INFINITE DIMENSIONAL UNIVERSAL FORMAL GROUP LAWS AND FORMAL A-MODULES.

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1. INTRODUCTION AND MOTIVATION.

Let B be a commutative ring with $1 \in B$. An n-dimensional commutative formal group law over B is an n-tuple of power series $F(X,Y)$ in 2n variables $X_1, \ldots, X_n; Y_1, \ldots, Y_n$ with coefficients in B such that $F(X,0) \equiv X$, $F(0,Y) \equiv Y$ mod degree 2, $F(F(X,Y),Z) = F(X,F(Y,Z))$ (associativity) and $F(X,Y) = F(Y,X)$ (commutativity). From now on all formal group laws will be commutative.

Let A be a discrete valuation ring with finite residue field k. Let $B \in \underline{\underline{Alg}}_A$, the category of commutative A-algebras with 1. A n-dimensional formal A-module over B is a formal group law $F(X,Y)$ over B together with a ring homomorphism $\rho_F: A \to \mathrm{End}_B(F(X,Y))$ such that $\rho_F(a) \equiv aX$ mod degree 2 for all $a \in A$. One would like to have a classification theory for formal A-modules which is parallel to the classification theory of formal group laws over $\mathbb{Z}_{(p)}$-algebras. Such a theory is sketched below and details can be found in [2], section 29. As in the case of formal group laws over $\mathbb{Z}_{(p)}$-algebras the theory inevitably involves infinite dimensional objects. Now two important operators for the formal A-module classification theory, viz. ε_q and $\underline{\underline{f}}_\pi$, the analogues of p-typification and Frobenius, are defined by lifting back to the universal case, and, for the moment at least, I know of no other way of defining them, especially if char(A) = p > 0. In case char(A) = 0, cf. also [1]. But by the very nature of the usual definition of infinite dimensional formal group law and formal A-module there cannot exist universal infinite dimensional formal group laws and formal A-modules, so that the definitions of ε_q and $\underline{\underline{f}}_\pi$ break down. In [2], this problem is surmounted by an ad hoc construction which works in the particular case needed (Witt vector like formal A-modules). But this method decidedly lacks elegance. It is the second and main purpose of the present paper to remedy this by showing that after all, in a suitable sense, universal infinite dimensional formal group laws and formal A-modules do exist (and have all the nice properties one could wish for). As a byproduct one obtains then of course such results as liftability to characteristic zero and existence and uniqueness of logarithms also for infinite dimensional formal group laws and formal A-modules.

2. SKETCH OF THE (COVARIANT) CLASSIFICATION THEORY FOR FORMAL GROUP LAWS OVER $\mathbb{Z}_{(p)}$-ALGEBRAS.

Let p be a fixed prime number. Let F(X,Y) be an m-dimensional formal group law over a $\mathbb{Z}_{(p)}$-algebra B. A curve in F over R is simply an m-tuple of power series $\gamma(t)$ in one variable t with coefficients in B such that $\gamma(0) = 0$.

Two curves can be added by means of the formula

$$\gamma(t) +_F \delta(t) = F(\gamma(t),\delta(t)) \tag{2.1}$$

giving us a complete topological abelian group $\mathcal{C}(F;B)$; the topology is defined by the subgroups of curves $\gamma(t)$ such that $\gamma(t) \equiv 0$ mod(degree n), n = 1,2,3, ... In addition one has operators $\underline{\underline{V}}_n$, $\underline{\underline{f}}_n$, $\langle b \rangle$, for $n \in \mathbb{N}$, $b \in B$.

These are defined as follows

$$\underline{\underline{V}}_n\gamma(t) = \gamma(t^n),\ \langle b\rangle\gamma(t) = \gamma(bt),\ \underline{\underline{f}}_n\gamma(t) = \sum_{i=1}^{n}{}^{F}\ \gamma(\zeta_n^i t^{1/n}) \tag{2.2}$$

where ζ_n is a primitive n-th root of unity. This last definition must be rewritten slightly in case n-th roots of unity make no particular sense over B, cf. [2], section 16 for details.

A curve $\gamma(t)$ is called p-typical if $\underline{\underline{f}}_q\gamma(t) = 0$ for all prime numbers $q \neq p$. The subgroup of p-typical curves, $\mathcal{C}_p(F;B)$, is complete in the induced topology and stable under $\underline{\underline{f}}_p$ and $\underline{\underline{V}}_p$ and the operators $\langle b \rangle$. Moreover using that B is a $\mathbb{Z}_{(p)}$-algebra there is a projector $\varepsilon_p\colon \mathcal{C}(F;B) \to \mathcal{C}_p(F;B)$ given by the formula

$$\varepsilon_p = \sum_{(n,p)=1} n^{-1}\mu(n)\underline{\underline{V}}_n\underline{\underline{f}}_n \tag{2.3}$$

where $\mu(n)$ is the Möbius function. We can assemble the operators $\underline{\underline{f}}_p$, $\underline{\underline{V}}_p$, $\langle b \rangle$ into a ring of operators $\mathrm{Cart}_p(B)$ consisting of all sums

$$\sum_{i,j=o}^{\infty} \underline{\underline{V}}_p^i\langle b_{i,j}\rangle\underline{\underline{f}}_p^j$$

with for all i only finitely many $b_{i,j} \neq 0$. For the calculation rules in $\mathrm{Cart}_p(B)$, cf. [2],16.2, 28.1. The subring $\{ \sum_{i=0}^{\infty} \underline{\underline{V}}_p^i \langle b_i \rangle \underline{\underline{f}}_p^i \}$ is naturally isomorphic to $W_{p,\infty}(B)$, the ring of Witt vectors over B of infinite length associated to the prime p. Using this we see that $\mathfrak{C}_p(F;B)$ is a module over $W_{p,\infty}(B)[\underline{\underline{f}},\underline{\underline{V}}]$ with calculation rules $\underline{\underline{f}}\underline{\underline{V}} = p$, $\underline{\underline{V}}\underline{\underline{f}} = (0,1,0,\ldots)$, $\underline{\underline{f}}x = x^{\sigma}\underline{\underline{f}}$, $x\underline{\underline{V}} = \underline{\underline{V}}x^{\sigma}$ for all $x \in W_{p,\infty}(R)$, where σ is the Frobenius endomorphism of $W_{p,\infty}(B)$. The functor $F(X,Y) \mapsto \mathfrak{C}_p(F;B)$ turns out to be faithful and its image can be described without much trouble.

3. A CARTIER-DIEUDONNE MODULE CLASSIFICATION THEORY FOR FORMAL A-MODULES (1).

Now let A be a discrete valuation ring with uniformizing element π and finite residue field k of q elements, $q = p^r$. Let K be the quotient field of A. We are going to describe a classification theory for formal A-modules which is completely analogous to the theory sketched in 2 above. In this theory $\underline{\underline{f}}$ gets replaced by $\underline{\underline{f}}_\pi$, $\underline{\underline{V}}$ by $\underline{\underline{V}}_q$, $W_{p,\infty}(B)$ by the appropriate ring of ramified Witt vectors $W^A_{q,\infty}(B)$, $B \in \underline{\underline{\mathrm{Alg}}}_A$, and $\mathfrak{C}_p(F;B)$ by $\mathfrak{C}_q(F;B)$. Of course we should have $\underline{\underline{f}}_\pi\underline{\underline{V}}_q = \pi$, $\underline{\underline{V}}_q\underline{\underline{f}}_\pi = (0,1,0,0,.)$, $x\underline{\underline{V}}_q = \underline{\underline{V}}_q x^{\sigma}$, $\underline{\underline{f}}_\pi x = x^{\sigma}\underline{\underline{f}}_\pi$. In case A is of characteristic zero, $p = u\pi^e$, this shows that $\underline{\underline{f}}_\pi$ and $\underline{\underline{f}}_p$ should be related as

$$(3.1) \qquad [u^{-1}]\underline{\underline{f}}_p = \underline{\underline{f}}_\pi^e \underline{\underline{V}}_p^{er-1}$$

Here we shall not discuss the ramified Witt vector functor $W^A_{q,\infty}: \underline{\underline{\mathrm{Alg}}}_A \to \underline{\underline{\mathrm{Alg}}}_A$, cf. [2[, [3], or [4]. It can be most easily obtained by taking q-typical curves in the Lubin-Tate formal group law over A, just as $W_{p,\infty}(-)$ can very nicely be described via the p-typical curves in $\underline{\underline{G}}_m$, the multiplicative formal group law. Alternatively $W^A_{q,\infty}(-)$ can be described via the polynomials

$$(3.2) \qquad X_0^{q^n} + \pi X_1^{q^{n-1}} + \ldots + \pi^n X_n, \quad n = 0,1,2, \ldots$$

exactly as $W_{p,\infty}(-)$ is constructed via the Witt polynomials $X_0^{p^n} + pX_1^{p^{n-1}} + \ldots + p^n X_n$.

We shall concentrate on the definition of $\underline{\underline{f}}_\pi$ and the "q-typification" projector $\varepsilon_q: \mathfrak{C}(F;B) \to \mathfrak{C}_q(F;B)$, partly also to illustrate the adagium "do everything first in the universal case", which appears to be particularly effective, in fact even necessary, when dealing with formal A-modules.

Now there seems to be no obvious analogues of the definitions for $\underline{\underline{f}}_p$ and ε_p given in (2.2) and (2.3). Things become better if we restate these definitions in terms of logarithms. Assume therefore that B is torsion free and let $f(X) \in B \otimes \mathbb{Q}[[X]]^m$ be the logarithm of $F(X,Y)$, i.e. $f(X)$ is the unique m-tuple of power series over $B \otimes \mathbb{Q}$ such that $f(X) \equiv X$ mod(degree 2), $F(X,Y) = f^{-1}(f(X) + f(Y))$. Setting

$$(3.3) \qquad f(\gamma(t)) = \sum_{i=1}^{\infty} x_i t^i, \; x_i \in B \otimes \mathbb{Q}^m$$

we then have

$$(3.4) \qquad f(\underline{\underline{f}}_n \gamma(t)) = \sum_{i=1}^{\infty} n x_{ni} t^i$$

$$(3.5) \qquad f(\varepsilon_p \gamma(t)) = \sum_{j=o}^{\infty} x_{p^j} t^{p^j}$$

Now let $(F(X,Y), \rho_F)$ be an m-dimensional formal A-module over $B \in \underline{\underline{\mathrm{Alg}}}_A$. Assume that B is A-torsion free. An A-<u>logarithm</u> for $(F(X,Y), \rho_F)$ is a power series $f(X) \in B \otimes_A K[[X]]^m$ such that $f(X) \equiv X$ mod degree 2 and such that $F(X,Y) = f^{-1}(f(X) + f(Y))$ and $\rho_F(a) = f^{-1}(af(X))$ for all $a \in A$. It is an immediate consequence of the construction of a universal formal A-module below in section 5 that A-logarithms exist. Uniqueness is then easy. Given A-logarithms there are obvious analogues of (3.4) and (3.5) viz.

$$(3.6) \qquad \underline{\underline{f}}_\pi \gamma(t) = f^{-1}(\sum_{i=1}^{\infty} \pi x_{qi} t^i)$$

$$(3.7) \qquad \varepsilon_q \gamma(t) = f^{-1}(\sum_{i=o}^{\infty} x_{q^i} t^{q^i})$$

It remains of course to prove that the m-tuples of power series thus defined are integral (i.e. that they have their coefficients in B not just in $B \otimes_A K$). This again will be done by proving this to be the case in the universal example, which, fortunately, is defined over the kind of algebra to which the functional equation (integrality) lemma applies. This lemma is our main tool for proving integrality statements. It is remarkably "universally" applicable, cf. also [3] for some other illustrations.

4. THE FUNCTIONAL EQUATION LEMMA.

The ingredients we need are the following

(4.1) $B \subset L$, $\mathfrak{a} \subset B$, $\sigma : L \to L$, p, q, s_1, s_2, ...

Here B is a subring of a ring L, $\mathfrak{a}$ is an ideal in B, σ a ring endomorphism of L, p is a prime number, q is a power of p and the s_i, $i = 1,2,3,\ldots$ are $m \times m$ matrices with coefficients in L. These ingredients are supposed to satisfy the following conditions

(4.2) $p \in \mathfrak{a}$, $\sigma(b) \equiv b^q \bmod \mathfrak{a}$ for all $b \in B$, $\sigma^r(s_i(j,k))\mathfrak{a} \subset B$ for all i,j,k,r

Here $s_i(j,k)$ is the (j,k) entry of the matrix s_i, $j,k \in \{1,\ldots,m\}$. If $g(X)$ is an m-tuple of power series in $X_1, \ldots, X_n$ with coefficients in L then we denote with $\sigma_* g(X)$ the m-tuple of power series obtained by applying σ to the coefficients of $g(X)$.

4.3. <u>Functional</u> <u>Equation</u> <u>Lemma</u>. Let $f(X) \in L[[X]]^m$ be an m-tuple of power series in m determinates $X_1, \ldots, X_m$ and $\bar{f}(\bar{X}) \in L[[\bar{X}]]^m$ an m-tuple of power series in n indeterminates $\bar{X}_1, \bar{X}_2, \ldots, \bar{X}_n$. Suppose that $f(X) \equiv b_1 X$ mod(degree 2) where b_1 is a matrix with coefficients in B which is invertible (over B). Suppose moreover that

(4.4) $f(X) - \sum_{i=1}^{\infty} s_i \sigma_*^i f(X^{q^i}) \in B[[X]]^m$, $\bar{f}(\bar{X}) - \sum_{i=1}^{\infty} s_i \sigma_*^i \bar{f}(\bar{X}^{q^i}) \in B[[\bar{X}]]^m$

where X^{q^i} and $\bar{X}^{q^i}$ are short for $(X_1^{q^i},\ldots,X_m^{q^i})$ and $(\bar{X}_1^{q^i},\ldots,\bar{X}_n^{q^i})$. Then we have

$$F(X,Y) = f^{-1}(f(X) + f(Y)) \in B[[X;Y]]^m \tag{4.5}$$

$$f^{-1}(\bar{f}(\bar{X})) \in B[[\bar{X}]]^m \tag{4.6}$$

Let $h(\hat{X}) \in B[[\hat{X}]]^m$, $\hat{f}(\hat{X}) = f(h(\hat{X}))$. Then

$$\hat{f}(\hat{X}) - \sum_{i=1}^{\infty} s_i \sigma_*^i \hat{f}(\hat{X}^{q^i}) \in B[[\hat{X}]]^m \tag{4.7}$$

Let $\alpha(\hat{X}) \in B[[\hat{X}]]^m$, $\beta(\hat{X}) \in L[[\hat{X}]]^m$ and $r \in \mathbb{N} = \{1,2,3, \ldots\}$. Then

$$(4.8) \qquad \alpha(\hat{X}) \equiv \beta(\hat{X}) \bmod \mathfrak{a}^r \iff f(\alpha(\hat{X})) \equiv f(\beta(\hat{X})) \bmod \mathfrak{a}^r$$

For a proof cf. [2], section 10.

5. A UNIVERSAL m-DIMENSIONAL FORMAL A-MODULE.

For each multiindex $\alpha = (n_1,\ldots,n_m)$ of length m, $n_i \in \mathbb{N} \cup \{0\}$ let $|\alpha| = n_1 + \ldots + n_m$ and $s\alpha = (sn_1,\ldots,sn_m)$ for all $s \in \mathbb{N} \cup \{0\}$. For each α such that $|\alpha| \geq 2$ and $i \in \{1,\ldots,m\}$ let $U(i,\alpha)$ be an indeterminate. We denote with $\varepsilon(i)$ the multiindex $(0,\ldots,0,1,0,\ldots,0)$ with 1 in the i-th spot. We set $U(i,\varepsilon(j)) = 0$ if $i \neq j$ and $U(i,\varepsilon(i)) = 1$. For each $\alpha \neq q^r\varepsilon(i)$ for all $r = 1,2, \ldots, i \in \{1,\ldots,m\}$ we let U_α denote the column vector $U(1,\alpha), \ldots, U(m,\alpha)$ and for each $r \in \mathbb{N}$, U_{q^r} denotes the m x m matrix $U_{q^r} = (U(i,q^r\varepsilon(j)))$. Finally let $X^\alpha = X_1^{n_1}. \ldots. X_m^{n_m}$. For each multiindex α such that $|\alpha| \geq 1$ we now define the m-vector $a_\alpha(U) \in K[U]^m$ by

$$(5.1) \qquad A_\alpha(U) = \sum_{(r_1,\ldots,r_t,\beta)} \pi^{-t} U_{q^{r_1}} U_{q^{r_2}}^{(q^{r_1})} \ldots U_{q^{r_t}}^{(q^{r_1+\ldots+r_{t-1}})} U_\beta^{(q^{r_1+\ldots+r_t})}$$

where the sum is over all $(r_1,\ldots,r_t,\underline{d})$, $r_i \in \mathbb{N}$, $t \in \mathbb{N} \cup \{0\}$ such that $q^{r_1}q^{r_2} \ldots q^{r_t}\beta = \alpha$ and $\beta \neq q^r\varepsilon(i)$ for all $r \in \mathbb{N}$, $i \in \{1,\ldots,m\}$. Here $U_{q^r}^{(q^i)}$ is the matrix obtained from U_{q^r} by raising each of its entries to the q^i-th power. We now define

$$(5.2) \qquad f_U^A(X) = \sum_{|\alpha| \geq 1} a_\alpha X^\alpha \in K[U][[X]]^m$$

Now let $L = K[U] \supset A[U] = B$, $\mathfrak{a} = \pi A[U]$, $s_i = \pi^{-1} U_{q^i}$ and $\sigma : L \to L$ the K-algebra endomorphism that sends each $U(i,\alpha)$ into its q-th power. Then the conditions (4.2) hold. Also we have

$$(5.3)\qquad f_U^A(X) \equiv X \bmod(\text{degree } 2),\ f_U^A(X) - \sum_{i=1}^{\infty} s_i \sigma_*^i f_U^A(X^{q^i}) \in A[U][[X]]^m.$$

It follows that if we define

$$(5.4)\qquad F_U^A(X,Y) = (f_U^A)^{-1}(f_U^A(X)+f_U^A(Y)),\ \rho_U^A(a) = (f_U^A)^{-1}(af_U^A(X))$$

then $(F_U^A(X,Y),\rho_U^A)$ is a formal A-module over A[U] (by parts (4.5) and (4.6) of the functional equation lemma 4.3).

5.5. Theorem.

$(F_U^A(X,Y),\rho_U^A)$ is a universal m-dimensional formal A-module.

I.e. if $(G(X,Y),\rho_G)$ is any m-dimensional formal A-module over an A-algebra B then there is a unique A-algebra homomorphism ϕ: $A[U] \to B$ such that $\phi_* F_U^A(X,Y) = G(X,Y)$ and $\phi_* \rho_U^A(a) = \rho_G(a)$ for all $a \in A$.
For a proof cf. [2], section 25.

6. A CARTIER-DIEUDONNE MODULE CLASSIFICATION THEORY FOR FORMAL A-MODULES (2).

For each $n \in \mathbb{N}$, $i \in \{1,\ldots,m\}$ let C(n,i) be an indeterminate. Let C_n be the columnvector (C(n,1), ..., C(n,m). Now consider the curve

$$(6.1)\qquad \gamma_C(t) = \sum_{n=1}^{\infty} C_n t^n$$

in the universal formal A-module $(F_U^A(X,Y),\rho_U^A)$ considered as a formal A-module over A[U;C]. This is again the sort of ring to which the functional equation lemma applies. It follows by part (4.7) of lemma 4.3 that the m-tuple of power series in one variable

$$(6.2)\qquad f_U^A(\gamma_C(t)) = \sum_{n=1}^{\infty} x_i t^i\ ,\ x_i \in K[U;C]^m$$

satisfies the functional equation condition (4.4). An easy check shows that then the m-tuples of power series

$$\sum_{j=o}^{\infty} x_{q^j} t^{q^j} \qquad\qquad \sum_{n=1}^{\infty} \pi x_{qn} t^n$$

also satisfy this condition. It now follows from part (4.6) of the functional equation lemma that

$$(6.3) \qquad \varepsilon_q\gamma_C(t) = (f_U^A)^{-1}(\sum_{j=o}^{\infty} x_{q^j} t^{q^j})$$

$$(6.4) \qquad \underline{\underline{f}}_\pi\gamma_C(t) = (f_U^A)^{-1}(\sum_{n=1}^{\infty} \pi x_{qn} t^n)$$

have in fact their coefficients in A[U;C].
Now $(F_U^A(X,Y),\rho_U^A,\gamma_C)$ over A[U;C] is (given theorem 5.5) clearly universal for m-dimensional formal A-modules together with a curve.
Let $(F(X,Y),\rho_F)$ be a formal A-module over $B \in \underline{\underline{\mathrm{Alg}}}_A$ and let $\gamma(t)$ be a curve in F(X,Y) over B. Let ϕ: A[U;C] $\to$ B be the unique A-algebra homomorphism taking (F_U^A,ρ_U^A) into (F,ρ_F) and $\gamma_C(t)$ into $\gamma(t)$. Then we define

$$(6.5) \qquad \varepsilon_q\gamma(t) = \phi_*\varepsilon_q\gamma_C(t)$$

$$(6.6) \qquad \underline{\underline{f}}_\pi\gamma(t) = \phi_*\underline{\underline{f}}_\pi\gamma_C(t)$$

It follows immediately that this agrees with the tentative definitions (3.6), (3.7) of section 3 above (if B is A-torsion free so that we have a unique A-logarithm available).
Let $\mathfrak{C}_q(F;B)$ be the image of ε_q: $\mathfrak{C}(F,B) \to \mathfrak{C}(F;B)$. One now easily proves that ε_q is the identity on $\mathfrak{C}_q(F;B)$ and that $\mathfrak{C}_q(F;B)$ is stable under $\underline{\underline{f}}_\pi$, $\underline{\underline{V}}_q$, $\langle b\rangle$ for all $b \in B$. (Recall that $\underline{\underline{V}}_q\gamma(t) = \gamma(t^q)$, $\langle b\rangle\gamma(t) = \gamma(bt)$). One checks that

$$(6.7) \qquad \underline{\underline{f}}_\pi\underline{\underline{V}}_q = [\pi]$$

where $[\pi]$ is the operator induced by the endomorphism $\rho_F(\pi)$ of F(X,Y).
Further

$$(6.8) \qquad \underline{\underline{f}}_\pi\langle b\rangle = \langle b^q\rangle\underline{\underline{f}}_\pi \;, \quad \langle b\rangle\underline{\underline{V}}_q = \underline{\underline{V}}_q\langle b^q\rangle$$

We can assemble all these operators into a ring $\mathrm{Cart}_A(B)$

$$(6.9) \qquad \mathrm{Cart}_A(B) = \{\sum_{i,j=o}^{\infty} \underline{\underline{V}}_q^i\langle b_{i,j}\rangle\underline{\underline{f}}_\pi^j\}$$

with for every i only finitely many $b_{i,j} \neq 0$. The subset $\{\sum_{i=o}^{\infty} \underline{\underline{V}}_q^i \langle b_i \rangle \underline{\underline{f}}_\pi^i\}$ turns out to be a subring naturally isomorphic to $W_{q,\infty}^A(B)$, the ring of ramified Witt vectors associated to A with coefficients in B. There results a classification theory of (finite dimensional) formal A-modules in terms of $W_{q,\infty}^A(B)[\underline{\underline{f}}_\pi, \underline{\underline{V}}_q]$ modules which, both in statements and proofs, is completely analogous to the theory for formal group laws over $\mathbb{Z}_{(p)}$-algebras. In particular there is an analogue of Cartier's first theorem. It states that the formal A-module $\hat{W}_{q,\infty}^A(X;Y)$ represents the functor $F \mapsto \mathcal{C}_q(F;B)$ going from formal A-modules over B to their modules of q-typical curves. Here $\hat{W}_{q,\infty}^A$ is the (infinite dimensional) formal A-module with as A-logarithm the column vector

$$(X_o,\ X_1 + \pi^{-1}X_o^q,\ X_2 + \pi^{-1}X_1^q + \pi^{-2}X_o^{q^2},\ \ldots)$$

As in the case of formal group laws this theorem is important for the proofs of the classification results. This makes it necessary to be able to define ε_q and $\underline{\underline{f}}_\pi$ also for curves in $\hat{W}_{q,\infty}^A$, which can be done by an ad hoc method. It would be nicer to be able to do it also for all other infinite dimensional formal A-modules. It would also be more elegant to be able to extend the classification theory sketched above to all formal A-modules. To do this it is necessary to define ε_q and $\underline{\underline{f}}_\pi$ also in those cases. This, judging from what we did in the finite dimensional case, will involve something like universal infinite dimensional formal A-modules, a gadget which, in terms of the usual definitions, obviously cannot exist. This, the main topic of this paper, is what I take up next.
Before I do so let me remark that the analogy: "formal group laws over $\mathbb{Z}_{(p)}$-algebra" - "formal A-modules" also extends to give a "tapis de Cartier" and related type results for lifting formal A-modules; cf. [2], section 30.

7. "CLASSICAL" INFINITE DIMENSIONAL FORMAL GROUP LAWS AND FORMAL A-MODULES.

Let $(X_i)_{i\in I}$ be a set of indeterminates indexed by an arbitrary index set I. The formal power series ring $B[[X_i;i\in I]]$ is now defined as the ring of all formal (infinite) sums $\Sigma\, c_\alpha X^\alpha$ where α runs through all functions

$\alpha: I \to \mathbb{N} \cup \{0\}$ with finite support, i.e. $\mathrm{supp}(\alpha) = \{i \in I | \alpha(i) \neq 0\}$ is finite. We shall call such functions multiindices. Here X^α is short for $\prod_{i \in \mathrm{supp}(\alpha)} X_i^{\alpha(i)}$.

One can now consider elements $F(i)(X,Y) \in B[[X_i,Y_i;i \in I]]$ and at first sight one could define an infinite dimensional commutative formal group law as a set of power series $F(i)(X,Y) \in B[[X;Y]]$ indexed by I such that $F(i)(X,Y) \equiv X_i + Y_i \bmod(\text{degree } 2)$, $F(i)(X,Y) = F(i)(Y,X)$ and such that

$$(7.1) \qquad F(i)(X,F(Y,Z)) = F(i)(F(X,Y),Z) \text{ for all } i \in I$$

However, in general this associativity condition (7.1) makes no sense because the calculation of the coefficient of a monomial $X^\alpha Y^\beta Z^\gamma$ in $F(i)(X,F(Y,Z))$ or $F(i)(F(X,Y),Z)$ involves infinite sums of elements of B. The "classical" solution is to require a finite support condition in the following sense.

7.2. Definition.

Let I and J be index sets. Let f(X) be an I-tuple of power series in the indeterminates $X_j, j \in J$. We say that f(X) satisfies the monomials have finite support condition if for all multiindices $\alpha: J \to \mathbb{N} \cup \{0\}$ there are only finity many $i \in I$ such that the coefficient of X^α in f(i)(X) is nonzero.

This property is stable under composition and taking inverses in the sense of the following lemma.

7.3. Lemma.

Let I, J, K be index sets. Let f(X) be an I-tuple of power series in the X_j, $j \in J$ and g(Y) a J-tuple of power series in the Y_k, $k \in K$. Suppose that f(X) and g(Y) both satisfy the monomials have finite support condition. Then f(g(Y)) is well defined and satisfies the same condition. Further if $f(X) \equiv X$ mod degree 2 then $f^{-1}(X)$ is well defined and also satisfies the monomials have finite support condition.

Proof. Write $f(i)(X) = \Sigma\, r_{i,\alpha} X^\alpha$, $g(j)(Y) = \Sigma\, s_{j,\beta} Y^\beta$. Formally one has

$$(7.4) \qquad f(i)(g(Y)) = \Sigma\, r_{i,\alpha} s_{j_1,\beta_1} \cdots s_{j_t,\beta_t} Y^{\beta_1+\cdots+\beta_t}$$

where the sum is over all α and sequences $(j_1,\ldots,j_t)$, $(\beta_1,\ldots,\beta_t)$ such that

$\beta_i \neq 0$, the zero multiindex, and $j_1 + \ldots + j_t = \alpha$, where $j \in J$ is identified with the multiindex $\varepsilon(j)$: $J \to \mathbb{N} \cup \{0\}$, $j \mapsto 1$, $j' \mapsto 0$ if $j \neq j'$. Given γ: $K \to \mathbb{N} \cup \{0\}$ there are only finitely many sequences $(\beta_1, \ldots, \beta_t)$ such that $\beta_i \neq 0$ and $\beta_1 + \ldots + \beta_t = \gamma$. For each β_i there are only finitely many j such that $s_{j,\beta_i} \neq 0$; finally $\alpha = j_1 + \ldots + j_t$. It follows that in the sum (7.4) only finitely many coefficients of Y^γ are nonzero (for a given γ). Thus $f(g(Y))$ is welldefined. Also for every γ there are only finitely many α, such that there exist nonzero $s_{j_1,\beta_1}, \ldots, s_{j_t,\beta_t}$ such that $\alpha = j_1 + \ldots + j_t$, $\beta_1 + \ldots + \beta_t = \gamma$. For each α there are only finitely many i such that $r_{i,\alpha} \neq 0$. It follows that the coefficient of Y^γ in $f(i)(g(Y))$ is nonzero for only finitely many i. The second statement of the lemma is proved similarly by comparing coefficients in $f^{-1}(f(X)) = X$. Using these ideas we can now give the "classical" definition of infinite dimensional formal group laws and formal A-modules as follows.

7.5. Definitions.

An (infinite) dimensional formal group law $F(X,Y)$ over B with index set I is an I-tuple of power series $F(X,Y) = (F(i)(X,Y))_{i\in I}$, $F(i)(X,Y) \in B[[X_i, Y_i; i \in I]]$ such that $F(X,Y)$ satisfies the monomials have finite support condition and such that $F(X,0) = X$, $F(0,Y) = Y$, $F(F(X,Y),Z) = F(X,F(Y,Z))$. If moreover $F(X,Y) = F(Y,X)$ the formal group law is said to be commutative. All formal group laws will be commutative from now on. A homomorphism from $F(X,Y)$ with index set I to $G(X,Y)$ with index set J is an J-tuple of power series $\alpha(X)$ in X_i, $i \in I$ with coefficients in B, which satisfies the monomials have finite support condition such that $\alpha(F(X,Y)) = G(\alpha(X),\alpha(Y))$. Finally a formal A-module over $B \in \underline{\underline{\mathrm{Alg}}}_A$ with index set I is a formal group law $F(X,Y)$ over B together with a ring homomorphism ρ_F: $A \to \mathrm{End}_B(F(X,Y))$ such that $\rho_F(a) \equiv aX \bmod(\text{degree } 2)$ for all $a \in A$. (This implies of course that all the $\rho_F(a)$ satisfy the monomials have finite support condition). Note that the various formulas above like $F(X,F(Y,Z)) = F(F(X,Y),Z)$ and $\alpha(F(X,Y)) = G(\alpha(X),\alpha(Y)$ make sense because of lemma 7.3.

7.6. It is now immediately obvious that a universal formal group law with infinite index set I cannot exist because there is no predicting for which finitely many $i \in I$ the coefficient of a given monomial $X^\alpha Y^\beta$ in $F(i)(X,Y)$ will have nonzero coefficient. The way to remedy this is to extend the definition a bit by considering complete topological rings B whose topology is defined by a (filtered) set of ideals $\mathfrak{a}_s$, $s \in S$ such that $\bigcap_s \mathfrak{a}_s = \{0\}$ (so that B is Hausdorff).

7.7. Definition.

Let B be as above in 7.6 and let I and J be index sets. An I-tuple of power series $f(X)$ in X_j, $j \in J$ with coefficients in B is said to be continuous if for all multiindices $\alpha: J \to \mathbb{N} \cup \{0\}$ and all $s \in S$ there are only finitely many $i \in I$ such that the coefficient of X^α in $f(i)(X)$ is not in $\mathfrak{a}_s$. It is an immediate consequence of lemma 7.3 that the composite of two continuous sets of power series is welldefined and continuous and that the inverse power series $f^{-1}(X)$ of a continuous power series $f(X)$ such that $f(X) \equiv X$ mod(degree 2) is also welldefined and continuous.

7.8. Definitions.

Let B be as above in 7.6 and let I be an index set. A commutative infinite dimensional formal group law over B is now a continuous I-tuple of power series over B in X_i, Y_i, $i \in I$ such that $F(X,0) = X, F(0,Y) = Y$, $F(F(X,Y),Z) = F(X,F(Y,Z))$, $F(X,Y) = F(Y,X)$. Note that the condition $F(F(X,Y),Z) = F(X,F(Y,Z))$ makes sense again (because it makes sense mod $\mathfrak{a}_s$ for all s and because B is complete). The definitions for homomorphisms and formal A-modules are similarly modified by requiring all I-tuples of power series to be continuous. The definitions of 7.5 correspond to the case of a discretely topologized ring B (defined by the single ideal 0).

8. CONSTRUCTION OF AN INFINITE DIMENSIONAL UNIVERSAL FORMAL GROUP LAW.

8.1. Let R be any ring. Let I be an index set. The first thing to do is to describe the appropriate ring "of polynomials" over which a universal formal group law with index set I will be constructed. For each multiindex $\alpha : I \to \mathbb{N} \cup \{0\}$ (with finite support) such that $|\alpha| \geq 2$ and each $i \in I$ let $U(i,\alpha)$ be an indeterminate. Consider the ring of polynomials $R[U(i,\alpha) \mid i \in I,\ \alpha: I \to \mathbb{N} \cup \{0\},\ |\alpha| \geq 2]$.
Let T be the set of all functions on the set of multiindices on I to the

set of finite subsets of I. For each $\tau \in T$ let $\mathfrak{a}'_\tau \subset R[U]$ be the ideal generated by all the $U(i,\alpha)$ such that $i \notin \tau(\alpha)$. We now denote with $R\langle U;I\rangle$ the completion of $R[U]$ with respect to the topology defined by these ideals, and with $\mathfrak{a}_\tau$ the closure of $\mathfrak{a}'_\tau$ in $R\langle U;I\rangle$ for all $\tau \in T$. If I is a finite set then $R\langle U;I\rangle$ is simply $R[U(i,\alpha)]$ because one of the possible functions τ in this case is $\tau(\alpha) = I$ for all α and then $\mathfrak{a}_\tau = 0$. For each finite subset $\kappa \subset I$ there is a natural surjection $\phi_\kappa : R\langle U;I\rangle \to R\langle U;\kappa\rangle = R[U(i,\alpha) \mid \mathrm{supp}(\alpha) \cup \{i\} \subset \kappa]$. In fact the kernel is the ideal $\mathfrak{a}_{\tau(\kappa)}$ defined by the function $\tau(\kappa) \in T$, $\tau(\kappa)(\alpha) = \emptyset$ if $\mathrm{supp}(\alpha) \not\subset \kappa$ and $\tau(\kappa)(\alpha) = \kappa$ if $\mathrm{supp}(\alpha) \subset \kappa$. The $\mathfrak{a}_{\tau(\kappa)}$, $\kappa \subset I$, κ finite define another, coarser, topology in $R[U(i,\alpha)]$ which is, however, still Hausdorff. This means that $R\langle U;I\rangle$ is a certain subalgebra of $\varprojlim R\langle U;\kappa\rangle$, which in turn is a proper subalgebra of the projective limit of the polynomial rings in finitely many $U(i,\alpha)$'s over R. For example if $I = \mathbb{N}$ then $\sum_{i=1}^{\infty} U(i,2\varepsilon(i))$ where $\varepsilon(i)$ is the multiindex $\varepsilon(i)(j) = 0$ if $j \neq i$, $\varepsilon(i)(i) = 1$ is an element of $\varprojlim R\langle U;\kappa\rangle$ because for each κ it is a polynomial mod $\mathfrak{a}_{\tau(\kappa)}$. But this element is not an element of $R\langle U;\mathbb{N}\rangle$ because it is not a polynomial modulo $\mathfrak{a}_\tau$ if τ is, e.g., the function $\tau(\alpha) = \mathrm{supp}(\alpha)$.

The R-algebra $R\langle U;I\rangle$ has an obvious freeness property with respect to topological R-algebras B as in 7.6. Let B be such an algebra. And for every $\alpha: I \to \mathbb{N} \cup \{0\}$, $|\alpha| \geq 2$ and $i \in I$ let $b(i,\alpha)$ be an element of B. Suppose that for every α and every $s \in S$ there are only finitely many $b(i,\alpha) \notin \mathfrak{a}_s$. Then there is a unique continuous R-algebra homomorphism $\phi: R\langle U;I\rangle \to B$ such that $\phi(U(i,\alpha)) = b(i,\alpha)$ for all i,α.

8.2. Finite Dimensional Universal Formal Group Laws.

We recall the construction of an m-dimensional universal formal group law in [2], section 11. Let I be a finite set. For each multiindex $\alpha : I \to \mathbb{N} \cup \{0\}$ such that $|\alpha| \geq 2$ and each $i \in I$ let $U(i,\alpha)$ be an indeterminate. Let $\mathbb{Z}[U]$ (= $\mathbb{Z}\langle U;I\rangle$ if I is finite) be the ring of polynomials in these indeterminates.

In addition we define $U(i,\varepsilon(j)) = 1$ if $i = j$ and $= 0$ if $i \neq j$, where $\varepsilon(j)$ is the multiindex $\varepsilon(j)(k) = 0$ if $k \neq j$, $\varepsilon(j)(k) = 1$ if $k = j$. For each multiindex α let U_α be the columnvector $U(i,\alpha)_{i \in I}$. For each prime power $q = p^r$, $r \in \mathbb{N}$, p a prime number we use U_q to denote the matrix

$U_q = (U(i,q\varepsilon(j)))_{i,j\in I}$. Using all this notation we now define for all $\alpha : I \to \mathbb{N} \cup \{0\}$ with $|\alpha| \geq 1$ a column vector a_α with entries in $\mathbb{Q}[U]$ by means of the formula

$$(8.3)\qquad a_\alpha = \sum_{(q_1,\ldots,q_t,\beta)} \frac{n(q_1,\ldots,q_t)}{p_1} \cdots \frac{n(q_{t-1},q_t)}{p_{t-1}} \frac{n(q_t)}{p_t} U_{q_1} U_{q_2}^{(q_1)} \cdots U_{q_t}^{(q_1\cdots q_{t-1})} U_\beta^{(q_1\cdots q_t)}$$

where the sum is over all sequences $(q_1,\ldots,q_t,\beta)$, $t \in \mathbb{N} \cup \{0\}$, $q_i = p_i^{s_i}$, $s_i \in \mathbb{N}$, p_i a prime number, β a multiindex not of the form $\beta = p^r\varepsilon(j)$, $r \in \mathbb{N}$, $j \in I$, p a prime number, such that $q_1\ldots q_t\beta = \alpha$; $U_q^{(r)}$ is the matrix obtained from U_q by raising each of its entries to the power r and $U_\beta^{(s)}$ has the obvious analogous meaning. The numbers $n(q_1,\ldots,q_t)$ are integers which can be chosen arbitrarily subject to the conditions

$$(8.4)\qquad \begin{aligned} n(q_1,\ldots,q_t) &\equiv 1 \bmod p_1^r \text{ if } p_1 = p_2 = \ldots = p_r \neq p_{r+1},\ 1 \leq r \leq t \\ n(q_1,\ldots,q_t) &\equiv 0 \bmod p_2^r \text{ if } p_1 \neq p_2 = \ldots = p_r \neq p_{r+1},\ 2 \leq r \leq t \end{aligned}$$

Sometimes, in order to have reasonable formula for the U's in terms of the a's it is useful to choose the $n(q_1,\ldots,q_t)$ in a very special way, cf. [2] section 5.6 and section 34.4.

We now define

$$(8.5)\qquad f_U(X) = \sum_{|\alpha|\geq 1} a_\alpha X^\alpha\ ,\ F_U(X,Y) = f_U^{-1}(f_U(X) + f_U(Y))$$

Then, as is proved in section 11 of [2], $F_U(X,Y)$ is a universal formal group law with finite index set I. The integrality of $F_U(X,Y)$ is a consequence of the functional equation lemma 4.3.

8.6. Construction of an Infinite Dimensional Universal Formal Group Law.

Now let I be an infinite index set. Let $\mathbb{Z}\langle U;I\rangle$ be the ring constructed above in 8.1. For each finite subset $\kappa \subset I$ let $\mathbb{Z}\langle U;\kappa\rangle$ be the natural quotient $\mathbb{Z}[U(\alpha,i) \mid \mathrm{supp}(\alpha) \cup \{i\} \subset \kappa]$ of $\mathbb{Z}\langle U;I\rangle$. For each κ let $f_{U,\kappa}(X)$ and $F_{U,\kappa}(X,Y)$ be the power series in X_i, $i \in \kappa$ and X_i, Y_i, $i \in \kappa$ defined by (8.5). Fix a choice of the $n(q_1,\ldots,q_t)$ for all sequences of prime powers

$(q_1,\dots,q_t)$, $t \in \mathbb{N}$.
For each pair of finite subsets $\kappa,\lambda \subset I$ such that $\kappa \subset \lambda$ we use $\phi_{\lambda,\kappa}$: $\mathbb{Z}\langle U;\lambda\rangle \to \mathbb{Z}\langle U;\kappa\rangle$, $\mathbb{Q}\langle U;\lambda\rangle \to \mathbb{Q}\langle U;\kappa\rangle$, $\mathbb{Q}\langle U;\lambda\rangle[[X_i;i\in\lambda]] \to \mathbb{Q}\langle U;\lambda\rangle[[X_i;i\in\kappa]]$, $\mathbb{Z}\langle U;\lambda\rangle[[X_i,Y_i;i\in\lambda]] \to \mathbb{Z}\langle U;\kappa\rangle[[X_i,Y_i;i\in\kappa]]$ to denote the natural projections ($U(\alpha,i)\mapsto 0$ if $\lambda \supset \mathrm{supp}(\alpha) \cup \{i\} \not\subset \kappa$, $U(\alpha,i)\mapsto U(\alpha,i)$ if $\mathrm{supp}(\alpha) \cup \{i\} \subset \kappa$, $Y_i,X_i \mapsto 0$ if $i \in \lambda \setminus \kappa$, $X_i,Y_i \mapsto X_i,Y_i$ if $i \in \kappa$).
Now note that $\phi_{\lambda,\kappa}F_{U,\lambda}(X,Y) = F_{U,\kappa}(X,Y)$ and $\phi_{\lambda,\kappa}f_{U,\lambda}(X) = f_{U,\kappa}(X)$. This means that we can define I-tuples of power series $f_U(X)$ and $F_U(X,Y)$ as follows. For each multiindex α: $I \to \mathbb{N} \cup \{0\}$ and pair of multiindices α,β: $I \to \mathbb{N} \cup \{0\}$ and element $i \in I$ consider the finite subsets κ such that $\kappa \supset \mathrm{supp}(\alpha) \cup \mathrm{supp}(\beta) \cup \{i\}$. Now consider the coefficients $e_{\alpha,\kappa}(i)$ and $e_{\alpha,\beta,\kappa}(i)$ of X^α and $X^\alpha Y^\beta$ in $f_{U,\kappa}(i)(X)$ and $F_{U,\kappa}(i)(X,Y)$ respectively. In virtue of the compatibility of the $f_{U,\kappa}(X)$ and $F_{U,\kappa}(X,Y)$ under the $\phi_{\lambda,\kappa}$ the systems of elements $e_{\alpha,\kappa}(i)$ and $e_{\alpha,\beta,\kappa}(i)$ determine welldefined elements $e_\alpha(i)$, $e_{\alpha,\beta}(i)$ in $\varprojlim \mathbb{Q}\langle U;\kappa\rangle$ and $\varprojlim \mathbb{Z}\langle U;\kappa\rangle$ respectively.
We now define $f_U(X)$ and $F_U(X,Y)$ by

$$f_U(i)(X) = \sum_{|\alpha|\geq 1} e_\alpha(i)X^\alpha \qquad F_U(i)(X,Y) = \sum_{\alpha,\beta} e_{\alpha,\beta}(i)X^\alpha Y^\beta$$

I claim that in fact $e_\alpha(i) \in \mathbb{Q}\langle U;I\rangle \subset \varprojlim \mathbb{Q}\langle U;\kappa\rangle$.
Indeed we clearly have

$$(8.7)\quad e_\alpha(i) = \sum U(i,q_1\varepsilon(i_1))U(i_1,q_2\varepsilon(i_2))^{q_1} \cdots U(i_{t-1},q_t\varepsilon(i_t))^{q_1\cdots q_{t-1}}U(i_t,\beta)^{q_1\cdots q_t}d(q_1,\dots,q_t)$$

where $d(q_1,\dots,q_t) = p_1^{-1}\cdots p_t^{-1}n(q_1,\dots,q_t)n(q_2,\dots,q_t)\cdots n(q_{t-1},q_t)n(q_t)$ and where the sum is over all sequences $(q_1,\dots,q_t,\beta)$ as in the sum 8.3 and all $i_1,\dots,i_t \in I$. Let $\tau \in T$. Because $q_1\cdots q_t\beta = \alpha$ we have that $\mathrm{supp}(\beta) = \mathrm{supp}(\alpha)$. So there are only finitely many i_t such that $U(i_t,\beta) \notin \mathfrak{a}_\tau$, for each of these i_t there are only finitely many i_{t-1} such that $U(i_t,q_t\varepsilon(i_t)) \notin \mathfrak{a}_\tau$, ... , and for each of the i_2 there are only finitely many i_1 such that $U(i_1,q_1\varepsilon(i_2)) \notin \mathfrak{a}_\tau$. Finally there are only finitely many factorizations $q_1\cdots q_t\beta = \alpha$. It follows that $e_\alpha(i)$ is a polynomial mod $\mathfrak{a}_\tau$ for all τ

proving that $e_\alpha(i) \in \mathbb{Q}\langle U;I\rangle$. Because for every i_1 there are but finitely many i such that $U(i,q_1\varepsilon(i_1)) \notin \mathfrak{a}_\tau$ it also follows that $e_\alpha(i) \equiv 0 \bmod \mathfrak{a}_\tau$ for all but finitely many i. It follows that $f_U(X)$ is a continuous I-tuple of power series in the sense of definition 7.7. This in turn means that $f_U^{-1}(f_U(X) + f_U(Y))$ makes sense and has its coefficients in $\mathbb{Q}\langle U;I\rangle$. And this finally means that

$$f_U^{-1}(f_U(X) + f_U(Y)) = F_U(X,Y) \tag{8.8}$$

so that $F_U(X,Y)$ has its coefficients in $\mathbb{Z}\langle U;I\rangle \subset \varprojlim \mathbb{Z}\langle U;\kappa\rangle \subset \varprojlim \mathbb{Q}\langle U;\kappa\rangle$.

9. PROOF OF THE UNIVERSALITY OF THE INFINITE DIMENSIONAL UNIVERSAL FORMAL GROUP LAW $F_U(X,Y)$ over $\mathbb{Z}\langle U;I\rangle$.

This proof is in it essentials exactly like the proof in [2], section 11.4 of the universality of the finite dimensional formal group law described in 8.2 above.

If β,α: $I \to \mathbb{N} \cup \{0\}$ are multiindices we write $\alpha > \beta$ if $\alpha(i) \geq \beta(i)$ for all $i \in I$ and $|\alpha| > |\beta|$. We use 0 to denote the multiindex $0(i) = 0$ all $i \in I$. We define $\nu(\alpha) = 1$ unless α is of the form $\alpha = p^r\varepsilon(j)$, $r \in \mathbb{N}$, $j \in I$, p a prime number and $\nu(p^r\varepsilon(j)) = p$. Then $\nu(\alpha)$ is the greatest common divisor of the $\binom{\alpha}{\beta} = \prod_{i\in \mathrm{supp}(\alpha)} \binom{\alpha(i)}{\beta(i)}$ for $0 < \beta < \alpha$.

For each $\alpha > \beta$ choose $\lambda_{\alpha,\beta} \in \mathbb{Z}$ as in [2], 11.3.5 such that

$$\sum_{0<\beta<\alpha} \lambda_{\alpha,\beta}\binom{\alpha}{\beta} = \nu(\alpha) \tag{9.1}$$

Then exactly as in [2], lemma 11.3.7 we have the following lemma

9.2. <u>Lemma</u>. Let α: $I \to \mathbb{N} \cup \{0\}$ be a multiindex, $|\alpha| \geq 2$. For each $0 < \beta < \alpha$ let X_β be an indeterminate and let $X_\beta = X_{\alpha-\beta}$. Then every X_β, $0 < \beta < \alpha$ can be written as a linear expression with coefficients in $\mathbb{Z}$ of the expressions

$$\sum_{0<\beta<\alpha} \lambda_{\alpha,\beta}X_\beta,\ \binom{\beta+\gamma}{\gamma}X_{\beta+\gamma} - \binom{\gamma+\delta}{\delta}X_{\gamma+\delta},\ \beta + \gamma + \delta = \alpha, \beta,\gamma,\delta > 0$$

9.3. Proof of the Universality of $F_U(X,Y)$.

From formula (8.7) above we see that

$$(9.4)\qquad f_U(X) \equiv X + \sum_{|\alpha|=n} \nu(\alpha)^{-1} U_\alpha X^\alpha \mod(\text{degree } n+1,\ U(\beta,j) \text{ with } |\beta| < n)$$

It follows that

$$(9.5)\qquad F_U(X,Y) \equiv X + Y + \sum_{|\alpha|=n} \nu(\alpha)^{-1} U_\alpha X^\alpha + \sum_{|\alpha|=n} \nu(\alpha)^{-1} U_\alpha Y^\alpha - \sum_{|\alpha|=n} \nu(\alpha)^{-1} (X+Y)^\alpha$$

mod(degree n+1, $U(\beta,j)$ with $|\beta| < n$). Now write

$$(9.6)\qquad F_U(i)(X,Y) = X_i + Y_i + \sum_{|\alpha|,|\beta|\geq 1} e_{\alpha,\beta}(i) X^\alpha Y^\beta$$

and define

$$(9.7)\qquad y(i,\alpha) = - \sum_{0<\beta<\alpha} \lambda_{\alpha,\beta} e_{\beta,\alpha-\beta}(i)$$

for all α: $I \to \mathbb{N} \cup \{0\}$, $|\alpha| \geq 2$, $i \in I$. It follows immediately from (9.6) that

$$(9.8)\qquad y(i,\alpha) \equiv U(i,\alpha) \mod(U(j,\beta) \text{ with } |\beta| < |\alpha|).$$

Also $y(i,\alpha)$ is a polynomial mod $\mathfrak{a}_\tau$ for all τ, i.e. $y(i,\alpha) \in \mathbb{Z}\langle U;I\rangle$, because (9.7) is a finite sum. From this it follows that we can, so to speak, describe $\mathbb{Z}\langle U;I\rangle$ also as $\mathbb{Z}\langle y;I\rangle$, or, in other words, the $y(i,\alpha)$ are a "free polynomial basis" for $\mathbb{Z}\langle U;I\rangle$ meaning that the images of the $y(i,\alpha)$, $i \in \tau(\alpha)$ are a free polynomial basis for $\mathbb{Z}\langle U;I\rangle/\mathfrak{a}_\tau$ for all τ. Now let $G(X,Y)$ over B, where B is as in 7.6, be any formal group law (in the sense of 7.8) with index set I. We write

$$(9.9)\qquad G(i)(X,Y) = X_i + Y_i + \sum_{|\alpha|,|\beta|\geq 1} b_{\alpha,\beta}(i) X^\alpha Y^\beta$$

We now define a continuous homomorphism $\mathbb{Z}\langle U;I\rangle \to B$ by requiring that

$$(9.10)\qquad \phi(y(i,\alpha)) = - \sum_{0<\beta<\alpha} \lambda_{\alpha,\beta} b_{\beta,\alpha-\beta}(i)$$

for all i,α. This ϕ is welldefined and determined uniquely because of 9.8 and the remarks just below 9.8. The homomorphism is continuous because G(X,Y) is continuous I-tuple of power series in the sense of definition 7.7, and because the sum on the right of (9.10) is finite.
Certainly ϕ is the only possible continuous homomorphism $\mathbb{Z}\langle U;I\rangle \to B$ such that $\phi_* F_U(X,Y) = G(X,Y)$. It remains to show that $\phi(e_{\alpha,\beta}(i)) = b_{\alpha,\beta}(i)$ for all α,β,i. This is obvious if $|\alpha+\beta| = 2$. So by induction let us assume that this has been proved for all α,β with $|\alpha+\beta| < n$. Commutativity and associativity of $F_U(X,Y)$ and G(X,Y) mean that we have relations

$$e_{\alpha,\beta}(i) = e_{\beta,\alpha}(i) \qquad\qquad b_{\alpha,\beta}(i) = b_{\beta,\alpha}(i)$$

$$\binom{\alpha+\beta}{\beta} e_{\alpha+\beta,\gamma}(i) - \binom{\beta+\gamma}{\gamma} e_{\beta+\gamma,\alpha}(i) = Q_{\alpha,\beta,\gamma,i}(e_{\delta,\varepsilon}(j))$$

$$\binom{\alpha+\beta}{\beta} b_{\alpha+\beta,\gamma}(i) - \binom{\beta+\gamma}{\gamma} b_{\beta+\gamma,\alpha}(i) = Q_{\alpha,\beta,\gamma,i}(b_{\delta,\varepsilon}(j))$$

where the $Q_{\alpha,\beta,\gamma,i}$ are certain universal expressions involving only the $e_{\delta,\varepsilon}(j)$, $b_{\delta,\varepsilon}(\gamma)$ with $|\delta+\varepsilon| < |\alpha+\beta+\gamma|$. By induction we therefore know that

$$\phi(\binom{\alpha+\beta}{\beta} e_{\alpha+\beta,\gamma}(i) - \binom{\beta+\gamma}{\gamma} e_{\beta+\gamma,\alpha}(i)) = \binom{\alpha+\beta}{\beta} b_{\alpha+\beta,\gamma}(i) - \binom{\beta+\gamma}{\gamma} b_{\beta+\gamma,\alpha}(i)$$

for all $\alpha,\beta,\gamma > 0$ with $|\alpha+\beta+\gamma| = n$. We also have by the definition of ϕ

$$\phi(\sum_{0<\beta<\alpha} \lambda_{\alpha,\beta} e_{\beta,\alpha-\beta}(i)) = \sum_{0<\beta<\alpha} \lambda_{\alpha,\beta} b_{\beta,\alpha-\beta}(i)$$

for all α, i with $|\alpha| = n$. Using lemma 9.2 it follows that $\phi(e_{\alpha,\beta}(i)) = b_{\alpha,\beta}(i)$ for all α,β,i with $|\alpha+\beta| = n$. With induction this finishes the proof.

9.11. Corollary.

Every infinite dimensional formal group law in the classical sense (cf. definition 7.5) can be lifted to characteristic zero.

Indeed these formal group laws correspond to continuous homomorphisms $\phi\colon \mathbb{Z}\langle U;I\rangle \to B$ where B has the discrete topology. This means that $\phi(\mathfrak{a}_\tau) = 0$ for a certain τ and $\mathbb{Z}\langle U;I\rangle/\mathfrak{a}_\tau$ is a ring of polynomials.

9.12. Corollary.

Every infinite dimensional formal group law over a torsion free ring has a unique logarithm.

10. INFINITE DIMENSIONAL UNIVERSAL FORMAL A-MODULES.

Let A be as in section 3 above. Let I be an index set. The construction of an infinite dimensional formal A-module is completely analogous to the constructions of section 8 above. For each finite subset κ let $f_{U,\kappa}(X)$ be the logarithm of the universal formal A-module with index set κ over A $[U(\alpha,i)\,|\,\mathrm{supp}(\alpha) \cup \{i\} \subset \kappa,\ |\alpha| \geq 2]$. By taking projective limits of the coefficients we obtain a formal power series $f_U^A(X)$ over $\varprojlim A[U(\alpha,i)\,|\,\mathrm{supp}(\alpha) \cup \{i\} \subset \kappa]$ and by making use of the explicit formula (5.1) one shows that in fact the coefficients of $f_U^A(X)$ are in the sub-A-algebra A<U;I> and that $f_U^A(X)$ is a continuous I-tuple of power series. Now let

$$(10.1)\quad F_U^A(X,Y) = (f_U^A)^{-1}(f_U^A(X) + f_U^A(Y)),\ \rho(a) = [a](X) = = (f_U^A)^{-1}(af_U^A(X)) \text{ all } a \in A$$

Then $(F_U^A(X,Y),\rho)$ is a formal A-module over A<U;I>. This can be shown either by performing the same projective limit construction with respect to the finite dimensional objects $F_{U,\kappa}^A(X,Y)$, $[a](X)_\kappa$ and observing that the relations (10.1) hold in $\varprojlim A[U(i,\alpha)\,|\,\mathrm{supp}(\alpha) \cup \{i\} \subset \kappa]$. This is what we used in section 8 above. Or one can state and prove an appropriate infinite dimensional version of the functional equation lemma. This version is simply obtained by requiring all I-tuples of power series to be continuous. The proof that the formal A-module (10.1) is indeed universal is an entirely straightforward adaptation of the proof in [2], section 25.4 that the finite dimensional formal A-modules described in section 5 above are universal.

10.2. Corollary.

Every infinite dimensional formal A-module in the sense of 7.5 above can be lifted to formal A-module over an A-torsion free A-algebra.

10.3. Corollary.

Every infinite dimensional formal A-module over an A-torsion free algebra B has a unique A-logarithm.

10.4. The A-logarithm $f_U^A(X)$ of the universal formal A-module $F_U^A(X,Y)$ over A<U;I> is of functional equation type, and there does exist a topological analogue of the functional equation lemma 4.3. In the case of A<U;I> and $\mathbb{Z}$ <U;I> this analogue is probably most easily proved by first remarking that the proofs in [2] also work in the infinite dimensional case provided that all the I-tuples of power series involved satisfy the monomials have compact support condition. The topological version alluded to above then results by proving things over A<U;I>/$\mathfrak{a}_\tau$ and $\mathbb{Z}$ <U;I>/$\mathfrak{a}_\tau$ for all τ.
This permits us to define ε_q and $\underline{\underline{f}}_\pi$ for curves in $F_U^A(X;Y)$ and hence by specialization for curves over arbitrary infinite dimensional formal A-modules.
The construction of the infinite dimensional formal group laws $F_U(X,Y)$ over $\mathbb{Z}$ <U;I> and the infinite dimensional universal formal A-modules over A<U;I> also permit us to extend the Cartier-Dieudonné module classification theory of [2], chapter V to cover infinite dimensional case. The proofs are entirely straightforward adaptations of the proofs given in [2].

REFERENCES.

1. V.G. Drinfel'd, Coverings of p-adic Symmetric Domains (Russian), Funk. Analiz i ego Pril. 10(1976), 29-40.
2. M. Hazewinkel, Formal Groups and Applications, Acad. Press (to appear, Sept. 1978).
3. M. Hazewinkel, On Formal Groups: The Functional Equation Lemma and some of its applications. In: Journées de géométrie algébrique, Rennes, 1978 (to appear, Astérisque).
4. M. Hazewinkel, Twisted Lubin-Tate Formal Group Laws, ramified Witt Vectors and ramified Artin-Hasse Exponentials, Report 7711, Econometric Inst., Erasmus Univ., Rotterdam 1977.