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ON FORMAL GROUPS. THE FUNCTIONAL EQUATION LEMMA AND SOME OF ITS APPLICATIONS.

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1. INTRODUCTION. Let R be a ring and let F(X,Y) be an n-dimensional commutative formal group law over R. Assume that R is torsion free and let f(X) over $R \otimes \mathbb{Q}$ be the logarithm of F(X,Y). Roughly, the functional equation lemma to be discussed below says what kind of regularity $f(X) \in R \otimes \mathbb{Q}[[X]]^n$ must exhibit in order that it be the logarithm of a formal group law with coefficients in R. The precise statement of the lemma is in section 2 below. The lemma turns out to have many more applications (then just the construction of universal formal group laws). It is the purpose of the present paper to outline a few of these and to try to convince the reader of the power of the lemma in proving a large variety of integrality statements. (Because commutative formal group laws over $\mathbb{Q}$-algebras are trivial, the theory of commutative formal group laws over torsion free rings is largely a matter of integrality statements). To cite of few instances: the integrality of the addition and multiplication polynomials of the Witt vectors, the Atkin-Swinnerton Dyer congruences, the construction of generalized Lubin-Tate formal group laws ("tapis de Cartier") can all be seen as applications of the functional equation lemma. Many more applications of the functional equation lemma can be found in [7] and [8]. This paper contains no new results or proofs which are not also in [7], with the exception of the proof of "$v(M,\eta)(X)$ reduces to $V(X)$" in section 6 below, which in [7] is done in a needlessly cumbersome fashion.

2. THE FUNCTIONAL EQUATION LEMMA. The ingredients we need are the following

(2.1) $B \subset L$, $\mathfrak{a} \subset B$, $\sigma : L \to L$, p, q, s_1, s_2, ...

Here B is a subring of a ring L, $\mathfrak{a}$ is an ideal in B, σ a ring endomorphism of L, p is a prime number, q is a power of p and the s_i, i = 1,2, ... are m x m matrices with their coefficients in L. These ingredients are supposed to satisfy the following conditions

(2.2) $p \in \mathfrak{a}$, $\sigma(b) \equiv b^q \bmod \mathfrak{a}$ for all $b \in B$, $\sigma^r(s_i(j,k))\,\mathfrak{a} \subset B$ for all i,j,k,r

Here $s_i(j,k)$ is the (j,k)- entry of the matrix s_i. For example if $\mathfrak{a} = B$ then the last condition means that $s_i(j,k) \in B$; and if e.g. $B = \mathbb{Z}$, $L = \mathbb{Q}$, $\sigma = \mathrm{id}$, $q = p$ then the conditions are satisfied iff $s_i(j,k) \in p^{-1}\mathbb{Z}$ for all i,j,k.

If g(X) is an m-tuple of power series in X_1, ..., X_n with coefficients in L then we denote with $\sigma_* g(X)$ the m-tuple of power series obtained by applying σ to the coefficients of g(X).

2.3. __Functional Equation Lemma__. _Let_ $f(X) \in L[[X]]^m$ _be an m-tuple of power series in_ m _indeterminates_ $X_1, \ldots, X_m$ _and_ $\bar{f}(\bar{X})$ _an m-tuple of power series in_ n _indeterminates_ $\bar{X}_1, \ldots, \bar{X}_n$. _Suppose that_ $f(X) \equiv b_1 X$ mod(degree 2) _where_ b_1 _is a matrix with coefficients in_ B _which is invertible_ (_over_ B). _Suppose moreover that_

$$(2.4)\quad f(X) - \sum_{i=1}^{\infty} s_i \sigma_*^i f(X^{q^i}) \in B[[X]]^m, \qquad \bar{f}(\bar{X}) - \sum_{i=1}^{\infty} s_i \sigma_*^i \bar{f}(\bar{X}^{q^i}) \in B[[\bar{X}]]^m$$

where X^{q^i} _and_ $\bar{X}^{q^i}$ _are short for_ $(X_1^{q^i}, \ldots, X_m^{q^i})$ _and_ $(\bar{X}_1^{q^i}, \ldots, \bar{X}_n^{q^i})$. _Then we have_

$$(2.5)\qquad F(X,Y) = f^{-1}(f(X) + f(Y)) \in B[[X;Y]]^m$$

$$(2.6)\qquad f^{-1}(\bar{f}(X)) \in B[[\bar{X}]]^m.$$

Let $h(\hat{X}) \in B[[\hat{X}]]^m$ _be an m-tuple of power series with coefficients in B in yet another set of indeterminates and let_ $\hat{f}(\hat{X}) = f(h(\hat{X}))$. _Then_

$$(2.7)\qquad \hat{f}(\hat{X}) - \sum_{i=1}^{\infty} s_i \sigma_*^i \hat{f}(\hat{X}^{q^i}) \in B[[\hat{X}]]$$

Finally let $\alpha(\hat{X}) \in B[[\hat{X}]]^m$, $\beta(\hat{X}) \in L[[\hat{X}]]^m$, $r \in \mathbb{N} = \{1,2,...\}$. Then

(2.8) $$\alpha(\hat{X}) \equiv \beta(\hat{X}) \bmod \mathfrak{a}^r \iff f(\alpha(\hat{X})) \equiv f(\beta(\hat{X})) \bmod \mathfrak{a}^r$$

For a proof cf. [7], sections 2 and 10.

3. SOME ALMOST TRIVIAL APPLICATIONS. Let $H(X) = X + p^{-1}X^p + p^{-2}X^{p^2} + \ldots$ and $\ell(X) = \log(1+X) = \sum_{n=1}^{\infty} (-1)^{n+1}n^{-1}X^n$. One notes that $H(X) - p^{-1}H(X^p) = X$ and $\ell(X) - p^{-1}\ell(X^p) \in \mathbb{Z}_{(p)}[X]$. So taking $B = \mathbb{Z}_{(p)}$, $\mathfrak{a} = pB$, $L = \mathbb{Q}$, $q = p$, $s_1 = p^{-1}$, $s_2 = s_3 = \ldots = 0$ and $\sigma = \mathrm{id}$, we obtain from (2.6) Hasse's old result that $\exp(H(X))$ has its coefficients in $\mathbb{Z}_{(p)}$.

More generally let $d(X) = d_oX + d_1X^p + \ldots$, $d_i \in \mathbb{Q}$. Using the same ingredients and combining (2.6) and (2.7) above one finds that $\exp(d(X)) \in \mathbb{Z}_{(p)}[[X]]$ if and only if $d_i - p^{-1}d_{i-1} \in \mathbb{Z}_{(p)}$ for all i (where one takes $d_{-1} = 0$). This a lemma of Dieudonné [3].

An easy application with σ non trivial is the following. Let B be the ring of integers of the completed maximal unramified extension T of $\mathbb{Q}_p$; let $L = T$, $p = q$, $s_1 = p^{-1}$, $s_2 = s_3 = \ldots = 0$, and σ the Frobenius automorphism of T. Let $h(X) = 1 + a_1X + a_2X^2 + \ldots \in T[[X]]$. In this setting the combination of (2.6) and (2.7) yields that $h(X) \in B[[X]]$ if and only if $\sigma_* h(X^p)/h(X)^p \in 1 + pXB[[X]]$, which is lemma 1 of Dwork [6].

For an easy more dimensional application consider the slightly modified Witt vector polynomials $\bar{w}_o(X) = X_o$, $\bar{w}_1(X) = X_1 + p^{-1}X_o^p$, ... , $\bar{w}_n(X) = X_n + p^{-1}X_{n-1}^p + \ldots + p^{-n}X_o^{p^n}$. Take $B = \mathbb{Z}$, $\mathfrak{a} = p\mathbb{Z}$, $L = \mathbb{Q}$, $\sigma = \mathrm{id}$, $q = p$, $s_2, s_3, \ldots = 0$ and let s_1 be the $(n+1) \times (n+1)$ matrix with p^{-1} on the first subdiagonal and zero's elsewhere; i.e. $s_1(j,k) = 0$ unless $j = k + 1$ and $s_1(k+1,k) = p^{-1}$, $k = 1,2, \ldots, n$. Let $\bar{w}(X)$ be the column vector $(\bar{w}_o(X),\ldots,\bar{w}_n(X))$. Then, obviously, $\bar{w}(X) = X + s_1\bar{w}(X^p)$. It now follows from (2.5) that $\Sigma(X) = \bar{w}^{-1}(\bar{w}(X) + \bar{w}(Y))$ has integral coefficients; or, multiplying both sides of $\bar{w}(\Sigma(X)) = \bar{w}(X) + \bar{w}(Y)$ with p^n, we see that we have shown that the addition polynomials of the Witt vectors have integral coefficients.

4. ATKIN-SWINNERTON DYER CONGRUENCES. Let E be an elliptic curve over $\mathbb{Q}$ and let $L(s) = \prod (1 - a_p p^{-s} + b_p p^{1-2s})^{-1}$ be its global L-function, where the local factors $(1 - a_p p^{-s} + b_p p^{1-2s})^{-1}$ are defined as follows in terms of the

reductions mod p of a global minimal model D over $\mathbb{Z}$ for E :

(i) if p is good, i.e. if $D \otimes \mathbb{Z}/(p)$ is nonsingular then $(1-a_p p^{-s}+b_p p^{1-2s})$ is the numerator of the zetafunction of the elliptic curve $D \otimes \mathbb{Z}/(p)$ over $\mathbb{Z}/(p)$;

(ii) if $D \otimes \mathbb{Z}/(p)$ has an ordinary doublepoint then $1 - a_p p^{-s} + b_p p^{1-2s} = 1 - \varepsilon_p p^{-s}$ where $\varepsilon_p = \pm 1$ depending on whether the tangents in the double point are rational over $\mathbb{Z}/(p)$ or not;

(iii) if $D \otimes \mathbb{Z}/(p)$ has a cusp $1 - a_p p^{-s} + b_p p^{1-2s} = 1$.

Now let $f_E(X) = \sum_{n=1}^{\infty} n^{-1} a_n X^n$ where $L(s) = \sum_{n=1}^{\infty} a_n n^{-s}$. Then an immediate and obvious consequence of the Euler product structure of L(s) is that for all p

$$f_E(X) - p^{-1} a_p f_E(X^p) + p^{-1} b_p f_E(X^{p^2}) \in \mathbb{Z}_{(p)}[X]. \tag{4.1}$$

It now follows from (2.5) that $F_E(X,Y) = f_E^{-1}(f_E(X) + f_E(Y))$ is a formal group law over $\mathbb{Z}$. Let $G_E(X,Y)$ be the formal completion along the identity of the minimal model D over $\mathbb{Z}$. The formal group law $G_E(X,Y)$ can be explicitly described as follows. Let D be given by $y^2 + c_1XY + c_3Y = X^3 + c_2X^2 + c_4X + c_6$; let $\omega = (2Y+c_1X+c_3)^{-1}dX$ be the invariant differential and $z = (2Y)^{-1}X$ a local parameter at zero. Let, locally, $\omega = \Sigma\beta(n)z^{n-1}dz$ and define $g_E(X) = \sum_{n=1}^{\infty} n^{-1}\beta(n)X^n$, then $G_E(X,Y) = g_E^{-1}(g_E(X) + g_E(Y))$. This comes from the fact that if f(X) is the logarithm of a formal group law F(X,Y) over a torsion free ring R then df(X) is an invariant differential for F(X,Y).

4.2. Theorem (Honda, Hill; [11] , [10] and [12]). <u>The formal group laws $F_E(X,Y)$ and $G_E(X,Y)$ are strictly isomorphic over $\mathbb{Z}$ (i.e. there exists a power series $\phi(X) = X + b_2X^2 + \ldots$, $b_i \in \mathbb{Z}$ such that $\phi(F_E(X,Y)) = G_E(\phi(X),\phi(Y))$.</u>

It follows that $g_E(X) = f_E(\phi^{-1}(X))$. So that by (2.7) we have that $g_E(X)$ also satisfies the integrality conditions (4.1). Writing this out in terms of coefficients one finds the Atkin Swinnerton-Dyer congruences.

$$\beta(np) - a_p\beta(n) + b_p\beta(n/\!/p) \equiv 0 \bmod p^s \text{ if } n \equiv 0 \bmod p^{s-1} \tag{4.3}$$

where $\beta(n/\!/p) = \beta(n/p)$ if $p|n$ and $\beta(n/\!/p) = 0$ otherwise.

5. LUBIN-TATE FORMAL GROUP LAWS. The socalled Lubin-Tate formal group laws are constructed as follows in [13]. Let K be a local field with finite residue field (i.e. K is a finite extension of $\mathbb{Q}_p$ or $\mathbb{F}_p(x)$); let A be the ring of integers of K, let π be a uniformizing element and let q be the number of

elements of k, the residue field of K. Let $e(X) \in A[[X]]$ be any power series in one variable such that

$$(5.1) \qquad e(X) \equiv \pi X \bmod(\text{degree } 2), \; e(X) \equiv X^q \bmod \pi$$

Then there is a unique power series $F_e(X,Y)$ such that $F_e(e(X),e(Y)) = e(F_e(X,Y))$ and $F_e(X,Y) \equiv X + Y \bmod(\text{degree } 2)$. This is a formal group law over A. Moreover for all $a \in A$ there is a unique power series $[a]_e(X)$ such that $e([a]_e(X)) = [a]_e(e(X))$ and $[a]_e(X) \equiv aX$ mod degree 2; the map $a \mapsto [a]_e(X)$ defines a ring homomorphism $A \to \mathrm{End}_A(F(X,Y))$ and $[\pi]_e(X) = e(X)$. Finally if both $e(X)$ and $e'(X)$ satisfy (5.1) (with respect to the same π) then $F_e(X,Y)$ and $F_{e'}(X,Y)$ are strictly isomorphic over A.

In the ingredients (2.1) for the functional equation lemma now take $B = A$, $L = K$, $\mathfrak{a} = \pi A$, $p = \mathrm{char}(k)$, $q = \# k$, $\sigma = \mathrm{id}$, $s_1 = \pi^{-1}$, $0 = s_2 = s_3 = \dots$ Then the conditions (2.2) are satisfied. Let $g(X) \in A[[X]]$ be any power series such that $g(X) \equiv X \bmod(\text{degree } 2)$, and consider $f(X) \in K[[X]]$ defined (recursively) by the functional equation

$$(5.2) \qquad f(X) = g(X) + \pi^{-1} f(X^q)$$

Then parts (2.5) and (2.6) of the functional equation lemma say that the power series

$$(5.3) \qquad F(X,Y) = f^{-1}(f(X) + f(Y)), \; [a](X) = f^{-1}(af(X)), \; a \in A$$

have their coefficients in A and hence define a formal A-module over A. (A formal A-module, where A is as above, over an A-algebra R is a formal group law F(X,Y) over R together with a ring endomorphism $\rho_F \colon A \to \mathrm{End}_R(F(X,Y))$ such that $\rho_F(a) \equiv aX \bmod(\text{degree } 2)$ for all $a \in A$). Now consider $[\pi](X)$. We have

$$(5.4) \qquad f([\pi](X)) = \pi f(X) = \pi g(X) + f(X^q) \equiv f(X^q) \bmod \pi$$

It follows by part (2.8) of the functional equation lemma that $[\pi](X) \equiv X^q \bmod \pi$ Also of course (cf. (5.3)) $F([\pi](X),[\pi](Y)) = [\pi](F(X,Y))$ so that F(X,Y) is a Lubin-Tate formal group law with $e(X) = [\pi](X))$. As all Lubin-Tate formal group laws constructed via the same uniformizing element π are strictly isomorphic, it follows from part (2.7) of the functional equation lemma that all Lubin-Tate formal group laws are obtained by the construction (5.2), (5.3) by varying g(X).

Finally we use the functional equation lemma to show that Lubin-Tate formal group laws constructed via different uniformizing elements π and $\bar{\pi}$ become isomorphic over $\hat{A}_{nr}$, the completion of the ring of integers of the completion $\hat{K}_{nr}$ of the maximal unramified extension K_{nr} of K. Let therefore $f(X)$, $\bar{f}(X) \in A[[X]]$ satisfy

$$(5.5) \qquad f(X) - \pi^{-1}f(X^q) \in A[[X]], \ \bar{f}(X) - \bar{\pi}^{-1}\bar{f}(X^q) \in A[[X]]$$

Now take as functional equation ingredients $B = \hat{A}_{nr}$, $\mathfrak{a} = \pi B$, $L = \hat{K}_{nr}$, σ the Frobenius substitution in $Gal(K_{nr}/K)$ extended by continuity to $\hat{K}_{nr}$, p, q, $s_1, s_2, \ldots$ as before. Let $u \in \hat{A}^*_{nr}$, the units of $\hat{A}_{nr}$, be such that $u^{-1}\sigma(u) = \pi^{-1}\bar{\pi}$. (Such a u exists). Then we have

$$(5.6) \quad uf(X) - \bar{\pi}^{-1}\sigma_*(uf(X^q)) = uf(X) - \bar{\pi}^{-1}\sigma(u)f(X^q) = \\ = u(f(X) - \pi^{-1}f(X^q)) \in \hat{A}_{nr}[[X]]$$

and also of course $\bar{f}(X) - \bar{\pi}^{-1}\sigma_*\bar{f}(X^q) = \bar{f}(X) - \pi^{-1}\bar{f}(X^q) \in A[[X]] \subset \hat{A}_{nr}[[X]]$, so that by part (2.6) of the functional equation lemma we have that

$$(5.7) \qquad \phi(X) = \bar{f}^{-1}(uf(X)) \in \hat{A}_{nr}[[X]]$$

which defines as an isomorphism $\phi(X)$ between the formal A-modules defined by $f(X)$ and $\bar{f}(X)$ as in (5.3).

6. TAPIS DE CARTIER. Let A be the ring of integers of an unramified extension K of $\mathbb{Q}_p$. Let $\sigma \in Gal(K/\mathbb{Q}_p)$ be the Frobenius automorphism. Now suppose we have given a free A-module M of finite rank $h < \infty$ together with a semilinear endomorphism $\eta : M \to M$ (i.e. $\eta(m+m') = \eta(m) + \eta(m')$, $\eta(am) = \sigma(a)\eta(m)$). To these data we associate a formal group law over A as follows. Let $D(\eta)$ be the matrix of η with respect to some basis for M. Define $g(M,\eta)(X) \in K[[X_1,\ldots,X_h]]^h$ by the equation

$$(6.1) \qquad g(M,\eta)(X) = X + p^{-1}D(\eta)\sigma_*g(M,\eta)(X^p)$$

By part (2.5) of the functional equation lemma (with B = A, L = K, $\mathfrak{a} = pA$, σ as above, q = p, $s_1 = p^{-1}D(\eta)$, $s_2 = s_3 = \ldots = 0$) it follows that $G(M,\eta)(X,Y) = g(M,\eta)^{-1}(g(M,\eta)(X) + g(M,\eta)(Y))$ is a formal group law over A. This construction is functorial in the following sense. Let $\alpha : (M,\eta) \to (M',\eta')$ be a morphism. This means that $\alpha : M \to M'$ is A-linear and that $\eta'\alpha = \alpha\eta$. Let $E(\alpha)$ be the matrix of α with respect to the chosen bases of M and M'. Then we have $E(\alpha)g(M,\eta)(X) - p^{-1}D(\eta')\sigma_*(E(\alpha)g(M,\eta)(X^p) = E(\alpha)X \in A[[X]]^{h'}$, because $\eta'\alpha = \alpha\eta$, together with the semilinearity of η and η', precisely means that $D(\eta')\sigma_*(E(\alpha)) = E(\alpha)D(\eta)$. It follows in particular that $G(M,\eta)(X,Y)$ does not depend (up to isomorphism) on the choice of a basis for M.

For each (M,η) as above let (M^σ,η) be the pair obtained by leaving the additive group M and the map η unchanged but by changing the A-action to $a.m = \sigma^{-1}(a)m$. One easily checks that $G(M^\sigma,\eta) = \sigma_*G(M,\eta)$. There is an obvious morphism $(M^\sigma,\eta) \to (M,\eta)$, viz. η itself. Let $v(M,\eta) : \sigma_*G(M,\eta) \to G(M,\eta)$ be the corresponding morphism of formal groups. We claim that $v(M,\eta)$ reduces mod p to the Verschiebung morphism $V(X)\colon \sigma_*\bar{G}(X,Y) \to \bar{G}(X,Y)$ over k where the bar denotes reduction mod p and where we omitted to write (M,η). (If $F(X,Y)$ is a formal group law over k, then $V(X)\colon \sigma_*F(X,Y) \to F(X,Y)$ is the power series over k defined by $V(X^p) = [p](X)$ (because char(k) = p, $[p](X)$ is necessarily a power series in X^p)). This is seen as follows. We have

$$g(M,\eta)v(X^q) = D(\eta)g(M^\sigma,\eta)(X^q) = D(\eta)\sigma_*g(M,\eta)(X^q) \equiv pg(M,\eta)(X) \bmod pA$$

It follows by part (2.8) of the functional equation lemma that $v(X^q) \equiv g(M,\eta)^{-1}(pg(M,\eta)(X)) = [p](X) \bmod pB$, proving our claim.

Thus we have a functor $(M,\eta) \mapsto (G(M,\eta), v(M,\eta))$. There is an obvious functor in the inverse direction, viz. taking Lie-algebras. And we clearly have $\mathrm{Lie}(G(M,\eta)) = M$, $\mathrm{Lie}(v(M,\eta)) = \eta$. The Tapis de Cartier ([1], [2], [7]) now says that these functors are inverse equivalence of categories. To prove this we have to show that every formal group law $F(X,Y)$ together with a morphism $v\colon \sigma_*F(X,Y) \to F(X,Y)$ over A which reduces to $V(X)$ mod pA comes from a pair (M,η).

To prove this we first remark that, because A is unramified, every $F(X,Y)$ over A is of functional equation type (Honda [12], cf. [7], section 20.3) i.e. if $f(X)$ is the logarithm of $F(X,Y)$ then there are $s_1,s_2,\dots$ such that $f(X) - \Sigma\, s_i\sigma_*^i f(X^{p^i}) \in A[[X]]^h$, where $h = \dim(F(X,Y))$. Now a homomorphism $v(X)\colon \sigma_*F(X,Y) \to F(X,Y)$ is necessarily of the form $v(X) = f^{-1}(E\sigma_*f(X))$ for some matrix E.
Hence $f^{-1}(pf(X)) = [p](X) \equiv v(X^p) = f^{-1}(E\sigma_*f(X^p))$. It follows by part (2.8) of the functional equation lemma that $pf(X) \equiv E\sigma_*f(X^p) \bmod pA$, i.e. that $f(X) - p^{-1}E\sigma_*f(X^p) \in A[[X]]$, so that by part (2.6) of the functional equation

lemma $F(X,Y)$ is strictly isomorphic to the formal group law with logarithm defined by $g(X) = X + p^{-1}Eg(X^p)$ which is of the form $g(M,n)(X)$.

For some details about the rôle which the tapis de Cartier plays in the theory of lifting formal group laws cf. [7], section 30, as well as for an analogous theory for formal A-modules, where A is a finite extension of $\mathbb{Q}_p$ or $\mathbb{F}_p(x)$.

7. RAMIFIED WITT VECTORS. Let A be the ring of integers of a finite (not necessarily unramified) extension K of $\mathbb{Q}_p$ or $\mathbb{F}_p(x)$. Let k be the residue field of K, $q = \# k = p^r$, π a uniformizing element. Consider the power series

$$(7.1) \qquad g_\pi(X) = X + \pi^{-1}X^q + \pi^{-2}X^{q^2} + \ldots, \quad G_\pi(X,Y) = g_\pi^{-1}(g_\pi(X) + g_\pi(Y))$$

Then $g_\pi(X) = X + \pi^{-1}g_\pi(X^q)$ so that by section 5 above, $G_\pi(X,Y)$ is a Lubin-Tate formal group law over A. For every A-torsion free A-algebra B let $W^A_{q,\infty}(B)$ be the following set of power series in one variable t

$$(7.2) \quad W^A_{q,\infty}(B) = \{\gamma(t) \in B[[t]] \mid \gamma(0) = 0,\ g_\pi\gamma(t) = \sum_{i=o}^{\infty} x_i t^{q^i} \text{ for certain } x_i \in B \otimes_A K\}$$

For arbitrary A-algebras B one can define $W^A_{q,\infty}(B) = \{\phi_*\gamma(t) \mid \gamma(t) \in W^A_{q,\infty}(B')\}$ where B' is any A-torsion free A-algebra with a surjective A-algebra homomorphism $\phi : B' \to B$. The sets $W^A_{q,\infty}(B)$ have a natural group structure defined by $\gamma(t) + \delta(t) = G_\pi(\gamma(t),\delta(t))$ and a topology defined by the subgroups $\{\gamma(t) \in W^A_{q,\infty}(B) \mid \gamma(t) \equiv 0 \bmod t^{q^n}\}$. There is an obvious morphism $W^A_{q,\infty}(B_1) \to W^A_{q,\infty}(B_2)$ attached to an A-algebra homomorphism $\phi : B_1 \to B_2$, viz. $\gamma(t) \mapsto \phi_*\gamma(t)$. So that we have a complete topological group valued functor $B \mapsto W^A_{q,\infty}(B)$.

We are now going to define a functorial ring structure on $W^A_{q,\infty}(B)$. The definition for A-torsion free A-algebras B is:

$$(7.3) \quad \text{if } g_\pi\gamma(t) = \sum x_i t^{q^i},\ g_\pi\delta(t) = \sum y_i t^{q^i}, \text{ then } \gamma(t)\delta(t) = g_\pi^{-1}(\sum \pi^i x_i y_i t^{q^i})$$

To show that this is welldefined we must show that the coefficients of $\gamma(t)\delta(t)$ are in B (and not just in $B \otimes_A K$). This is seen as follows.

Assume that B is A-torsion free and admits an A-algebra endomorphism σ such that $\sigma(b) \equiv b^q \bmod \pi B$ for all $b \in B$. By part (2.7) of lemma 2.3 we then have $x_i - \pi^{-1}x_{i-1} = a_i \in B$, $y_i - \pi^{-1}x_{i-1} = b_i \in B$ for all i (with $x_{-1} = y_{-1} = 0$). Hence $\pi^i x_i, \pi^i y_i \in B$ for all i. It follows that $\pi^i x_i y_i - \pi^{-1}(\pi^{i-1}x_{i-1}y_{i-1}) =$ $= \pi^i a_i b_i + \pi^{i-1}a_i y_{i-1} + \pi^{i-1}b_i x_{i-1} \in B$, so that by part (2.6) of lemma 2.3 we have indeed that $g_\pi^{-1}(\Sigma\pi^i x_i y_i t^{q^i})$ has its coefficients in B. To extend this definition to the case of arbitrary A-algebras B use an argument similar as just below (7.2) using that every A-algebra B is a quotient of an A-algebra B' which satisfies our assumptions, e.g. $B' = A[Z_b | b \in B]$. There is also a natural A-module structure on $W^A_{q,\infty}(B)$ defined by $\gamma(t) \mapsto [a](\gamma(t))$ where $[a](X) = g_\pi^{-1}(ag_\pi(X))$, $a \in A$, cf. also section 5. All in all this defines a functor $W^A_{q,\infty}: \underline{\underline{Alg}}_A \to \underline{\underline{Alg}}_A$, which, we claim, possibly deserves the name "ramified Witt vector functor". To bolster this claim we remark the following

- There is an additive Verschiebung morphism $\underline{\underline{V}}_q$ defined by $\underline{\underline{V}}_q\gamma(t) = \gamma(t^q)$ and a Frobenius A-algebra functor endomorphism $\underline{\underline{f}}_\pi$. The latter is defined for A-torsion free A-algebras B by the formula $\underline{\underline{f}}_\pi\gamma(t) = g_\pi^{-1}(\sum_{i=o}^{\infty} \pi x_{i+1}t^{q^i})$ where the x_i are as in (7.3). Of course the integrality of $\underline{\underline{f}}_\pi\gamma(t)$ is proved by means of the functional equation lemma. We have $\underline{\underline{f}}_\pi\underline{\underline{V}}_q = [\pi]$, $\underline{\underline{f}}_\pi\gamma(t) \equiv \gamma(t)^q \bmod [\pi]W^A_{q,\infty}(B)$.
- Let A' be the ring of integers of an unramified extension K' of K. Let k' be the residue field of K' and let $\sigma \in Gal(K'/K)$ be the Frobenius automorphism. For each $a' \in A'$ let $\Delta(a') = g_\pi^{-1}(\sum_{i=o}^{\infty} \pi^{-i}\sigma^i(a')t^{q^i}) \in W^A_{q,\infty}(B)$. (Integrality of $\Delta(a')$ is of course proved by means of the functional equation lemma). Then $a' \mapsto \Delta(a')$ is a homomorphism of A-algebras and the composite $A' \overset{\Delta}{\to} W^A_{q,\infty}(A') \to W^A_{q,\infty}(k')$ is an isomorphism. In particular $W^A_{q,\infty}(k') = A'$ with σ corresponding to $\underline{\underline{f}}_\pi$, generalizing a wellknown property of the Witt vectors.
- There is an A-algebra homomorphism $\Delta : W^A_{q,\infty}(-) \longrightarrow W^A_{q,\infty}(W^A_{q,\infty}(-))$, the ramified Artin-Hasse exponential, characterized by $w^A_{q,i} \circ \Delta = \underline{\underline{f}}^i_\pi$, where $w^A_{q,i} : W^A_{q,\infty}(B) \longrightarrow B$ is the functorial A-algebra homomorphism $w^A_{q,i}(\gamma(t)) = \pi^i$ times the coefficient of t^{q^i} in $g_\pi(\gamma(t))$.

For more details concerning this construction cf. [7], section 25 ; for a twisted version of these constructions which also works for local fields with not necessarily finite residue field cf. also [9]. Another construction of the functors $W^A_{q,\infty}$ has independently been given by Ditters [4] and Drinfel'd [5].

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