not have a free, or even projective, state module, however, if the family $\Sigma(\mathfrak{p})$ has constant MacMillan degree than the realization obtained by the methods of section 5 above is minimal and the realization obtained by the constructions described above has a projective state space module.

6.3. *$2-d$ and $n-d$ systems.* Consider a linear discrete time system with direct feed-through term

$$(6.3.1) \qquad x(t+1) = Fx(t) + Gu(t)\,, \qquad y(t) = Hx(t) + Ju(t)\,.$$

The associated input/output operator is a convolution operator, viz.

$$(6.3.2) \qquad y(t) = \sum_{i=0}^{t} A_i u(t-i)\,,\ A_0 = J\,,\ A_i = HF^{i-1}G\,,\ i = 1, 2, \ldots\,.$$

Now there is an obvious more dimensional (north-east causal) generalization of such a convolution operator, *viz.*

$$(6.3.3) \qquad y(h, k) = \sum_{i=0}^{h} \sum_{j=0}^{k} A_{ij} u(h-i, k-j)\,, \qquad h, k = 0, 1, 2, \ldots\,.$$

A (Givone-Roesser) realization of such an operator is a « $2-d$ system »

$$(6.3.4) \qquad \begin{cases} x_1(h+1, k) = F_{11}x_1(h, k) + F_{12}x_2(h, k) + G_1 u(h, k)\,, \\ x_2(h, k+1) = F_{21}x_1(h, k) + F_{22}x_2(h, k) + G_2 u(h, k)\,, \\ y(h, k) \qquad = H_1 x_1(h, k) + H_2 x_2(h, k) + Ju(h, k)\,, \end{cases}$$

which yields an input/output operator of the form (6.3.3) with the $A_{i,j}$ determined by the power series development of the $2-d$ transfer function $T(s_1, s_2)$

$$\sum_{j,i} A_{i,j} s_1^{-i} s_2^{-j} = T(s_1, s_2)\,, \tag{6.3.5}$$

$$T(s_1, s_2) = (H_1 \quad H_2)\left(\begin{pmatrix} s_1 I_{n_1} & 0 \\ 0 & s_2 I_{n_2}\end{pmatrix} - \begin{pmatrix} F_{11} & F_{12} \\ F_{21} & F_{22}\end{pmatrix}\right)^{-1}\begin{pmatrix} G_1 \\ G_1 \end{pmatrix} + J \tag{6.3.6}$$

where I_r is the $r\times r$ unit matrix and where n_1 and n_2 are the dimensions of the state space vectors x_1 and x_2. There are obvious generalizations to $n-d$ systems. The question now arises whether every proper (cf. e.g. [Eis 1] for a definition) $2-d$ transfer function can indeed be realized by a set of « processing equations » like (6.3.4).

One way to approach this is to treat one of the s_i as a parameter which then gives us a realization problem over a ring (or a realization problem with parameters).

More precisely let R_g be the ring of all proper rational functions in s_1. Now consider $T(s_1, s_2)$ as a proper rational function in s_2 with coefficients in R_g. This transfer function can be realized over R_g, giving us a quadruple of matrices $(F(s_1), G(s_1)), H(s_1), J(s_1))$. Each of these matrices is proper as a function of s_1 and hence can be realized by a quadruple of matrices with coefficients in whatever field we happen to work over. Suppose that

(F_F, G_F, H_F, J_F) realizes $F(s_1)$, $\quad (F_G, G_G, H_G, J_G)$ realizes $G(s_1)$,

(F_H, G_H, H_H, J_H) realizes $H(s_1)$, $\quad (F_J, G_J, H_J, J_J)$ realizes $J(s_1)$.

Then, as is easily checked, a realization in the sense of (6.3.4) is defined by

$$F = \begin{pmatrix} F_{11} & F_{12} \\ F_{21} & F_{22}\end{pmatrix} = \left(\begin{array}{c|cccc} J_F & H_F & H_G & 0 & 0 \\ \hline G_F & F_F & 0 & 0 & 0 \\ 0 & 0 & F_G & 0 & 0 \\ G_H & 0 & F_H & 0 & 0 \\ 0 & 0 & 0 & 0 & F_J \end{array}\right), \qquad G = \begin{pmatrix} G_1 \\ G_2 \end{pmatrix} = \left(\begin{array}{c} J_G \\ \hline 0 \\ G_G \\ 0 \\ G_J \end{array}\right),$$

$$H = (H_1 \quad H_2) = (J_H \mid 0 \quad 0 \quad H_H \quad H_J)\,, \qquad J = J_J\,.$$

This is the procedure followed in [Eis 1]; a somewhat different approach with essentially the same first step and also based on realization over rings is used in [So 2].

7. Output feedback, blending and Stein spaces.

7.1. *Dynamic output feedback.* Consider a scalar (for simplicity) transfer function $T(s) = p(s)/q(s)$. Then the introduction of a dynamic output feedback loop with transfer function $L(s) = a(s)/b(s)$ results, as was mentioned in 3.4 above, in a new system with transfer function

$$\frac{T(s)}{1 - T(s)\,L(s)} = \frac{p(s)b(s)}{b(s)q(s) - a(s)p(s)}\,. \tag{7.1.1}$$

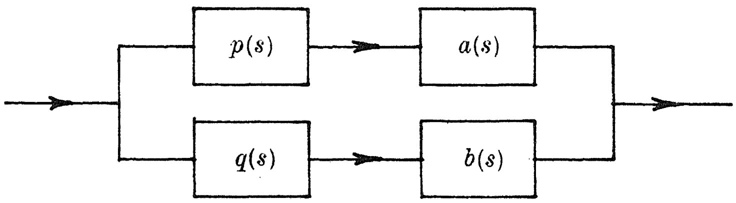
The system described by $T(s) = p(s)/q(s)$, where $p(s)$ and $q(s)$ are without common factors, is *stable* if $q(s)$ has all its roots in the left half plane.

Now suppose that the system $T(s)$ depends on some only approximately known parameters c varying in some compact set C; i.e. we have a certain amount of parameter uncertainty. And suppose that we want to stabilize $T_c(s) = p_c(s)/q_c(s)$ by means of a dynamic output feedback loop $L(s)$ for all c simultaneously. Then our problem is to find polynomials $a(s)$ and $b(s)$ such that all the roots of

$$b(s)\,q_c(s) - a(s)\,p_c(s) \tag{7.1.2}$$

are in the left halfplane for all $c \in C$.

7.2. *The blending problem.* Consider the single input/single output control system represented by



where the transfer polynomials $p(s)$ and $q(s)$ are given, but there is some uncertainty about their parameters, and where it is desired to find polynomials $a(s)$ and $b(s)$ such that the total system has only left halfplane zero's, a property which is sometimes called minimum phase. Thus it is desired to find $a(s)$ and $b(s)$ such that

$$a(s)\,p_c(s) + b(s)\,q_c(s) \tag{7.2.1}$$

has only left halfplane zero's. This has been called the blending problem and mathematically it is the same problem as the dynamic output stabilization problem of 7.1 above.

If it is required that $b(s)$ is minimum phase also one speaks about the strong blending problem. For the dynamic output feedback stabilization variant this corresponds to the requirement that the feedback loop system $L(s)$ be itself stable.

The (strong) blending problem can not always be solved. For instance if there are points d, e in the right half plane such that $p_c(d) = p_c(e) = 0$ for all c and such that $q_c(d)$ cicles around zero as c varies, while $q_c(e)$ is a fixed constant, then the blending problem has no solution ([Ta]).

7.3. *Connection with Stein spaces.* Let E be the right halfplane: then we want to find polynomials $a(s)$, $b(s)$ such that $a(s)p_c(s) + b(s)q_c(s) \neq 0$ for all $s \in E$ and $c \in C$. Let $T_c(s) = p_c(s)/q_c(s)$ and $L(s) = a(s)/b(s)$. Then we want to find a rational $L(s)$ such that $T_c(s) \neq -L(s)$ for all $c \in C$ and $s \in E$. For a fixed c let

$$Z_c = \{(s, T_c(s)) | s \in E\} \subset E \times \boldsymbol{P}^1(\boldsymbol{C}),$$

$$Z'_c = \{(s, T_c(s)) | s \in E,\ T_c(s) \neq \infty\} \subset E \times \boldsymbol{C},$$

and let $Z = \bigcup_c Z_c$, $Z' = \bigcup_c Z'_c$, $Y = E \times \boldsymbol{P}^1(\boldsymbol{C}) \setminus Z$, $Y' = E \times \boldsymbol{C} \setminus Z'$. We have the natural mappings $Y \to E$, $Y' \to E$, induced by $(s, w) \mapsto s$ Solving the blending problem now consists of finding a meromorphic section of $Y \to E$ and a holomorphic section of $Y' \to E$ gives a solution of the strong blending problem. Now it turns out that (op. cit.) Y' is a Stein space, which helps in obtaining some positive results for the blending problems, [Ta].

I should add that in the case that the uncertainty in $T_c(s)$ is of the form $T_c(s) = cT(s)$, where $T(s)$ is a fixed rational function, so that the undertainty is just a gain factor, Tannenbaum in op. cit. gives a complete solution using very different methods (complex interpolation).

8. **Matrix polynomials.**

In this section I briefly discuss a few variations on the theme matrix polynomials. It will be clear, I hope, that the various «morceaux» mentioned below are intimatedly related, though the overall picture does not seem, as yet, to be completely clear.

8.1. *Preliminary remarks concerning matrix polynomials.* Let k be a field. We denote with $k^{p\times m}[s]$ (resp. $k^{p\times m}(s)$) the module of all $p\times m$ matrices with entries in $k[s]$ (resp. $k(s)$) and with $k^m[s]$ (resp. $k^m(s)$) the module of column m-vectors of polynomials (resp. rational functions) in s over k. Matrix multiplication makes $k^{p\times p}[s]$ a ring. An element $U(s)$ of this ring is called *unimodular* if it is invertible in this ring; i.e. if $\det(U(s)) \in k^*$. An element $D(s)$ in $k^{p\times p}[s]$ is called nonsingular if $\det(D(s)) \not\equiv 0$.

A first most useful fact about the ring $k^{p\times p}[s]$ is that it is a left and right principal ideal ring. Thus in particular any two elements A, B have a greatest right common divisor D (that is, there are C, C' such that $A = CD$, $B = C'D$, and if D' is any other common right divisor of A and B then D is a left multiple of D', i.e. of the form $D = ED'$ for some E in $k^{p\times p}[s]$). This greatest common right divisor is simply any generator of the left ideal generated by A and B, and is of course determined up to a left unimodular factor. Similarly there are left greatest common divisors. As an immediate consequence one has:

8.1.1. PROPOSITION: Let $0 \not\equiv T(s) \in k^{p\times m}(s)$ be a matrix of rational functions. Then there are $N(s) \in k^{p\times m}[s]$ and a nonsingular $D(s) \in k^{m\times m}[s]$ such that $T(s) = N(s)D(s)^{-1}$ and such that there are $A(s) \in k^{m\times p}[s]$, $B(s) \in k^{m\times m}[s]$ with $A(s)N(s) + B(s)D(s) = I_m$. These $N(s)$ and $D(s)$ are unique up to a common right unimodular factor.

One interesting fact in this connection is that if $T(s)$ is a strictly proper rational matrix function and $T(s) = N(s)D(s)^{-1}$ is the factorization of 8.1.1 above, then the MacMillan degree of $T(s)$ is the degree of $\det(D(s))$.

8.2. *The disturbance decoupling problem.* Suppose we have a control system with an extra noise input; i.e. we have a set of equations

$$\dot{x} = Fx + Gu + G'w\,, \qquad y = Hx \tag{8.2.1}$$

(or the discrete time version of this). One now tries to find a state space feedback matrix L (cf. also the picture in 2.4 above), such that for the system with this feedback loop

$$\dot{x} = (F + GL)x + Gu + G'w\,, \qquad y = Hx \tag{8.2.2}$$

the output no longer depends on the noise w. In terms of matrix formulas this means that one tries to find a matrix L such that $H(F + GL)^iG' = 0$ for all i.

8.3. *The model matching problem.* The model matching problem is defined as follows: given transfer function matrices $T(s)$, $T'(s)$, find a strictly proper $Q(s)$ such that $T'(s)Q(s) = T(s)$.

I.e. by first processing our inputs by means of $Q(s)$ and then by $T'(s)$ we match exactly the input/output behaviour defined by $T(s)$.

This problem (MMP) and the disturbance decoupling problem (DDP) have been shown to be equivalent in [EH], in the sense that each DDP gives rise to an MMP and vice versa and that the one is solvable iff the other is.

8.4. *F mod G invariant subspaces*, [Wo 1]. Let (F, G, H) be a system of dimension n over a field k. A subspace $V \subset k^n$ is called an F mod G invariant subspace if

$$FV \subset V + \langle G \rangle \tag{8.4.1}$$

where $\langle G \rangle = Gk^m$ is the subspace of k^n spanned by the columns of G. These subspaces are naturally called A mod B invariant subspaces by those who write their equations $\dot{x} = Ax + Bu$, $y = Cx$ rather than $\dot{x} = Fx + Gu$, $y = Hx$; a less notation dependant name is sorely needed.

8.4.2. PROPOSITION, ([Wo 1]): A given DDP has a solution iff there is an F mod G invariant subspace V such that $\langle G' \rangle \subset V \subset \operatorname{Ker} H$.

This rests on the observation that V is an F mod G invariant subspace iff there is a matrix L such that $(F + GL)V \subset V$.

Obviously the sum of two F mod G invariant subspaces is an F mod G invariant subspace. Thus there is a largest F mod G invariant subspace contained in any subspace.

There are still a number of (largely) open problems concerning F mod G invariant subspaces. For instance a description of all of them (of a given dimension r) as, say, a subset of the Grassmann variety $G_{r,n}(k)$. Also open is the problem of finding a good minimal F mod G invariant subspace which contains a given space. (There need not be a smallest one as the intersection of two F mod G invariant subspaces need not be F mod G invariant.)

Geometrically F mod G invariant subspaces V of k^n are those subspaces with the property that once one is in it one can stay in it by a judicious choice of controls. This gives a natural notion of an almost F mod G invariant subspace (as a subspace for which once one is in it one can stay arbitrarily close to it), and this notion then solves an approximate DDP ([Wi 2]).

8.5. *Matrix polynomial factorization.* Consider a matrix polynomial

$$D(s) = A_r s^r + \ldots + A_1 s + A_0 \tag{8.5.1}$$

where the A_i are $m\times m$ matrices. Two such matrix polynomials are said to be equivalent if there exist polynomial unimodular matrices $U(s)$, $V(s)$ such that $D(s) = U(s)E(s)V(s)$.

A *linearization* of $D(s)$ is an $(m+l)\times(m+l)$ matrix L such that $sI_{m+l}-L$ and $D(s)\oplus I_l$ are equivalent. If A_r is invertible such a linearization always exists. One particular one is obtained as follows. Let $A_i' = A_r^{-1}A_i$, $i = 0, 1, \ldots, r-1$, and substitute $T_{r-i} = -A_i'$ in the F matrix of (5.1.3) above to obtain a matrix $F(D)$. Then this matrix $F(D)$ is a linearization of dimension rm. Of course equivalent matrix polynomials have the same sets of linearizations, but here it is also true that all linearizations of $D(s)$ of dimension rm are similar ([GLR 1]). Gohberg a.o. ([GLR 1-5, GMR, GKV, GKL]) make this notion of linearization a cornerstone of their (spectral) analysis of operator polynomials and in their study of factors and multiples of such polynomials. E.g. by theorem 8 of [GLR 1] there is a nice correspondence between monic factors of $D(s)$ (still assuming A_r to be invertible) and certain $F(D)$ invariant subspaces.

It is not true however, that every matrix polynomial is linearizable in this sense. For instance if A is nilpotent then a contradiction is obtained by taking determinants on both sides of the equation

$$(I_m + sA)\oplus I_l = U(s)(sI_{m+l}-L)V(s)\,.$$

(But it is true that one can always find L, M such that $(D(s)\oplus I_l)$ is equivalent to $L - sM$, cf. [GKL].)

Now this linearization described above (by a block companion matrix) is a special case of what has been called the Fuhrmann model of a matric polynomial ([Fu 1]), which is what we describe next.

For each rational function $f(s)\in k(s)$ let $\pi f(s)$ be its strictly proper part; i.e. if $f(s) = p(s)/q(s)$, $p(s), q(s)\in k[s]$, write $p(s) = n(s)q(s) + r(s)$ with degree $r(s) <$ degree $q(s)$ and define $\pi f(s) = r(s)/q(s)$. We use the same notation for the analogous map $k^m(s)\to k^m(s)$. Now let $D(s)$ be a nonsingular matrix polynomial (with $m\times m$ matrices as coefficients) and define

$$\pi_D\colon k^m[s]\to k^m[s]\,, \qquad \pi_D f = D\pi(D^{-1}f) \tag{8.5.2}$$

(If $n(f)$ is the integral part of $D^{-1}f$, then $\pi_D f = f - Dn(f)$, showing that $\pi_D f$ is indeed polynomial again.) This map is a projection with kernel $Dk^m[s]$. Its image $V(D)$ is a vectorspace of dimension degree det $(D(s))$. Now define

$$(8.5.3) \qquad F(D)\colon V(D) \to V(D)\,, \qquad f \mapsto \pi_D(sf)$$

which gives $V(D)$ a $k[s]$ module structure for which $V(D) \simeq k^m[s]/ /Dk^m[s]$. (Of course, abstractly $(V(D), F(D))$ is simply this quotient module.)

8.5.4. PROPOSITION ([Fu1, Theorem 8.8]): Let $D(s)$, $D'(s)$ be $m\times m$ matrix polynomials. Then $F(D)$ and $F(D')$ are similar if and only if $D(s)$ and $D'(s)$ are equivalent.

Thus it is not unreasonable to expect that the invariant subspaces of $F(D)$ and the polynomial factors of $D(s)$ correspond. This does indeed turn out to be the case ([Ant, EH]). The Fuhrmann model of $D(s)$ is also closely related to realization theory. In fact if $D(s)^{-1}$ is proper (and by changing, if necessary, $D(s)$ by a unimodular factor this can always be assured) then $F(D)$ is the F matrix of a minimal dimensional realization (F, G, H, J) of $D(s)^{-1}$. This fact, together with the remark that the F mod G invariant subspaces are the $(F+GL)$ invariant subspaces for some L, lies at the basis of a correspondence between factors of $D(s)$ and F mod G invariant subspaces ([EH, FW]).

9. The feedback group and its invariants.

9.1. *The feedback group and the Kronecker indices.* In this and the following subsection we consider control systems $\dot{x} = Fx + Gu$ rather than input/output systems $\dot{x} = Fx + Gu$, $y = Hx$, and we consider a larger group of transformations than just state space isomorphisms viz. the socalled feedback group, which is generated by « base change in state space », base change in input space and « state space feedback ». More precisely let $L_{m,n}(k)$ be the set of all pairs of matrices over k of dimensions $n\times n$ and $n\times m$, and let $L^{cr}_{m,n}(k)$ be the subset of all completely reachable pairs. Then the feedback group acting on these spaces, is generated by the transformations

$$(9.1.1) \quad (F, G) \mapsto (S^{-1}FS, SG)\,, \quad S\in \boldsymbol{GL}^n(k) \text{ (state space base change)}\,,$$

$$(9.1.2) \quad (F, G) \mapsto (F, GT^{-1})\,, \quad T\in \boldsymbol{GL}_m(k) \text{ (input space base change)}\,,$$

$$(9.1.3) \quad (F, G) \mapsto (F + GL, G)\,, \quad L\in k^{mn} \text{ (state space feedback)}\,.$$

This group is readily seen to be a linear algebraic group, viz. the closed subgroup of GL_{m+n} of all matrices of the form

$$\begin{pmatrix} S & 0 \\ L & T \end{pmatrix}$$

acting as follows

$$\left(\begin{pmatrix} S & 0 \\ L & T \end{pmatrix}, (F, G)\right) \mapsto (SFS^{-1} + SGT^{-1}LS^{-1}, SGT^{-1}) .$$

Let $\tilde{\varkappa}(F, G)$ be the Kronecker nice selection defined in 4.3 above (which was independant of the matrix H). Now let $\varkappa(F, G) = (\varkappa_1(F, G), \ldots, \varkappa_m(F, G))$ be the set of numbers $\tilde{\varkappa}(F, G)$ arranged according to magnitude with the largest one first. So in the example of 4.3 above we have $\varkappa_1 = 3$, $\varkappa_2 = 2$, $\varkappa_3 = 1$, $\varkappa_4 = 0$.

We claim that the $\varkappa_i(F, G)$ are invariant under the feedback group. This can be seen as follows. Let d_i be the dimension of the subspace of k^n generated by the columns of the matrices $G, FG, \ldots, F^{i-1}G$, $i = 1, 2, \ldots, n$. Then the d_i are clearly invariant under the transformations (9.1.1)-(9.1.3). But the d_i determine the $\varkappa_i$ as follows. Let $e_i = d_i - d_{i-1}$, $i = 2, \ldots, n$, $e_1 = d_1$. Then $\varkappa_1$ is the number of e_i which are $\geqslant 1$, $\varkappa_2$ is the number of e_i which are $\geqslant 2, \ldots, \varkappa_m$ is the number of e_i which are $\geqslant m$. (An inversely the $\varkappa_i$ determine the e_i by analogous rules and hence the d_i.) Thus the $\varkappa_i$ are indeed invariants.

9.2. *The block companion canonical form.* In this subsection we show that all the elements in $O(\tilde{\varkappa})$, which is the set of all (F, G) such that $\tilde{\varkappa}(F, G) = \tilde{\varkappa}$, can be brought into a certain special form by transformations which vary continuously with the parameters of (F, G) (as long as (F, G) varies within a fixed $O(\tilde{\varkappa})$), a result which we shall also need in section 10 below. We shall assume that $\tilde{\varkappa}_1 + \ldots + \tilde{\varkappa}_m = n$, which is equivalent to $O(\tilde{\varkappa}) \subset L^{cr}_{m,n}$, and which is necessary for the arguments below. The «proof» is by a, hopefully, sufficiently complicated example. For even more details cf. [Haz 3, Ka 1]. In fact below there are already more details than is normally appropriate for a survey type paper, for which I apologize. We shall need, however, the fact that this construction is continuous in section 10 below to give a new proof of a theorem of Chris Byrnes. In view of the plethora of constructions in the field which are discontinuous it seemed worthwhile to make it absolutely clear that this one is continuous for a change.

For the sufficiently complicated example we shall take $m = 4$, $n = 6$ and $\tilde{\varkappa} = (2, 3, 0, 1)$, so that the corresponding pattern of dots and crosses looks like

$$(9.2.1)\qquad \begin{array}{cccccccc} \times & \times & \cdot & \cdot & \cdot & \cdot & \cdot \\ \times & \times & \times & \cdot & \cdot & \cdot & \cdot \\ \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot \\ \times & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot \end{array}$$

By the definition of the pattern $\tilde{\varkappa}$ we have for each $i = 1, 2, 3, 4$ a relation

$$(9.2.2)\qquad F^{\tilde{\varkappa}_i} g_i + \sum a_{ij}^k F^k g_j = 0$$

where the sum on the left runs over all $(k, j) \in \tilde{\varkappa}$ such that $(\tilde{\varkappa}_i, i) > (k, j)$ in the lexicographic order on $J_{m,n}$, cf. 4.3 above. ($J_{m,n}$ is the set of all pairs (i, j), $i = 0, 1, \ldots, n$; $j = 1, \ldots, m$.)

A first preliminary step is now to find an $m \times m$ matrix $T(F, G)$ which is upper diagonal (with ones on the diagonal) such that if we write down the corresponding relations for the pair $(F, G') = (F, GT(F, G))$ then (9.2.2) has $a_{ij}^k = 0$ for all $k \geqslant \tilde{\varkappa}_i$. In our example the relevant four relations are

$$(9.2.3)\qquad \begin{cases} F^2 g_1 + (a_{11}^1 F g_1 + a_{12}^1 F g_2) + (a_{11}^0 g_1 + a_{12}^0 g_2 + a_{14}^0 g_4) = 0 \\ F^3 g_2 + (a_{22}^2 F^2 g_2) + (a_{21}^1 F g_1 + a_{22}^1 F g_2) + \\ \qquad\qquad + (a_{21}^0 g_1 + a_{22}^0 g_2 + a_{24}^0 g_4) = 0 \\ g_3 + (a_{31}^0 g_1 + a_{32}^0 g_2) = 0\,, \\ F g_4 + (a_{41}^1 F g_1 + a_{42}^1 F g_2) + (a_{41}^0 g_1 + a_{42}^0 g_2 + a_{44}^0 g_4) = 0\,. \end{cases}$$

Note that for example the third relation does not involve g_4 by the definition of $\tilde{\varkappa}$. In this case $T(F, G)$ is the matrix

$$(9.2.4)\qquad T(F, G) = \begin{pmatrix} 1 & 0 & a_{31}^0 & a_{41}^1 \\ 0 & 1 & a_{32}^0 & a_{42}^1 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix}.$$

Note that we are only using those a_{ij}^k for which $k = \tilde{\varkappa}_i$, note that $T(F, G)$ comes out to be upper diagonal because in (9.2.2) $a_{ij}^k = 0$ if $(k, j) \geqslant (\tilde{\varkappa}_i, i)$, and finally note that a transformation $(F, G) \mapsto (F, GT)$ does not change $\tilde{\varkappa}$ provided T is upper diagonal (even though in general a base change transformation in input space does change the Kronecker selection $\tilde{\varkappa}$ even it if leaves the Kronecker indices $\varkappa$ unchanged). Now let $G' = GT(F, G)$, then an easy check shows that in the relations for the pair (F, G') corresponding to

(9.2.2) we have $a'^k_{ij} = 0$ if $k = \tilde{\varkappa}_i$, so that

$$F^{\varkappa_i} g'_i + \sum a'^k_{ij} F^k g_j = 0 \tag{9.2.5}$$

where now the sum runs over all $(j, k) \in$ [illegible] for which $k < \tilde{\varkappa}_i$. We now define a new basis $(b_1, \dots, b_n)$ of k^n such that with respect to this basis F and G' look like

$$F'' = \left(\begin{array}{cc|ccc|c} 0 & 1 & 0 & 0 & 0 & 0 \\ * & * & * & * & * & * \\ \hline 0 & 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 & 0 \\ * & * & * & * & * & * \\ \hline * & * & * & * & * & * \end{array}\right), \qquad G'' = \begin{pmatrix} 0 & 0 & 0 & 0 \\ 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix}. \tag{9.2.6}$$

To this end we use the relations (9.2.5) which written out in our example result from the formulas (9.2.3) by replacing g_i with g'_i, and a^k_{ij} with a'^k_{ij} for $k < \tilde{\varkappa}_i$, and by setting $a^0_{31} = a^0_{32} = a^1_{41} = a^1_{42} = 0$. Now define

$$\begin{aligned}
b_1 &= Fg'_1 + a'^1_{11} g'_1 + a'^1_{12} g'_2 , \\
b_2 &= g'_1 , \\
b_3 &= F^2 g'_2 + a'^2_{22} F g'_2 + a'^1_{21} g'_1 + a'^1_{22} g'_2 , \\
b_4 &= Fg'_2 + a'^2_{22} g'_2 , \\
b_5 &= g'_2 , \\
b_6 &= g'_4 .
\end{aligned}$$

Note that the three groups of basis vectors b_1, b_2; b_3, b_4, b_5; b_6 (corresponding to the three nonzero $\tilde{\varkappa}_1, \tilde{\varkappa}_2, \tilde{\varkappa}_4$) are obtained by « dividing as best as one can » the left hand sides of the first; second; fourth equation of (9.2.3) by F, F^2; F, F^2, F^3; F.

Now let L be the 4×6 matrix whose first row is the second row of F'', whose second row is the fifth row of F'', whose third row is zero, and whose fourth row is equal to the sixth row of F''. Then $(F'' - G''L, G'')$ looks like (9.2.6) with all the $*$'s replaced by zero's.

Finally let S be the permutation matrix consisting of the columns $e_3, e_4, e_1, e_2, e_3, e_6$ where the e_i are the standard basis vectors in k^6, and let T be the 4×4 permutation matrix formed by the standard basis vectors of k^4 in the order 1, 2, 4, 3. Then

$$(9.2.7)\quad \left\{\begin{aligned} SF''S^{-1} &= \left(\begin{array}{ccc|cc|c} 0 & 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \\ \hline 0 & 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \\ \hline 0 & 0 & 0 & 0 & 0 & 0 \end{array}\right), \\ SG''T &= \begin{pmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \end{pmatrix}, \end{aligned}\right.$$

which matrices depend only on the Kronecker indices $\varkappa_1, \varkappa_2, \varkappa_3, \varkappa_4$.

9.2.8. Corollary ([Bru 2, WM, Ros, Kal]). The Kronecker indices $\varkappa_i$ are the only invariants of the feedback group acting on $L^{cr}_{m,n}$.

For results concerning the feedback group acting on $L^{cr}_{m,n,p}$ cf. e.g. [WD]. The form (9.2.7) has been called Brunovsky canonical form.

9.2.9. Remark: Note that on $O(\tilde{\varkappa})$, the set of all pairs (F, G) such that $\tilde{\varkappa}(F, G) = \tilde{\varkappa}$ the construction is clearly continuous. On $O(\varkappa)$, the orbit of the feedback group labelled by $\varkappa$, the construction is in fact not continuous in general.

9.2.10. Remark: The quotient map $L^{cr}_{m,n} \to \{\varkappa\}$ is continuous if the set of Kronecker indices $\{\varkappa\}$ is given the topology belonging to the partial order $(\varkappa \geqslant \varkappa') \Leftrightarrow (\varkappa_1 \leqslant \varkappa'_1$ and $\varkappa_1 + \varkappa_2 \leqslant \varkappa_1 + \varkappa'_2$ and ... and $\varkappa_1 + \dots + \varkappa_m \leqslant \varkappa'_1 + \dots + \varkappa'_m)$, and this is then in fact the quotient topology. This is the same order of partitions of n as turns up in the study of degeneration of vectorbundles over algebraic varieties ([Sh, Theorem 3]), which fact is explained by what comes next in subsection 9.3; it is also the same order which turns up in the theory of the representations of the symmetric groups ([Sn, LVi]), an « accident », which still needs explaining (1) and it is also the degeneration order

(1) This has meanwhile been done: Hazewinkel and Martin, Jan 1980, to appear. (Footnote added March 1980).

among the orbits whose closure contains zero for SL_n acting on its Lie algebra by the adjoint action, ([Ger, Hes]). Cf. [Bry] for yet more occurences of this partial order in various parts of mathematics.

9.3. *The Martin-Hermann vectorbundle of a system.* Now let $\Sigma = (F, G, H) \in L^{\mathrm{cr,co}}_{m,n,p}(\boldsymbol{C})$ be a cr and co input/output system, and let $T_\Sigma(s)$ be its transfer function, and write

$$T_\Sigma(s) = N(s)\,D(s)^{-1} \tag{9.3.1}$$

with $N(s)$ and $D(s)$ right coprime matrices of respective dimensions $p\times m$ and $m\times m$, $D(s)$ nonsingular; cf. 8.1 above.

Let $G_{m,m+p}$ be the complex Grassmann variety of complex m-planes in complex $m+p$ space. Define

$$\varphi_\Sigma\colon \boldsymbol{P}^1(\boldsymbol{C}) \to G_{m,m+p} \tag{9.3.2}$$

by the formula

$$\begin{cases} \varphi_\Sigma(s) &= \{(N(s)u,\, D(s)u) \mid u \in \boldsymbol{C}\}\,, \\ \varphi_\Sigma(\infty) &= \{(0, u) \mid u \in \boldsymbol{C}\}\,. \end{cases} \tag{9.3.3}$$

This defines a continuous, and in fact a holomorphic, morphism.

9.3.4. Proposition, ([HM 3]): The MacMillan degree of $T_\Sigma(s)$, i.e. the degree of $\det D(s)$, i.e. the dimension of Σ, is equal to the intersection number of $\varphi_\Sigma(\boldsymbol{P}^1(\boldsymbol{C}))$ with the hyperplane at infinity in $G_{m,m+p}$.

Let $E' \to G_{m,m+p}$ be the canonical m-dimensional bundle over the Grassmann variety whose fibre over x is the m-plane represented by x, and let E over $G_{m,m+p}$ be the dual bundle to E'. Define $E(\Sigma)$ over $\boldsymbol{P}^1(\boldsymbol{Z})$ as the pullback of E by means of φ_Σ. Now by [Gro] every holomorphic m-dimensional bundle E over the Riemann sphere $\boldsymbol{P}^1(\boldsymbol{C})$ splits as a sum of line bundles and is classified (up to isomorphism) by m integers $K(E) = (K_1(E), \ldots, K_m(E))$, $K_1(E) \geqslant \ldots \geqslant K_m(E)$, where the $K_i(E)$ are the degrees of the line bundles in question; i.e. up to isomorphism a holomorphic bundle on $\boldsymbol{P}^1(\boldsymbol{C})$ is a direct sum $\oplus\, \mathcal{O}(\varkappa_i)$.

9.3.5. Theorem, ([HM 3]): $\varkappa(\Sigma) = K(E(\Sigma))$.

9.4. *The Kronecker matrix pencil of a control system.* A pencil of matrices over a field k is a polynomial matrix of degree 1

$$K(s) = A + Bs\,. \tag{9.4.1}$$

Two such pencils K, K' are said to be equivalent if there exist invertible matrices $P \in k^{m\times m}$, $Q \in k^{r\times r}$ such that $K' = PKQ$. Kronecker ([Kro]) classified such pencils, cf. also [Ga, Her 1]. Now let $\Sigma = (F, G)$ be a control system and associate to it the $n\times(m+n)$ pencil

$$K_\Sigma(s) = (G \vdots sI - F) . \tag{9.4.2}$$

Let $\Sigma' = (F', G')$ be a second control system. Partitioning Q as indicated below and considering the equation

$$(G' \vdots sI - F') = P(G \vdots sI - F)\begin{pmatrix} Q_{11} & Q_{12} \\ Q_{21} & Q_{22} \end{pmatrix} \tag{9.4.3}$$

it readily follows that $Q_{21} = 0$, $Q_{22} = P^{-1}$ so that $G' = PGQ_{11}$, $F' = PFP^{-1} - PGQ_{12}$, so that the pencils $K_\Sigma(s)$ and $K_{\Sigma'}(s)$ are equivalent iff the control systems Σ and Σ' are feedback equivalent, i.e. equivalent under the feedback group.

Most of the invariants of Kronecker for the classification of matrix pencils are zero for pencils of the form (9.4.2). The remaining ones are certain nonnegative integers which are precisely the numbers $\varkappa_1(\Sigma), \ldots, \varkappa_m(\Sigma)$ ([Kal]), whence the names « Kronecker indices » for $\varkappa(\Sigma)$ and « Kronecker selection » for $\tilde{\varkappa}(\Sigma)$.

10. Pole placement and coefficient assignability.

10.1. *Coefficient assignability over a field.* Let R be a ring and let $\Sigma = (F, G, H)$ be a system over R. Let $\chi(\Sigma) = \chi(F) = \det(sI_n - F)$ (the characteristic polynomial of Σ). The system is said to be *coefficient assignable* if for all $a_1, \ldots, a_n \in R$ there is a state feedback matrix L such that

$$\chi(F + GL) = s^n + a_1 s^{n-1} + \ldots + a_n .$$

A slightly weaker property is *pole assignability* which means that for all $b_1, \ldots, b_n \in R$ there is an L such that

$$\chi(F + GL) = (s - b_1) \ldots (s - b_n) .$$

Because $T_\Sigma(s) = H(sI - F)^{-1}G$ these properties (and their weaker variants of which stabilizability, cf. 7.1 above, is one) say things about how the poles of the transfer function can be shifted. Over a field things are quite clear.

10.1.1. Proposition, ([Wo 2]): Let k be a field, then a system over k is pole assignable iff it is coefficient assignable iff if it is cr.

This follows fairly immediately from the Brunovsky canonical form discussed above in 9.2.

There are of course entirely straightforward definitions of pole assignability and coefficient assignability for families of systems, which fit with the ones for systems over rings when a system over a ring is viewed as a family.

10.2. *Pole placement over a ring.* Over a ring R things are not so simple, and in fact largely unsettled. Two easy facts are

10.2.1. Lemma: If $m = 1$ then coefficient assignability is equivalent to cr (meaning that $R(F, G)$ defines a surjective map $R^r \to R^n$, $r = m(n+1)$).

10.2.2. Lemma. If Σ over R is pole assignable then Σ is cr.

In general it is not known whether cr implies pole assignability but over a ring with only finitely many maximal ideals it is still true that cr implies coefficient assignability ([So 1]), which takes care of the case of linear sequential circuits (where R is finite). For $R = k[\sigma]$, polynomials in *one* variable over a field, Steve Morse ([Mo]) has shown that cr implies pole assignability, a result which then (cf. section 2 above) also says things about the stabilization of delay-differential systems with only one delay operator. Morse's result holds more generally over principal ideal domains. There is also a simple example that shows that over $k[\sigma]$ cr need not imply coefficient assignability, [BS].

Apart from a result for polynomial families (and more generally for systems over rings which are projective free) which we describe below, this is about all that is known. Let me remark though that when $m = 1$ and Σ is not cr, Wyman in [Wy] describes the extent to which the system fails to be coefficient assignable in terms of a certain Ext group.

10.3. *Coefficient assignability for polynomial families.* In this subsection I give a new proof of the following theorem of Chris Byrnes.

10.3.1. Theorem, [By 4]: Let $\Sigma(\sigma)$ be a polynomial family of systems over a field k parametrized by $\sigma_1, \ldots, \sigma_r$ (or, equivalently) let $\Sigma(\sigma)$ be a cr system over $k[\sigma_1, \ldots, \sigma_r]$ (Quillen-Suslin theorem). Suppose that the sets of Kronecker indices of $\Sigma(\sigma)$ are constant as functions of σ for all values of $\sigma \in \bar{k}^r$, where $\bar{k}$ is the algebraic closure of k. Then $\Sigma(\sigma)$ is coefficient assignable.

Proof: Let $\Sigma = (F, G, H)$ and let $d_i(\sigma)$ for all $\sigma \in \bar{k}^r$ be the dimension of the subspace of $\bar{k}^n$ spanned by the columns of the matrices $G(\sigma)$, $F(\sigma)G(\sigma)$, ..., $F(\sigma)^{i-1}G(\sigma)$. Then, cf. 9.1 above, the hypothesis that the $\varkappa_i(\sigma) = \varkappa_i(\Sigma(\sigma))$ are constant implies that the $d_i(\sigma)$ are also constant. For $i = 1$ this means that $E_1 = \{(\sigma, \langle G(\sigma)\rangle)\}$, where $\langle M\rangle$ is the subspace spanned by the columns of the matrix M, is a vector sub*bundle* of the trivial n dimensional bundle over affine r space. By the Quillen-Suslin theorem this means that there is an invertible matrix T_1 with coefficients in $k[\sigma]$ such that the first d_1 columns of $G(\sigma)T_1$ are linearly independent for all σ. Because $d_2(\sigma)$ is also constant $E_2 = \{(\sigma, \langle G(\sigma), F(\sigma)G(\sigma)\rangle)\}$ is also a vectorbundle and applying the Quillen-Suslin theorem again we have that the quotient bundle E_2/E_1 is free. This one is generated fibre-wise by the first d_1 columns of $F(\sigma)G(\sigma) \bmod \langle G(\sigma)\rangle$, which means that there is a matrix T_2 with coefficients in $k[\sigma]$ of the form

$$T_2 = \begin{pmatrix} T_2' & 0 \\ 0 & I \end{pmatrix}$$

where T_2' is a $d_1 \times d_1$ matrix, such that the first $d_2 - d_1$ columns of $F(\sigma)G(\sigma)T_1T_2$ generate the fibre at σ of E_2/E_1, and because T_2, so to speak, only acts on the first d_1 columns it is still true that the first d_1 columns of $G(\sigma)$ generate the fibres of E_1. In terms of the Kronecker *selection* this means that after two base changes in input space we have arranged things in such a way that the first two columns of the Kronecker selection $\tilde{\varkappa}(\Sigma(\sigma))$ for all $\sigma \in \bar{k}^r$ look like

$$\begin{array}{l} d_1\left\{\begin{array}{l} \left.\begin{array}{cc} \times & \times \\ \vdots & \vdots \\ \times & \times \end{array}\right\} d_2 - d_1 \\ \begin{array}{cc} \times & \bullet \\ \vdots & \vdots \\ \times & \bullet \end{array} \end{array}\right. \\ \qquad \begin{array}{cc} \bullet & \bullet \\ \vdots & \vdots \\ \bullet & \bullet \end{array} \end{array}$$

Continuing in this way (the next matrix, T_3, is of the form

$$T_3 = \begin{pmatrix} T_3' & 0 \\ 0 & I \end{pmatrix}$$

with T_3' a $(d_2 - d_1) \times (d_2 - d_1)$ matrix we see that by a polynomial base change T in input space we can see to it that the Kronecker selection of $(F(\sigma), G(\sigma)T)$ is constant. But then, by means of the construction which we so elaborately described in 9.2 above we can bring $\Sigma(\sigma)$ in the Brunovsky canonical form (9.2.7) by means of polynomial base changes and polynomial feedback. A further polynomial feedback operation then puts precisely those polynomials in the $*$-spots in (9.2.6) which we need, proving the theorem.

The original proof of this theorem ([By 4]) relies instead of on the Quillen-Suslin theorem on results of Hanna ([Han]) on decompositions of vector bundles which are applied to the family of Martin-Hermann bundles (cf. 9.3 above) which is defined by the family $\Sigma(\sigma)$. Of course the proof given above works over any ring over which all finitely generated projective bundles are free; the same proof also gives, of course, results for continuous (differentiable) families over homotopically trivial spaces (manifolds).

By the interpretation of delay-differential systems as polynomial families of systems Theorem 10.3.1 tells us things about the stabilization of delay systems (which are in principle infinite dimensional gadgets, showing the power of the family interpretation). For these systems the proof of the theorem has the following corollary.

10.3.2. Corollary: If $\Sigma(\sigma)$ is a delay-differential system such that the conditions of the theorem hold for the associated polynomial family of systems, then the system $\Sigma(\sigma)$ is up to feedback equivalent to a system involving no delays.

10.4. *Pole placement for delay systems.* Let $\Sigma(\sigma)$ be a delay-differential system. Assume, which is reasonable and even customary in many cases, that all the functions $x(t)$, $u(t)$, $y(t)$ are zero for t far enough in the past. Then it makes perfect sense to talk about base changes and feedback by means of matrices which are power series over the real numbers in the delay operators $\sigma_1, \ldots, \sigma_r$. Now this ring of power series is local and hence certainly projective free so that the proof of Theorem 10.3.1 gives coefficient assignability and stabilization results for delay systems for which the two Kronecker indices $\varkappa_Q(\Sigma)$ and $\varkappa(\Sigma_0)$ are equal. Here $\varkappa_Q(\Sigma)$ is the set of Kronecker indices of $\Sigma(\sigma)$ considered as a system over the quotient field $\boldsymbol{R}(\sigma_1, \ldots, \sigma_r)$ and $\varkappa(\Sigma_0)$ is the set of Kronecker indices of the residual system over $\boldsymbol{R}$ obtained from $\Sigma(\sigma)$ by setting all the σ_i equal to 0.

11. The (canonical) completely reachable subsystem.

11.1. Σ^{cr} *for systems over fields.* Let $\Sigma = (F, G, H)$ be a system over a field k. Let X^{cr} be the image of $R(F, G)\colon k^r \to k^n$, $r = m \cdot (n+1)$. Then obviously $F(X^{cr}) \subset X^{cr}$, $G(k^m) \subset X^{cr}$, so that there is an induced subsystem $\Sigma^{cr} = (X^{cr}; F', G', H')$ which is called the canonical cr subsystem of Σ. In terms of matrices this means that there is an $S \in \boldsymbol{GL}_n(k)$ such that Σ^S has the form

$$\Sigma^s = \left(\begin{pmatrix} G_1 \\ 0 \end{pmatrix}, \begin{pmatrix} F_{11} & F_{12} \\ 0 & F_{22} \end{pmatrix}, (H_1 \quad H_2)\right) \tag{11.1.1}$$

with $(F_{11}, G_1, H_1) = \Sigma^{cr}$, the canonical cr subsystem. The words Kalman « decomposition » are also used in this context. There is a dual construction relating to co and combining these two constructions « decomposes » the system into four parts.

In this section we examine whether this construction can be globalized, i.e. we ask whether this construction is continuous, and we ask whether something similar can be done for time varying linear dynamical systems.

11.2. Σ^{cr} *for time varying systems.* Now let $\Sigma = (F, G, H)$ be a time varying system, i.e. the coefficients of the matrices F, G, H are alowed to vary, say continuously, with time. For time varying systems the controlability matrix $R(\Sigma) = R(F, G)$ must be redefined as follows

$$R(F, G) = \big(G(0) \vdots G(1) \vdots \ldots \vdots G(n)\big) \tag{11.2.1}$$

where

$$G(0) = G\,; \qquad G(i) = FG(i-1) - \dot{G}(i-1) \tag{11.2.2}$$

where the $\dot{}$ denotes differentiation with respect to time, as usual. Note that this gives back the old $R(F, G)$ if F, G do not depend on time. The system is said to be cr if this matrix $R(\Sigma)$ has full rank. These seem to be the appropriate notions for time varying systems; cf. e.g. [We, Haz 5] for some supporting results for this claim.

A time variable base change $x' = Sx$ changes Σ to Σ^S with

$$\Sigma^S = (SFS^{-1} + \dot{S}S^{-1}, SG, HS^{-1})\,. \tag{11.2.3}$$

Note that $R(\Sigma)$ hence transforms as

$$R(\Sigma^S) = SR(\Sigma)\,. \tag{11.2.4}$$

11.2.5. THEOREM: Let Σ be a time varying system with continuously varying parameters. Suppose that rank $R(\Sigma)$ is constant as a function of t. Then there exists a continuous time varying matrix S, invertible for all t, such that Σ^S has the form (11.1.1) with (F_{11}, G_1, H_1) cr.

Proof: Consider the submodule of the trivial $(n+1)m$ dimensional bundle over the real line generated by the rows of $R(\Sigma)$. This is a vectorbundle because of the rank assumption. This bundle is trivial. It follows that there exist r sections of the bundle, where $r = \operatorname{rank} R(\Sigma)$, which are linearly independant everywhere. The continuous sections of the bundle are of the form $\sum a_i(t) z_i(t)$, where $z_1(t), \ldots, z_n(t)$ are the rows of $R(\Sigma)$ and the $a_i(t)$ are continuous functions of t. Let $b_1(t), \ldots, b_r(t)$ be the r everywhere linearly independant sections and let $b_j(t) = \sum a_{ji}(z) z_i(t)$, $j = 1, \ldots, r$; $i = 1, \ldots, n$.

Let E' be the r dimensional subbundle of the trivial bundle E of dimension n over the real line generated by the r row vectors $a_j(t) = (a_{j1}(t), \ldots, a_{jn}(t))$. Because the quotient bundle E/E' is trivial we can complete the r vectors $a_1(t), \ldots, a_r(t)$ to a set of n vectors $a_1(t), \ldots, a_n(t)$ such that the determinant of the matrix formed by these vectors is nonzero for all t. Let $S_1(t)$ be the matrix formed by these vectors, then $S_1R(\Sigma)$ has the property that for all t its first r rows are linearly independent and that it is of rank r for all t. It follows that there are unique continuous functions $c_{ki}(t)$, $k = r+1, \ldots, n$; $i = 1, \ldots, r$ such that $z'_k(t) = \sum c_{ki}(t) z'_i(t)$, where $z'_j(t)$ is the j-th row of $S_1R(\Sigma)$. Now let

$$S_2(t) = \begin{pmatrix} I_r & 0 \\ -C(t) & I_{n-r} \end{pmatrix}$$

where $C(t)$ is the $(n-r)\times r$ matrix with entries $c_{ki}(t)$.

Then $S(t) = S_2(t)S_1(t)$ is the desired transformation matrix (as follows from the transformation formula (11.2.4)).

Virtually the same arguments give a smoothly varying $S(t)$ if the coefficients of Σ vary smoothly in time, and give a polynomial $S(t)$ if the coefficients of Σ are polynomials in t (where in the latter case we need the constancy of the rank also for all complex values of t and use that projective modules over a principal ideal ring are free).

11.3. Σ^{cr} *for families.* For families of systems these techniques give

11.3.1. Theorem: Let Σ be a continuous family parametrized by a contractable topological space (resp. a differentiable family parametrized by a contractible manifold; resp. a polynomial family). Suppose that the rank of $R(\Sigma)$ is constant as a function of the parameters. Then there exists a continuous (resp. differentiable; resp. polynomial) family of invertible matrices S such that Σ^S has the form (11.1.1) with (F_{11}, G_1, H_1) a family of cr systems.

The proof is virtually the same as the one given above of theorem 11.2.5; in the polynomial case one of course relies on the Quillen-Suslin theorem again to conclude that the appropriate bundles are trivial. Note also that, inversely, the existence of an S as in the theorem implies that the rank of $R(\Sigma)$ is constant.

For delay-differential systems this gives a « Kalman decomposition » provided the relevant, obviously necessary, rank condition is met. There is also again a power series version of this result (as in 10.4) which requires a far weaker hypothesis.

Another way of proving Theorem 11.3.1 for systems over certain rings rests on the following lemma which is also a basic tool in the study of isomorphisms of families in [HP] and which implies a generalization of the main lemma of [OS] concerning, the solvability of sets of linear equations over rings.

11.3.2. Lemma: Let R be a reduced ring (i.e. there are no nilpotents $\neq 0$) and let A be a matrix over R. Suppose that the rank of $A(\mathfrak{p})$ over the quotient field of $R/\mathfrak{p}$ is constant as a function of $\mathfrak{p}$ for all prime ideals $\mathfrak{p}$. Then $\operatorname{Im}(A)$ and $\operatorname{Coker}(A)$ are projective modules.

Now let Σ over R be such that $\operatorname{rank} R(\Sigma(\mathfrak{p}))$ is constant and let R be projective free (i.e. all finitely generated projective modules over R are free). Then $\operatorname{Im} R(\Sigma) \subset R^n$ is projective and hence free. Taking a basis of $\operatorname{Im} R(\Sigma)$ and extending it to a basis of all of R^n, which can be done because $R^n/\operatorname{Im} R(\Sigma) = \operatorname{Coker} R(\Sigma)$ is projective and hence free, now gives the desired matrix S.

There is a complete set of dual theorems concerning co.

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