Uniform Proofs of Order Independence for Various Strategy Elimination Procedures

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Abstract

We provide elementary and uniform proofs of order independence for various strategy elimination procedures for finite strategic games, both for dominance by pure and by mixed strategies. The proofs follow the same pattern and focus on the structural properties of the dominance relations. They rely on Newman's Lemma (see Newman [1942]) and related results on the abstract reduction systems.

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1 Introduction

1.1 Preliminaries

To properly discuss the background for this research we need to recall a number of concepts commonly used in the study of strategic games. We follow here a standard terminology of the game theory, see, e.g., Myerson [1991] or Osborne and Rubinstein [1994]. We stress the fact that we deal here only with *finite* games. Given n players we represent a strategic game by a sequence

$$(S_1, \ldots, S_n, p_1, \ldots, p_n),$$

where for each $i \in [1..n]$

- S_i is the *finite*, non-empty, set of ***strategies*** (sometimes called ***pure strategies***) available to player i,
- p_i is the payoff function for the player i, so

  $$p_i : S_1 \times \ldots \times S_n \rightarrow \mathcal{R},$$

  where $\mathcal{R}$ is the set of real numbers.

We assume that $S_i \cap S_j = \emptyset$ for $i \neq j$.

Given a sequence of non-empty sets of strategies $S_1, \ldots, S_n$ and $s \in S_1 \times \ldots \times S_n$ we denote the ith element of s by s_i and use the following standard notation:

- $s_{-i} := (s_1, \ldots, s_{i-1}, s_{i+1}, \ldots, s_n)$,
- $(s'_i, s_{-i}) := (s_1, \ldots, s_{i-1}, s'_i, s_{i+1}, \ldots, s_n)$, where we assume that $s'_i \in S_i$. In particular $(s_i, s_{-i}) = s$,
- $S_{-i} := S_1 \times \ldots \times S_{i-1} \times S_{i+1} \times \ldots \times S_n$,
- $(S'_i, S_{-i}) := S_1 \times \ldots \times S_{i-1} \times S'_i \times S_{i+1} \times \ldots \times S_n$.

We denote the strategies of player i by s_i, possibly with some superscripts.

Next, given a game $G := (S_1, \ldots, S_n, p_1, \ldots, p_n)$ and non-empty sets of strategies $S'_1, S''_1, \ldots, S'_n, S''_n$ such that $S'_i \subseteq S_i$ and $S''_i \subseteq S_i$ for $i \in [1..n]$ we say that $G' := (S'_1, \ldots, S'_n, p_1, \ldots, p_n)$ and $G'' := (S''_1, \ldots, S''_n, p_1, \ldots, p_n)$ are ***restrictions***[1] of G and denote by $G' \cap G''$ the restriction $(S'_1 \cap S''_1, \ldots, S'_n \cap$

[1]Sometimes the name ***reduction*** is used. In Gilboa, Kalai and Zemel [1990] a restriction is called a ***subgame***.

$S''_n, p_1, \ldots, p_n)$. In each case we identify each payoff function p_i with its restriction to the Cartesian product of the new strategy sets.

Fix a game $(S_1, \ldots, S_n, p_1, \ldots, p_n)$. We now introduce a number of well-known binary dominance relations on strategies. We say that a strategy s_i is ***weakly*** (***strictly***) ***dominated*** by a strategy s'_i, or equivalently, a strategy s'_i ***weakly*** (***strictly***) ***dominates*** a strategy s_i, if

$$p_i(s_i, s_{-i}) \leq p_i(s'_i, s_{-i})$$

for all $s_{-i} \in S_{-i}$, with some inequality (all inequalities) being strict. We denote the weak dominance relation by W and the strict dominance relation by S.

Further, we say that the strategies s_i and s'_i of player i are ***compatible*** if for all $j \in [1..n]$ and $s_{-i} \in S_{-i}$

$$p_i(s_i, s_{-i}) = p_i(s'_i, s_{-i}) \text{ implies } p_j(s_i, s_{-i}) = p_j(s'_i, s_{-i}).$$

We then say that s_i is ***nicely weakly dominated by*** s'_i if s_i is weakly dominated by s'_i and s_i and s'_i are compatible. This notion of dominance, that we denote by NW, was introduced in Marx and Swinkels [1997].

Finally, recall that two strategies s_i and s'_i of player i are called ***payoff equivalent*** if

$$p_j(s_i, s_{-i}) = p_j(s'_i, s_{-i})$$

for all $j \in [1..n]$ and all $s_{-i} \in S_{-i}$. We denote this binary relation on the strategies by PE.

These notions have natural counterparts for mixed strategies that will be introduced later in the paper.

Each binary dominance relation R, so in particular W, S, NW or PE, induces the following binary relation on strategic games $G := (S_1, \ldots, S_n, p_1, \ldots, p_n)$ and $G' := (S'_1, \ldots, S'_n, p_1, \ldots, p_n)$:

> $G \Rightarrow_R G'$ iff $G \neq G'$ and for all $i \in [1..n]$ each $s_i \in S_i \setminus S'_i$ is R-dominated in G by some $s'_i \in S'_i$.

If all iterations of $\Rightarrow_R$ starting in an initial game G yield the same final outcome, we say that R is ***order independent***.

1.2 Background

In the literature on dominance relations in finite strategic games several order independence results were established, to wit:

- Gilboa, Kalai and Zemel [1990] and Stegeman [1990] proved it for strict dominance by pure strategies,
- Börgers [1990,1993] established it for his notion of (unary) dominance,
- Osborne and Rubinstein [1994] proved it for strict dominance by mixed strategies,
- Marx and Swinkels [1997,2000]) proved it for nice weak dominance up to the addition or removal of the payoff equivalent strategies and a renaming of strategies.

  This implies the same form of order independence for weak dominance by pure strategies for the games $(S_1, \ldots, S_n, p_1, \ldots, p_n)$ satisfying the following ***transference of decisionmaker indifference (TDI)*** condition:

  for all $i, j \in [1..n]$, $r_i, t_i \in S_i$ and $s_{-i} \in S_{-i}$
  $p_i(r_i, s_{-i}) = p_i(t_i, s_{-i})$ implies $p_j(r_i, s_{-i}) = p_j(t_i, s_{-i})$.

  Informally, this condition states that whenever for player i two of its strategies r_i and t_i are indifferent w.r.t. some joint strategy s_{-i} of the other players, then r_i and t_i are also indifferent w.r.t. s_{-i} for all players.

  They also established analogous results for nice weak dominance and weak dominance by mixed strategies.

These results were established by different methods and techniques. In particular, the proof of order independence given in Börgers [1990] proceeds through a connection between the rationalizability notion of Pearce [1984] and the survival of a strategy under the iterated dominance. In turn, the original proof of order independence for strict dominance by mixed strategies given in Osborne and Rubinstein [1994, pages 61-62] involves in an analogous way a modification of the rationalizability notion and relies on the existence of Nash equilibrium for strictly competitive games.

It is useful to point out that the assumption that the games are finite is crucial. In fact, in an interesting paper Dufwenberg and Stegeman [2002] showed that in case of infinite games order independence for strict dominance does not hold. They also provided natural conditions under which the unique outcome is guaranteed.

1.3 Motivation

In this paper we provide uniform and elementary proofs of the abovementioned and related order independence results. The table in Figure 1 should clarify the scope of the paper. So we deal both with unary and binary dominance relations and with pure and mixed strategies. While binary dominance relations, such as the ones introduced in the previous subsection, are more known, the unary ones, introduced in Börgers [1990,1993], allow us to characterize a specific form of the rational strategies.

dominance \ strategies	pure	mixed
unary		
binary		

Figure 1: Classification of the order independence results

Further, we also consider combinations of binary dominance relations, both for pure and for mixed strategies.

Having in mind such a plethora of possibilities it is difficult to expect a single 'master result' that would imply all the discussed order independence results. Still, as we show, it is possible to provide uniform proofs of these results based on the same principles. Notably, our presentation focuses on so-called abstract reduction systems (see, e.g., Terese [2003]) in particular on Newman's Lemma (see Newman [1942]) and some of its natural refinements.

Newman's Lemma offers a simple but highly effective and versatile tool for proving order independence results. We discuss it and its consequences in detail in the next section and later, in Section 7. Let us just mention here that it deals with the properties of a binary relation $\rightarrow$ on an arbitrary set A. Below $\rightarrow^*$ denotes the transitive reflexive closure of $\rightarrow$. We say that $\rightarrow$ is ***weakly confluent*** if for all $a, b, c \in A$

$$\begin{array}{ccc} & a & \\ \swarrow & & \searrow \\ b & & c \end{array}$$

implies that for some $d \in A$

$$\begin{array}{ccc} b & & c \\ \searrow^{*} & & {}^{*}\swarrow \\ & d & \end{array}$$

Then Newman's lemma simply states that whenever

- no infinite $\rightarrow$ sequences exist,
- $\rightarrow$ is weakly confluent,

then for each element $a \in A$ all $\rightarrow$ sequences starting in a have a unique 'end outcome'.

It turns out that to prove order independence of a (binary or unary) dominance relation R it suffices to establish weak confluence of the corresponding reduction relation $\Rightarrow_R$ and apply Newman's lemma. In fact, since only finite games are considered, no infinite $\Rightarrow_R$ sequences exist.

To deal with combinations of two dominance relations, in particular the combination of nice weak dominance *NW* and payoff equivalence *PE*, a relativized version of Newman's lemma is helpful, where one only claims unique 'end outcome' up to an equivalence relation. In the game-theoretic setting this equivalence relation is an 'equivalence up to strategy renaming' relation on strategic games.

Further, the following notion involving a relative dependence between two binary relations $\rightarrow_1$ and $\rightarrow_2$ on some set A turns out to be useful. We say that $\rightarrow_1$ ***left commutes with*** $\rightarrow_2$ if

$$\rightarrow_1 \circ \rightarrow_2 \subseteq \rightarrow_2 \circ \rightarrow_1^*,$$

i.e., if for all $a, b, c \in A$ $a \rightarrow_1 b \rightarrow_2 c$ implies that for some $d \in A$ $a \rightarrow_2 d \rightarrow_1^* c$.

Now, one can prove that $\Rightarrow_{PE}$ left commutes with $\Rightarrow_{NW}$. This allows us to 'push' the removal of the payoff equivalent strategies to the 'end' and prove a 'structured' form of the order independence of *NW* combined with *PE*, a result originally established in Marx and Swinkels [1997].

Our presentation is also influenced by Gilboa, Kalai and Zemel [1990] where order independence for strict dominance was proved by establishing this result first for arbitrary (binary) dominance relations that are strict partial orders and hereditary.

In our approach we isolate other useful properties of dominance relations, both for the case of pure and mixed strategies. In particular, we identify conditions that allow us to conclude order independence up to a renaming of strategies for a combination of two reduction relations. This allows us to identify the relevant properties of nice weak dominance that lead to the results of Marx and Swinkels [1997].

Of course, each strategy elimination procedure needs to be motivated, either by clarifying the reasoning used by the players or by clarifying its effect on the structure of the game, for example on its set of Nash equilibria. In our exposition we ignore these issues since we focus on the dominance relations and the entailed elimination procedures that were introduced and motivated in the cited references.

1.4 Organization of the paper

The paper is organized as follows. In the next section we discuss Newman's lemma. Then in Section 3, following Gilboa, Kalai and Zemel [1990], we set the stage by discussing (binary) dominance relations for strategic games and their natural properties, in particular *hereditarity*. Intuitively, a dominance relation is hereditary if it is inherited from a game to any restriction. Some dominance relations are hereditary, while others not. Usually, for non-hereditary dominance relations order independence does not hold.

Then, in Section 4, we generalize the approach of Börgers [1990,1993] to deal with arbitrary non-hereditary dominance relations. Informally, each such binary dominance relation R can be modified to a unary dominance relation for which under some natural assumption both entailed reduction relations $\rightarrow_{inh-R}$ and $\Rightarrow_{inh-R}$ are order independent.

Next, in Section 5 we study dominance relations where the dominating strategies are mixed. We mimic here the development of Section 3 by identifying natural properties and establishing a general result on order independence. We apply then these results to show order independence for strict dominance by mixed strategies. In Section 6 we generalize the approach of Section 4 to the case when the dominating strategies are mixed.

To prepare the ground for results involving game equivalence we discuss in Section 7 a modification of Newman's Lemma in presence of an equivalence relation. Then in Section 8 we resume the discussion of dominance relations by focusing on the payoff equivalence. For this dominance relation order independence does not hold, but order independence up to a renaming of strategies does hold. Analogous results hold in case of equivalence to a mixed strategy and are discussed in Section 9.

Then in Section 10 we study conditions under which order dominance up to a renaming of strategies can be proved for a combination of two dominance relations. Such a combination is useful to study when one of these two relations is not hereditary. Then in Section 11 we apply the obtained general result to get a simple and informative proof of a result of Marx and Swinkels [1997] that nice weak dominance is order independent up to the removal of the payoff equivalent strategies and a renaming of strategies. In the next two Sections, 12 and 13, we mimic these developments for the case of equivalence to and dominance by a mixed strategy. Finally, in the concluding section we summarize the results in a tabular form and explain why each of the discussed order independence results has to be established separately.

2 Abstract Reduction Systems

We provide first completely general results concerning abstract reduction systems. An ***abstract reduction system***, see, e.g., Huet [1980], (and Terese [2003] for a more recent account, where a slightly different terminology is used) is a pair $(A, \rightarrow)$ where A is a set and $\rightarrow$ is a binary relation (a *reduction*) on A. Let $\rightarrow^+$ denote the transitive closure of $\rightarrow$ and $\rightarrow^*$ the transitive reflexive closure of $\rightarrow$. So in particular, if $a = b$, then $a \rightarrow^* b$. Further, $a \rightarrow^\epsilon b$ means $a = b$ or $a \rightarrow b$.

- We say that b is a $\rightarrow$***-normal form of*** a if $a \rightarrow^* b$ and no c exists such that $b \rightarrow c$, and omit the reference to $\rightarrow$ if it is clear from the context. If every element of A has a unique normal form, we say that $(A, \rightarrow)$ (or just $\rightarrow$ if A is clear from the context) satisfies the ***unique normal form property***. [2]

- We say that $\rightarrow$ is ***weakly confluent*** if for all $a, b, c \in A$

$$\begin{array}{ccc} & a & \\ \swarrow & & \searrow \\ b & & c \end{array}$$

  implies that for some $d \in A$

$$\begin{array}{ccc} b & & c \\ \searrow^{*} & & {}^{*}\swarrow \\ & d & \end{array}$$

- Following Gilboa, Kalai and Zemel [1990] we say that $\rightarrow$ is ***one step closed*** if for all $a \in A$ some $a' \in A$ exists such that

[2] We stress the fact that this notion of a normal form, standard in the theory of abstract reduction systems, has no relation whatsoever to the notion of a game in normal form, another name used for strategic games. In particular, the reader should bear in mind that later we shall consider *strategic* games that are *normal forms* of specific reduction relations on strategic games.

– $a \to^{\epsilon} a'$,

– if $a \to b$, then $b \to^{\epsilon} a'$.

In all proofs of weak confluence given in the paper we shall actually establish that for some $d \in A$ we have $b \to^{\epsilon} d$ and $c \to^{\epsilon} d$.

In the sequel, as already mentioned, we shall repeatedly rely upon the following lemma established in Newman [1942].

Lemma 2.1 (Newman) *Consider an abstract reduction system* $(A, \to)$ *such that*

- *no infinite* $\to$ *sequences exist,*
- $\to$ *is weakly confluent.*

Then $\to$ *satisfies the unique normal form property.*

Proof. (Taken from Terese [2003, page 15].)
By the first assumption every element of A has a normal form. To prove uniqueness, call an element a *ambiguous* if it has at least two different normal forms. We show that for every ambiguous a some ambiguous b exists such that $a \to b$. This proves absence of ambiguous elements by the first assumption.

So suppose that some element a has two distinct normal forms n_1 and n_2. Then for some b, c we have $a \to b \to^* n_1$ and $a \to c \to^* n_2$. By weak confluence some d exists such that $b \to^* d$ and $c \to^* d$. Let n_3 be a normal form of d. It is also a normal form of b and of c. Moreover $n_3 \neq n_1$ or $n_3 \neq n_2$. If $n_3 \neq n_1$, then b is ambiguous and $a \to b$. And if $n_3 \neq n_2$, then c is ambiguous and $a \to c$. □

Note that if $\to$ is not irreflexive, then the first condition is violated. So this lemma can be applicable only to the relations $\to$ that are irreflexive. All reduction relations on games here considered are by definition irreflexive. Moreover, because the games are assumed to be finite, these reduction relations automatically satisfy the first condition of Newman's lemma.

Also, the following simple observation will be helpful.

Note 2.2 (Unique Normal Form) *Consider two abstract reduction systems* $(A, \to_1)$ *and* $(A, \to_2)$ *such that*

- $\to_1$ *satisfies the unique normal form property,*
- $\to_1^+ = \to_2^+$.

Then $\rightarrow_2$ *satisfies the unique normal form property.* □

In the remainder of the paper we shall study abstract reduction systems that consist of the set of all restrictions of a game and a reduction relation on them. Since we limit ourselves to finite games, in such abstract reduction systems $(A, \rightarrow)$ no infinite $\rightarrow$ sequences exist.

In this context there are three natural ways of establishing that $(A, \rightarrow)$ satisfies the unique normal form property:

- by showing that $\rightarrow$ is one step closed: this directly implies weak confluence, and then Newman's Lemma can be applied;
- by showing that $\rightarrow$ is weakly confluent and applying Newman's Lemma;
- by finding a 'more elementary' reduction relation $\rightarrow_1$ such that
  - no infinite $\rightarrow_1$ sequences exist,
  - $\rightarrow_1$ is weakly confluent,
  - $\rightarrow_1^+ = \rightarrow^+$,

  and applying Newman's Lemma and the Unique Normal Form Note 2.2.

For some reduction relations all three results are equally easy to establish, while for some others only one.

3 Dominance Relations

We now study (binary) dominance relations in full generality. A ***dominance relation*** is a function that assigns to each game $G := (S_1, \ldots, S_n, p_1, \ldots, p_n)$ a subset R_G of $\bigcup_{i=1}^{n}(S_i \times S_i)$. Instead of writing that $s_i \; R_G \; s'_i$ holds we write that $s_i \; R \; s'_i$ ***holds for*** G. We say then that s_i ***is*** R***-dominated by*** s'_i ***in*** G or that that s'_i R***-dominates*** s_i ***in*** G. When G is clear from the context we drop a reference to it and view a dominance relation as a binary relation on the strategies of G.

Given a dominance relation R we introduce two notions of reduction between a game $G := (S_1, \ldots, S_n, p_1, \ldots, p_n)$ and its restriction $G' := (S'_1, \ldots, S'_n, p_1, \ldots, p_n)$.

- We write $G \rightarrow_R G'$ when $G \neq G'$ and for all $i \in [1..n]$

$$\text{each } s_i \in S_i \setminus S'_i \text{ is } R\text{-dominated in } G \text{ by some } s'_i \in S_i.$$

- We write $G \Rightarrow_R G'$ when $G \neq G'$ and for all $i \in [1..n]$

$$\text{each } s_i \in S_i \setminus S'_i \text{ is } R\text{-dominated in } G \text{ by some } s'_i \in S'_i.$$

So the relations $\rightarrow_R$ and $\Rightarrow_R$ differ in just one symbol (spot the difference). Namely, in the case of $\rightarrow_R$ we require that each strategy removed from S_i is R-dominated in G by a strategy in S_i, while in case of $\Rightarrow_R$ we require that each strategy removed from S_i is R-dominated in G by a strategy in S'_i. So in the latter case the dominating strategy should not be removed at the same time.

In the literature both reduction relations were considered. In our subsequent presentation we shall focus on the second one, $\Rightarrow_R$, since

- for most of the reduction relations studied here $\rightarrow_R$ and $\Rightarrow_R$ coincide,
- for payoff equivalence these relations do not coincide and only the second reduction relation is meaningful.

On the other hand, the first reduction relation, $\rightarrow_R$, allows us to define the 'maximal' elimination strategy according to which in each round all R-dominated strategies are deleted. Such a natural strategy is in particular of interest when order independence fails, see, e.g., Gilli [2002].

Further, note when $G \rightarrow_R G'$, the game G' can be 'degenerated' in the sense that some of the strategy sets of G' can be empty. However, this cannot happen when $\rightarrow_R$ and $\Rightarrow_R$ coincide, since then $G \Rightarrow_R G'$ implies that G' is not 'degenerated'.

Finally, let us mention that for various type of dominance relations R (unary or binary, for pure and mixed strategies) studied here the equivalence between the corresponding $\rightarrow_R$ and $\Rightarrow_R$ reduction relations plays a crucial role in the proofs of the order independence results.

So each reduction relation has some advantages and it is natural to introduce both of them.

Recall that a strict partial order is an irreflexive transitive relation. We say now that a dominance relation R is a ***strict partial order*** if for each game G the binary relation R_G is a strict partial order and reuse in a similar way other typical properties of binary relations. The following observation clarifies the first item above and will be needed later.

Lemma 3.1 (Equivalence) *If a dominance relation R is a strict partial order, then the relations $\rightarrow_R$ and $\Rightarrow_R$ coincide.*

Proof. It suffices to show that if $G \rightarrow_R G'$, then $G \Rightarrow_R G'$.

Let $G := (S_1, \ldots, S_n, p_1, \ldots, p_n)$ and $G' := (S'_1, \ldots, S'_n, p_1, \ldots, p_n)$. Suppose that some $s_i \in S_i \setminus S'_i$ is R-dominated in G by some $s'_i \in S_i$. Since R is a strict partial order and S_i is finite, a strategy $s'_i \in S_i$ exists that R-dominates s_i in G and is not R-dominated in G. So this s'_i is not eliminated in the step $G \rightarrow_R G'$ and consequently s_i is R-dominated in G by some $s'_i \in S'_i$. □

In what follows we establish a general 'order independence' result for the reduction relation $\Rightarrow_R$ for the dominance relations R that are strict partial orders and satisfy the following natural assumption due to Gilboa, Kalai and Zemel [1990]. We say that a dominance relation R is ***hereditary*** if for every game G, its restriction G' and two strategies s_i and s'_i of G'

s_i is R-dominated by s'_i in G implies s_i is R-dominated by s'_i in G'.

Each reduction relation $\Rightarrow_R$ can be specialized by stipulating that a *single strategy* is removed. We denote the corresponding reduction relation by $\Rightarrow_{1,R}$. A natural question when the reduction relation $\Rightarrow_R$ can be modeled using the iterated application of the $\Rightarrow_{1,R}$ reduction relation does not turn out to be interesting.

In fact, for most dominance relations that are of importance such a modeling is not possible. The reason is that when removing strategies in an iterated fashion, in particular in the one-at-a-time fashion, some previously undominated strategies can become eligible for removal. So this process can yield a different outcome than a single removal of several strategies.

In contrast, the following definition seems to capture a relevant property. We say that that a reduction relation $\Rightarrow_R$ satisfies the ***one-at-a-time*** property if

$$\Rightarrow^+_{1,R} = \Rightarrow^+_R .$$

Obviously, if $\Rightarrow^+_{1,R} = \Rightarrow^+_R$, then also $\Rightarrow^*_{1,R} = \Rightarrow^*_R$. The following result clarifies when the one-at-a-time property holds.

Theorem 3.2 (One-at-a-time Elimination) *For a dominance relation R that is hereditary the $\Rightarrow_R$ relation satisfies the one-at-a-time property.*

Proof. Note that always $\Rightarrow_{1,R} \subseteq \Rightarrow_R$, so $\Rightarrow^+_{1,R} \subseteq \Rightarrow^+_R$ always holds.

To prove the inverse inclusion it suffices to show that $\Rightarrow_R \subseteq \Rightarrow^+_{1,R}$. So suppose that $G \Rightarrow_R G'$. We prove that $G \Rightarrow^+_{1,R} G'$ by induction on the number k of strategies deleted in the transition from G to G'. If $k = 1$, then $G \Rightarrow_{1,R} G'$ holds.

Suppose now that claim holds for some $k > 1$. Assume that $G := (S_1, \ldots, S_n, p_1, \ldots, p_n)$ and $G' := (S'_1, \ldots, S'_n, p_1, \ldots, p_n)$. For each $i \in [1..n]$ let $S_i \setminus S'_i := \{t_i^1, \ldots, t_i^{k_i}\}$. So for all $i \in [1..n]$ and all $j \in [1..k_i]$ the strategy t_i^j is R-dominated in G by some $s_i^j \in S'_i$. Choose some strategy $t_{i_0}^{j_0}$ and let G'' be the game resulting from G by removing $t_{i_0}^{j_0}$ from S_{i_0}. Then $G \Rightarrow_{1,R} G''$.

Since $t_{i_0}^{j_0} \not\in \cup_{i=1}^n S'_i$ each strategy s_i^j is in G''. So by the hereditarity of R each strategy t_i^j, where $(i,j) \neq (i_0, j_0)$, is R-dominated in G'' by s_i^j. This means that $G'' \Rightarrow_R G'$. By the induction hypothesis $G'' \Rightarrow_{1,R}^+ G'$, hence $G \Rightarrow_{1,R}^+ G'$. □

Now, given a dominance relation R that is hereditary and is a strict partial order we can establish that the $\Rightarrow_R$ reduction relation on the set of all restrictions of a game H satisfies the unique normal form property (in short: ***is UN***) in one of the following three ways:

- by showing that $\Rightarrow_R$ is one step closed; this is the argument provided by Gilboa, Kalai and Zemel [1990],
- by proving that $\Rightarrow_R$ is weakly confluent,
- by proving that $\Rightarrow_{1,R}$ is weakly confluent.

In the last case one actually proceeds by showing that $\Rightarrow_{1,R}$ satisfies the diamond property, where we say that $\rightarrow$ satisfies the ***diamond property*** if for all $a, b, c \in A$ such that $b \neq c$

$$\begin{array}{ccc} & a & \\ \swarrow & & \searrow \\ b & & c \end{array}$$

implies that for some $d \in A$

$$\begin{array}{ccc} b & & c \\ & \searrow \ \swarrow & \\ & d & \end{array}$$

All three proofs are straightforward. As an illustration we provide the proof for the second approach as its pattern will be repeated a number of times.

Lemma 3.3 (Weak Confluence) *Consider a dominance relation R that is hereditary and is a strict partial order. Then the $\Rightarrow_R$ relation on the set of all restrictions of a game H is weakly confluent.*

Proof. Suppose

$$
\begin{array}{ccc}
 & G & \\
\swarrow_R & & {}_R\searrow \\
G' & & G''
\end{array}
$$

We prove that then

$$
\begin{array}{ccc}
G' & & G'' \\
{}_R\searrow^{\epsilon} & & {}^{\epsilon}\swarrow_R \\
 & G' \cap G'' &
\end{array}
$$

Recall that $a \Rightarrow_R^{\epsilon} b$ means $a \Rightarrow_R b$ or $a = b$.

If G' is a restriction of $G' \cap G''$, then $G' = G' \cap G''$ and consequently $G' \Rightarrow_R^{\epsilon} G' \cap G''$. Otherwise suppose

$$G' := (S'_1, \ldots, S'_n, p_1, \ldots, p_n),$$

$$G'' := (S''_1, \ldots, S''_n, p_1, \ldots, p_n).$$

Then

$$G' \cap G'' = (S'_1 \cap S''_1, \ldots, S'_n \cap S''_n, p_1, \ldots, p_n).$$

Fix $i \in [1..n]$ and consider a strategy $s_i \in S'_i$ such that $s_i \notin S'_i \cap S''_i$. So s_i is eliminated in the step $G \Rightarrow_R G''$. Hence some $s'_i \in S_i$ R-dominates s_i in G.

Case 1. $s'_i \in S'_i$.

G' is a restriction of G and R is hereditary so s'_i also R-dominates s_i in G'.

Case 2. $s'_i \notin S'_i$.

So s'_i is eliminated in the step $G \Rightarrow_R G'$. Hence a strategy $s''_i \in S'_i$ exists that R-dominates s'_i in G. By the transitivity of R, s''_i R-dominates s_i in G and hence, by hereditarity, in G'.

This proves $G' \rightarrow_R G' \cap G''$ and hence, by the Equivalence Lemma 3.1, $G' \Rightarrow_R G' \cap G''$.

By symmetry $G'' \Rightarrow_R^{\epsilon} G' \cap G''$. □

This brings us to the following result of Gilboa, Kalai and Zemel [1990].

Theorem 3.4 (Elimination) *For a dominance relation R that is hereditary and a strict partial order the $\Rightarrow_R$ relation is UN.* □

To illustrate a direct use of the above results consider the strict dominance relation S. It entails the reduction relation $\Rightarrow_S$ on games obtained by instantiating R in $\Rightarrow_R$ by the strict dominance relation. As already noted by Gilboa, Kalai and Zemel [1990] strict dominance is clearly hereditary and is a strict partial order. So we get the following conclusion.

Theorem 3.5 (Strict Elimination)

(i) The $\Rightarrow_S$ relation is UN.

(ii) The $\Rightarrow_S$ relation satisfies the one-at-a-time property. □

In other words, the process of iterated elimination of strictly dominated strategies yields a unique outcome and coincides with the outcome of the iterated elimination of a single dominated strategy.

4 Pure Strategies: Inherent Dominance

In this section we introduce and study a natural generalization of the binary dominance notion, due to Börgers [1990,1993]. Consider a game $(S_1, \ldots, S_n, p_1, \ldots, p_n)$. Let R be a dominance relation and $\tilde{S}_{-i}$ a non-empty subset of S_{-i}. We say that a strategy s_i is ***R-dominated given*** $\tilde{S}_{-i}$ by a strategy s'_i if s_i is R-dominated by s'_i in the game $(S_i, \tilde{S}_{-i}, p_1, \ldots, p_n)$. Then we say that a strategy s_i is ***inherently R-dominated*** if for every non-empty subset $\tilde{S}_{-i}$ of S_{-i} it is R-dominated given $\tilde{S}_{-i}$ by some strategy s'_i. So we turned in this way the binary relation R to a unary relation on the strategies.

Note that in the definition of inherent R-dominance for each subset $\tilde{S}_{-i}$ of S_{-i} a different strategy of player i can R-dominate the considered strategy s_i. This can make this notion of dominance stronger than R-dominance. Börgers [1990,1993] studied this notion of dominance for R being weak dominance and established for it the order independence. The resulting dominance relation, inherent weak dominance, is an intermediate notion between strict and weak dominance. Indeed, it is clearly implied by strict dominance and implies in turn weak dominance. The converse implications do not hold as the following two examples show. In the game

	L	R
T	$2, -$	$1, -$
M	$1, -$	$2, -$
B	$1, -$	$3, -$

the strategy M is weakly dominated by T given $\{L\}$ and weakly dominated by B given $\{R\}$ or given $\{L, R\}$. So M is inherently weakly dominated but is not strictly dominated by any strategy.

In turn in the game

	L	R
T	$2, -$	$1, -$
B	$1, -$	$1, -$

the strategy B is not inherently weakly dominated but is weakly dominated.

It is well-known that weak dominance is not order independent. We shall return to this matter in Section 11. The intuitive reason is that weak dominance is not hereditary. As a consequence the proof of the corresponding weak confluence property does not go through.

The notion of inherent R-dominance does not fit into the framework developed in Section 3, since it is a unary relation. However, when studying reduction by means of it we can proceed in a largely analogous fashion. So first we introduce two notions of reduction between a game $G := (S_1, \ldots, S_n, p_1, \ldots, p_n)$ and its restriction $G' := (S'_1, \ldots, S'_n, p_1, \ldots, p_n)$, this time involving the inherent R-dominance notion.

- We write $G \rightarrow_{inh-R} G'$ when $G \neq G'$ and for all $i \in [1..n]$

$$\text{each } s_i \in S_i \setminus S'_i \text{ is inherently } R\text{-dominated in } G.$$

- We write $G \Rightarrow_{inh-R} G'$ when $G \neq G'$ and for all $i \in [1..n]$ for every non-empty subset $\tilde{S}_{-i}$ of S_{-i}

$$\text{each } s_i \in S_i \setminus S'_i \text{ is } R\text{-dominated in } G \text{ given } \tilde{S}_{-i} \text{ by some } s'_i \in S'_i.$$

So in the $\rightarrow_{inh-R}$ relation for every non-empty subset $\tilde{S}_{-i}$ of S_{-i} we require R-dominance in G given $\tilde{S}_{-i}$ by some $s'_i \in S_i$, while in the $\Rightarrow_{inh-R}$ relation for every non-empty subset $\tilde{S}_{-i}$ of S_{-i} we require R-dominance in G given $\tilde{S}_{-i}$ by some $s'_i \in S'_i$. Börgers [1990,1993] considered the first relation, $\rightarrow_{inh-R}$, for R being weak dominance. We introduce the second one, $\Rightarrow_{inh-R}$, to streamline the presentation. As in Section 3 under a natural assumption both notions turn out to be equivalent.

Lemma 4.1 (Equivalence) *For a dominance relation R that is a strict partial order the relations $\rightarrow_{inh-R}$ and $\Rightarrow_{inh-R}$ coincide.*

Proof. The proof is similar to that of the Equivalence Lemma 3.1. It suffices to show that if $G \rightarrow_{inh-R} G'$, then $G \Rightarrow_{inh-R} G'$.

Let $G := (S_1, \ldots, S_n, p_1, \ldots, p_n)$ and $G' := (S'_1, \ldots, S'_n, p_1, \ldots, p_n)$. Suppose that some $s_i \in S_i \setminus S'_i$ is inherently R-dominated in G. Let $\tilde{S}_{-i}$ be a non-empty subset of S_{-i}. Some strategy $s'_i \in S_i$ R-dominates s_i in G given $\tilde{S}_{-i}$. R is a strict partial order and S_i is finite, so a strategy $s'_i \in S_i$ exists that R-dominates s_i in G given $\tilde{S}_{-i}$ and is not R-dominated in G given $\tilde{S}_{-i}$ by any strategy in S_i. So this s'_i is not eliminated in the step $G \rightarrow_{inh-R} G'$ and consequently s_i is R-dominated in G given $\tilde{S}_{-i}$ by some $s'_i \in S'_i$. □

The following simple observation relates the $\Rightarrow_{inh-R}$ reduction relation to the previously introduced relation $\Rightarrow_R$.

Note 4.2 (Comparison) *Consider a dominance relation R. Then*

(i) $\Rightarrow_{inh-R} \subseteq \Rightarrow_R$.

(ii) If R is hereditary, then the relations $\Rightarrow_{inh-R}$ and $\Rightarrow_R$ coincide. □

So for hereditary dominance relations no new reduction relations were introduced here. Further, it is easy to provide examples of a non-hereditary R, for instance weak dominance, for which the reduction relations $\Rightarrow_{inh-R}$ and $\Rightarrow_R$ differ.

We now establish order independence for specific $\Rightarrow_{inh-R}$ reduction relations. Following Gilboa, Kalai and Zemel [1990] we say that a dominance relation R satisfies the ***individual independence of irrelevant alternatives*** condition (in short, ***IIIA***) if for every game $(S_i, S_{-i}, p_1, \ldots, p_n)$ the following holds:

for all $i \in [1..n]$, all non-empty $S'_i \subseteq S_i$ and $s_i, s'_i \in S'_i$
$s_i \ R \ s'_i$ holds in $(S_i, S_{-i}, p_1, \ldots, p_n)$ iff it holds in $(S'_i, S_{-i}, p_1, \ldots, p_n)$.

IIIA is a very reasonable condition. All specific dominance relations considered in this paper satisfy it.

Lemma 4.3 (Weak Confluence) *For a dominance relation R that satisfies the IIIA condition and is a strict partial order the $\Rightarrow_{inh-R}$ relation on the set of all restrictions of a game H is weakly confluent.*

Proof. We proceed as in the proof of the Weak Confluence Lemma 3.3. Suppose $G \Rightarrow_{inh-R} G'$ and $G \Rightarrow_{inh-R} G''$. We prove that then $G' \Rightarrow^{\epsilon}_{inh-R} G' \cap G''$ and $G'' \Rightarrow^{\epsilon}_{inh-R} G' \cap G''$.

If G' is a restriction of $G' \cap G''$, then $G' = G' \cap G''$ and consequently $G' \Rightarrow^{\epsilon}_{inh-R} G' \cap G''$. Otherwise suppose $G' := (S'_1, \ldots, S'_n, p_1, \ldots, p_n)$, $G'' := (S''_1, \ldots, S''_n, p_1, \ldots, p_n)$. Then $G' \cap G'' = (S'_1 \cap S''_1, \ldots, S'_n \cap S''_n, p_1, \ldots, p_n)$.

Fix $i \in [1..n]$. Consider a strategy $s_i \in S'_i$ such that $s_i \notin S'_i \cap S''_i$. So s_i is eliminated in the step $G \Rightarrow_{inh-R} G''$. Take now a non-empty subset $\tilde{S}_{-i}$ of S'_{-i} (and hence of S_{-i}). The strategy s_i is R-dominated given $\tilde{S}_{-i}$ in G by some strategy $s'_i \in S_i$.

Case 1. $s'_i \in S'_i$.

Then, since R satisfies the IIIA condition, $s_i \ R \ s'_i$ holds in the game $(S'_i, \tilde{S}_{-i}, p_1, \ldots, p_n)$, i.e., s_i is R-dominated given $\tilde{S}_{-i}$ in G'.

Case 2. $s'_i \notin S'_i$.

So s'_i is eliminated in the step $G \Rightarrow_{inh-R} G'$. Hence a strategy $s''_i \in S'_i$ exists that R-dominates s'_i in G given $\tilde{S}_{-i}$. By the transitivity of R the strategy s''_i R-dominates s_i in G given $\tilde{S}_{-i}$ and hence, since R satisfies the IIIA condition, s_i is R-dominated given $\tilde{S}_{-i}$ in G'.

So we showed that each strategy s_i of player i eliminated in the transition from G' to $G' \cap G''$ is inherently R-dominated in G'. This proves $G' \rightarrow_{inh-R} G' \cap G''$ and hence, by the Equivalence Lemma 4.1 $G' \Rightarrow_{inh-R} G' \cap G''$.

By symmetry $G'' \Rightarrow^{\epsilon}_{inh-R} G' \cap G''$. □

We can now draw the desired conclusion using Newman's Lemma 2.1.

Theorem 4.4 (Inherent Elimination) *For a dominance relation R that satisfies the IIIA condition and is a strict partial order the $\Rightarrow_{inh-R}$ relation is UN.* □

As in Section 3 we introduce the $\Rightarrow_{1,inh-R}$ reduction relation that removes exactly one strategy, and as before we say that $\Rightarrow_{inh-R}$ satisfies the ***one-at-a-time*** property when

$$\Rightarrow^{+}_{1,inh-R} = \Rightarrow^{+}_{inh-R}.$$

The following counterpart of the One-at-a-time Elimination Theorem 3.2 then holds.

Theorem 4.5 (One-at-a-time Elimination) *For a dominance relation R that satisfies the IIIA condition the relation $\Rightarrow_{inh-R}$ satisfies the one-at-a-time property.*

Proof. Analogous to the proof of the One-at-a-time Elimination Theorem 3.2 and omitted. □

Since the weak dominance relation W satisfies the IIIA condition and is a strict partial order, by the above results we get the following counterpart of the Strict Elimination Theorem 3.5.

Theorem 4.6 (Inherent Weak Elimination)

(i) The $\Rightarrow_{inh-W}$ relation is UN.

(ii) The $\Rightarrow_{inh-W}$ relation satisfies the one-at-a-time property. □

The first item was established in Börgers [1990]. In Börgers [1993] it was shown that a strategy is inherently weakly dominated iff it is not rational, in the sense that it is not a best response to a belief formed over the pure strategies of other players when their payoff functions are not known — it is only assumed that their payoff functions are compatible with their publicly known preferences. So the $\Rightarrow_{inh-W}$ relation allows us to model iterated removal of strategies that are not rational in this sense.

5 Mixed Dominance Relations

The notion of dominance studied in Section 3 involved two pure strategies. In this section we study the dominance relations in which the dominating strategies are mixed and develop the appropriate general results.

Let us recall first the definitions. Given a set of strategies S_i available to player i, by a ***mixed strategy*** we mean a probability distribution over S_i and denote this set of mixed strategies by M_i.

Given a mixed strategy m_i we define

$$support(m_i) := \{s_i \in S_i \mid m_i(s_i) > 0\}.$$

Consider a game $(S_1, \ldots, S_n, p_1, \ldots, p_n)$. Each payoff function p_i is generalized to a function

$$p_i : M_1 \times \ldots \times M_n \rightarrow \mathcal{R}$$

by putting for a sequence $(m_1, \ldots, m_n)$ of mixed strategies from $M_1 \times \ldots \times M_n$

$$p_i(m_1, \ldots, m_n) := \sum_{s \in S} m_1(s_1) \ldots m_n(s_n)\, p_i(s).$$

As usual, we identify a mixed strategy for player i of a restriction G' of G with a mixed strategy of G by assigning the probability 0 to the strategies of player i that are present in G but not in G'. Further, we can view a mixed strategy for player i in G as a mixed strategy in G' if its support is a subset of the set of all strategies of player i in G'. Also, we can identify each pure strategy s_i with the mixed strategy that assigns to s_i the probability 1.

A ***mixed dominance relation*** is a function that assigns to each game $G := (S_1, \ldots, S_n, p_1, \ldots, p_n)$ a subset R_G of $\bigcup_{i=1}^{n}(S_i \times M_i)$. When $s_i \; R_G \; m'_i$ holds we say that $s_i \; R \; m'_i$ ***holds for*** G and also say that s_i ***is*** R***-dominated by*** m'_i ***in*** G, or that m'_i R***-dominates*** s_i ***in*** G.

As in Section 3 we introduce now two notions of reduction between a game $G := (S_1, \ldots, S_n, p_1, \ldots, p_n)$ and its restriction $G' := (S'_1, \ldots, S'_n, p_1, \ldots, p_n)$, this time involving a mixed dominance relation R.

- We write $G \rightarrow_R G'$ when $G \neq G'$ and for all $i \in [1..n]$

  each $s_i \in S_i \setminus S'_i$ is R-dominated in G by some $m'_i \in M_i$.

- We write $G \Rightarrow_R G'$ when $G \neq G'$ and for all $i \in [1..n]$

  each $s_i \in S_i \setminus S'_i$ is R-dominated in G by some $m'_i \in M'_i$.

So, as before, the difference between the $\rightarrow_R$ and $\Rightarrow_R$ lies in the requirement we put on the R-dominating —this time mixed— strategy. In $\rightarrow_R$ we require that each strategy removed from S_i is R-dominated in G by a mixed strategy in M_i, while in $\Rightarrow_R$ we require that it is R-dominated in G by a mixed strategy in M'_i. So in the latter case no strategy from the support of the R-dominating mixed strategy should be removed at the same time.

To establish equivalence between both reduction relations we need a counterpart of the notion of a strict partial order. Below, we occasionally write each mixed strategy m' over the set of strategies S_i as the sum $\sum_{t \in S_i} p_t t$, where each $p_t = m'(t)$. Then given two mixed strategies m_1, m_2 and a strategy t_1 we mean by $m_2[t_1/m_1]$ the mixed strategy obtained from m_2 by substituting the strategy t_1 by m_1 and by 'normalizing' the resulting sum.

We now say that a mixed dominance relation R is ***regular*** if in every game

- for all $\alpha \in (0, 1]$, $s \; R \; (1 - \alpha)s + \alpha \, m$ implies $s \; R \; m$,
- $t_1 \; R \; m_1$ and $t_2 \; R \; m_2$ implies $t_1 \; R \; m_1[t_2/m_2]$.

Lemma 5.1 (Equivalence) *For a mixed dominance relation R that is regular the relations $\rightarrow_R$ and $\Rightarrow_R$ coincide.*

Proof. We only need to show that $G \rightarrow_R G'$ implies $G \Rightarrow_R G'$.

Let $G := (S_1, \ldots, S_n, p_1, \ldots, p_n)$ and $G' := (S'_1, \ldots, S'_n, p_1, \ldots, p_n)$. Take some $s''_i \in S_i \setminus S'_i$. Let $S_i \setminus S'_i := \{t_1, \ldots, t_k\}$ with $t_k = s''_i$. By definition for all $j \in [1..n]$ some $m_j \in M_i$ exists such that $t_j \ R \ m_j$ (holds in G). We prove by complete induction that in fact for all $j \in [1..k]$ some $m'_j \in M_i$ exists such that $t_j \ R \ m'_j$ and $support(m'_j) \cap \{t_1, \ldots, t_j\} = \emptyset$.

For some $\alpha \in (0, 1]$ and a mixed strategy m'_1 with $t_1 \notin support(m'_1)$ we have

$$m_1 = (1 - \alpha)t_1 + \alpha \, m'_1.$$

Since R is regular, $t_1 \ R \ m_1$ implies $t_1 \ R \ m'_1$, which proves the claim for $k = 1$.

Assume now the claim holds for all $\ell \in [1..j]$. We have $t_{j+1} \ R \ m_{j+1}$. As in the case of $k = 1$ a mixed strategy m''_{j+1} exists such that $t_{j+1} \notin support(m''_{j+1})$ and $t_{j+1} \ R \ m''_{j+1}$. Let

$$m'_{j+1} := m''_{j+1}[t_1/m'_1]\ldots[t_j/m'_j].$$

Then for all $\ell \in [1..j]$ we have $support(m''_{j+1}[t_1/m'_1]\ldots[t_\ell/m'_\ell]) \cap \{t_1, \ldots, t_\ell, t_{j+1}\} = \emptyset$, so $support(m'_{j+1}) \cap \{t_1, \ldots, t_{j+1}\} = \emptyset$, i.e., $support(m'_{j+1}) \subseteq S'_i$.

Also $t_{j+1} \ R \ m''_{j+1}$ and $t_\ell \ R \ m'_\ell$ for all $\ell \in [1..j]$ imply by the regularity of R that $t_{j+1} \ R \ m'_{j+1}$. Hence s''_i (which equals t_k) is R-dominated by the mixed strategy $m'_k \in M'_i$. □

The second condition of the regularity notion appears in Lemma 1 of Robles [2003] under the name 'transitivity'. In that paper order independence of conditional dominance is established, a notion introduced in Shimoji and Watson [1998]. Establishing 'transitivity' for a specialized form of conditional dominance (called a robust demi-replacement) turns out to be a crucial step in the proof of the order independence. In our case regularity allows us to focus our representation on the second reduction relation, $\Rightarrow_R$.

In analogy to the case of dominance relations we say that a mixed dominance relation R is ***hereditary*** if for every game G, its restriction G', a strategy s_i of G' and a mixed strategy m'_i of G'

s_i is R-dominated by m'_i in G implies s_i is R-dominated by m'_i in G'.

Also, as in the case of the dominance relations, given a mixed dominance relation R we can specialize the reduction relation $\Rightarrow_R$ to $\Rightarrow_{1,R}$ in which a single strategy is removed. The following counterpart of the One-at-a-time Elimination Theorem 3.2 then holds.

Theorem 5.2 (One-at-a-time Elimination) *For a mixed dominance relation R that is hereditary the $\Rightarrow_R$ satisfies the one-at-a-time property.*

Proof. Analogous to the proof of the One-at-a-time Elimination Theorem 3.2 and left to the reader. $\square$

As in Section 3 for a mixed dominance relation R that is hereditary and regular we have three ways of proving that the reduction relation $\Rightarrow_R$ is UN. Here, for a change, we provide a proof for the first approach.

Lemma 5.3 (One Step Closedness) *For a mixed dominance relation R that is hereditary and regular the $\Rightarrow_R$ relation is one step closed.*

Proof. Given a game $G := (S_1, \ldots, S_n, p_1, \ldots, p_n)$, let $G'' := (S''_1, \ldots, S''_n, p_1, \ldots, p_n)$ be the game obtained from G by removing all the strategies that are R-dominated by a mixed strategy in G. Then $G \rightarrow^{\epsilon}_R G''$, so by the Equivalence Lemma 5.1 $G \Rightarrow^{\epsilon}_R G''$.

Suppose now that $G \Rightarrow_R G'$ for some $G' := (S'_1, \ldots, S'_n, p_1, \ldots, p_n)$. Then clearly $S''_i \subseteq S'_i$ for all $i \in [1..n]$. If G' and G'' coincide, then $G' \Rightarrow^{\epsilon}_R G''$.

Otherwise fix $i \in [1..n]$ and consider a strategy s_i such that $s_i \in S'_i \setminus S''_i$. So s_i is eliminated in the step $G \Rightarrow_R G''$. Hence s_i is R-dominated in G by a mixed strategy $m'_i \in M''_i$. By the hereditarity of R s_i is R-dominated in G' by m'_i. This proves $G' \Rightarrow_R G''$. $\square$

The reader may note a 'detour' in this proof through the $\rightarrow$ reduction, justified by the Equivalence Lemma 5.1. The above lemma brings us to the following conclusion.

Theorem 5.4 (Mixed Elimination) *For a mixed dominance relation R that is hereditary and regular the $\Rightarrow_R$ relation is UN.*

Proof. We noted already in Section 2 that one step closedness implies weak confluence. So Newman's Lemma 2.1 applies. $\square$

In other words, when R is a mixed dominance relation that is hereditary and regular, the process of iterated elimination of R-dominated strategies yields a unique outcome.

We can directly apply the results of this section to strict dominance by mixed strategies. Let us recall first the definition. Consider a game $(S_1, \ldots, S_n, p_1, \ldots, p_n)$. We say that a strategy s_i is ***strictly dominated*** by a mixed strategy m'_i, or equivalently, that a mixed strategy m'_i ***strictly dominates*** a strategy s_i, if

$$p_i(s_i, s_{-i}) < p_i(m'_i, s_{-i})$$

for all $s_{-i} \in S_{-i}$.

This mixed dominance relation entails the reduction relation $\Rightarrow_{SM}$ on games obtained by instantiating the mixed dominance relation R in $\Rightarrow_R$ by the strict dominance in the above sense. Clearly, strict dominance by a mixed strategy is hereditary and regular, so by virtue of the above results we get the following counterpart of the Strict Elimination Theorem 3.5.

Theorem 5.5 (Strict Mixed Elimination)

(i) The $\Rightarrow_{SM}$ relation is UN.

(ii) The $\Rightarrow_{SM}$ relation satisfies the one-at-a-time property. □

The first item states that strict dominance by means of mixed strategies is order independent.

6 Mixed Strategies: Inherent Dominance

The concepts and results of Section 4 can be naturally modified to the case of mixed dominance relations. Consider such a relation R and a game $(S_1, \ldots, S_n, p_1, \ldots, p_n)$ and let $\tilde{S}_{-i}$ be a non-empty subset of S_{-i}. We say that a strategy s_i is ***R-dominated given*** $\tilde{S}_{-i}$ by a mixed strategy m'_i if s_i is R-dominated by m'_i in the game $(S_i, \tilde{S}_{-i}, p_1, \ldots, p_n)$ and say that a strategy s_i is ***inherently R-dominated*** if for every non-empty subset $\tilde{S}_{-i}$ of S_{-i} it is R-dominated given $\tilde{S}_{-i}$ by some mixed strategy m'_i.

As before, each mixed dominance relation R entails two reduction relations $\rightarrow_{inh-R}$ and $\Rightarrow_{inh-R}$ on games and their 'one-at-a-time' versions, $\rightarrow_{1,inh-R}$ and $\Rightarrow_{1,R}$.

The ***individual independence of irrelevant alternatives*** condition (***IIIA***) now holds for a mixed dominance relation R if for every game $(S_i, S_{-i}, p_1, \ldots, p_n)$

for all $i \in [1..n]$, all non-empty $S'_i \subseteq S_i$, $s_i \in S'_i$ and $m_i \in M'_i$
$s_i \; R \; m'_i$ holds in $(S_i, S_{-i}, p_1, \ldots, p_n)$ iff it holds in $(S'_i, S_{-i}, p_1, \ldots, p_n)$.

By analogy we obtain the following results concerning the introduced reduction relations.

Lemma 6.1 (Equivalence) *For a mixed dominance relation R that is regular the relations $\rightarrow_{inh-R}$ and $\Rightarrow_{inh-R}$ coincide.*

Proof. Analogous to the proof of the Equivalence Lemma 5.1 and omitted. □

Lemma 6.2 (One Step Closedness) *For a mixed dominance relation R that satisfies the IIIA condition and is regular the $\Rightarrow_{inh-R}$ relation is one step closed.*

Proof. Analogous to the proof of the One Step Closedness Lemma 5.3, using the Equivalence Lemma 6.1, and omitted. □

Theorem 6.3 (Inherent Mixed Elimination) *For a mixed dominance relation R that satisfies the IIIA condition and is regular the $\Rightarrow_{inh-R}$ relation is UN.* □

Proof. By the One Step Closedness Lemma 6.2 and Newman's Lemma 2.1. □

Theorem 6.4 (One-at-a-time Elimination) *For a mixed dominance relation R that satisfies the IIIA condition the $\Rightarrow_{inh-R}$ relation satisfies the one-at-a-time property.*

Proof. Analogous to the proof of the One-at-a-time Elimination Theorem 3.2 and omitted. □

These results can be directly applied to weak dominance by a mixed strategy. Recall that given a game $(S_1, \ldots, S_n, p_1, \ldots, p_n)$ we say that a strategy s_i is ***weakly dominated*** by a mixed strategy m'_i, and write s_i *WM* m'_i, if

$$p_i(s_i, s_{-i}) \leq p_i(m'_i, s_{-i})$$

for all $s_{-i} \in S_{-i}$, with some disequality being strict.

It is straightforward to check that *WM* satisfies the IIIA condition and is regular. However, somewhat unexpectedly, we do not get now any new results, since as shown by Börgers [1990] the reduction relations $\rightarrow_{inh-WM}$ and $\rightarrow_{SM}$ (and hence $\Rightarrow_{inh-WM}$ and $\Rightarrow_{SM}$) coincide.

7 More on Abstract Reduction Systems

We shall soon deal with the elimination of payoff equivalent strategies and to this end we shall need a refinement of Newman's Lemma 2.1. Consider an abstract reduction system $(A, \rightarrow)$ and assume an equivalence relation $\sim$ on A. We now relativize the previously introduced notions to $\sim$ and introduce one new concept linking $\rightarrow$ and $\sim$.

- If every element of A has a unique up to $\sim$ normal form, we say that $(A, \rightarrow)$ (or simply $\rightarrow$) satisfies the ***$\sim$-unique normal form property***.

- We say that $\rightarrow$ is ***$\sim$-weakly confluent*** if for all $a, b, c \in A$

$$\begin{tikzcd} & a \arrow[dl] \arrow[dr] & \\ b & & c \end{tikzcd}$$

implies that for some $d_1, d_2 \in A$

$$\begin{tikzcd} b \arrow[dr, "*"] & & c \arrow[dl, "*"'] \\ & d_1 \sim d_2 & \end{tikzcd}$$

- We say that $\rightarrow$ is ***$\sim$-bisimilar*** if for all $a, b, c \in A$

$$\begin{tikzcd} a \arrow[r, phantom, "\sim"] \arrow[d] & b \\ c & \end{tikzcd}$$

implies that for some $d \in A$

$$\begin{tikzcd} a \arrow[r, phantom, "\sim"] \arrow[d] & b \arrow[d] \\ c \arrow[r, phantom, "\sim"] & d \end{tikzcd}$$

The following lemma is then a relativized version of Newman's Lemma 2.1. It is a special case of Lemma 2.7 from Huet [1980, page 803], with a more direct proof.

Lemma 7.1 ($\sim$-Newman) *Consider an abstract reduction system $(A, \rightarrow)$ and an equivalence relation $\sim$ on A such that*

- *no infinite $\rightarrow$ sequences exist,*
- *$\rightarrow$ is $\sim$-weakly confluent,*
- *$\rightarrow$ is $\sim$-bisimilar.*

Then $\rightarrow$ satisfies the $\sim$-unique normal form property.

Proof. We modify the proof of Newman's Lemma 2.1. We call now an element a *ambiguous* if it has at least two normal forms that are not equivalent w.r.t. $\sim$. As before we show that for every ambiguous a some ambiguous b exists such that $a \rightarrow b$. This proves absence of ambiguous elements by the first assumption.

So suppose that some element a has two distinct normal forms n_1 and n_2 such that $n_1 \not\sim n_2$. Then for some b, c we have $a \rightarrow b \rightarrow^* n_1$ and $a \rightarrow c \rightarrow^* n_2$. By the $\sim$-weak confluence some d_1 and d_2 exist such that $b \rightarrow^* d_1$, $c \rightarrow^* d_2$ and $d_1 \sim d_2$. Let n_3 be a normal form of d_1. Then it is a normal form of b, as well.

By the repeated use of the $\sim$-bisimilarity of $\rightarrow$

$$\begin{array}{ccc} d_1 & \sim & d_2 \\ \downarrow_* & & \\ n_3 & & \end{array}$$

implies that for some $n_4 \in A$

$$\begin{array}{ccc} d_1 & \sim & d_2 \\ \downarrow_* & & \downarrow_* \\ n_3 & \sim & n_4 \end{array}$$

Since n_3 is a normal form, by the $\sim$-bisimilarity of $\rightarrow$ so is n_4. So n_4 is a normal form of c. Moreover $n_3 \not\sim n_1$ or $n_3 \not\sim n_2$, since otherwise $n_1 \sim n_2$ would hold. If $n_3 \not\sim n_1$, then b is ambiguous and $a \rightarrow b$. And if $n_3 \not\sim n_2$, then also $n_4 \not\sim n_2$ and then c is ambiguous and $a \rightarrow c$. $\Box$

Also, we have the following relativized version of the Unique Normal Form Note 2.2.

Note 7.2 ($\sim$-Unique Normal Form) *Consider two abstract reduction systems $(A, \rightarrow_1)$ and $(A, \rightarrow_2)$ and an equivalence relation $\sim$ on A such that*

- $\rightarrow_1$ *satisfies the* $\sim$*-unique normal form property,*
- $\rightarrow_1^+ = \rightarrow_2^+$.

Then $\rightarrow_2$ *satisfies the* $\sim$*-unique normal form property.* $\Box$

We shall also study the combined effect of two forms of elimination. In what follows we abbreviate $\rightarrow_1 \cup \rightarrow_2$ to $\rightarrow_{1\vee 2}$. (The use of $\cup$ instead of $\vee$ would clash with the notation used in Section 10.) Given two abstract reduction systems $(A, \rightarrow_1)$ and $(A, \rightarrow_2)$ we say that $\rightarrow_1$ ***left commutes with*** $\rightarrow_2$ if

$$\rightarrow_1 \circ \rightarrow_2 \subseteq \rightarrow_2 \circ \rightarrow_1^*,$$

i.e., if for all $a, b, c \in A$ $a \rightarrow_1 b \rightarrow_2 c$ implies that for some $d \in A$ $a \rightarrow_2 d \rightarrow_1^* c$.

Note 7.3 (Left Commutativity) *If* $\rightarrow_1$ *left commutes with* $\rightarrow_2$*, then so does* $\rightarrow_1^+$. $\Box$

Then we shall rely on the following result.

Lemma 7.4 (Normal Form) *Consider two abstract reduction systems* $(A, \rightarrow_1)$ *and* $(A, \rightarrow_2)$ *and an equivalence relation* $\sim$ *on* A *such that*

- $(A, \rightarrow_{1\vee 2})$ *satisfies the* $\sim$*-unique normal form property,*
- $\rightarrow_1$ *left commutes with* $\rightarrow_2$.

Then for all $a \in A$*, if*

$$\begin{array}{ccc} & a & \\ {}^{*}\swarrow_{2} & & {}_{2}\searrow^{*} \\ b & & c \end{array}$$

for some $\rightarrow_2$*-normal forms* b *and* c*, then for some* $\rightarrow_{1\vee 2}$*-normal forms* $d_1, d_2 \in A$

$$\begin{array}{ccc} b & & c \\ {}_{1}\searrow^{*} & & {}^{*}\swarrow_{1} \\ d_1 & \sim & d_2. \end{array}$$

Proof. Suppose that $a \rightarrow_2^* b$ and $a \rightarrow_2^* c$ where b and c are $\rightarrow_2$-normal forms. By the first assumption for some $\rightarrow_{1\vee 2}$-normal forms $d_1, d_2 \in A$ we have $b \rightarrow_{1\vee 2}^* d_1$, $c \rightarrow_{1\vee 2}^* d_2$ and $d_1 \sim d_2$.

If for some $e_1, e_2 \in A$ we have $b \rightarrow_1^+ e_1 \rightarrow_2 e_2 \rightarrow_{1\vee 2}^* d_1$, then by the second assumption and the Left Commutativity Note 7.3 for some $e_3 \in A$ we have $b \rightarrow_2 e_3 \rightarrow_1^* e_2$, which contradicts the choice of b. So in the path $b \rightarrow_{1\vee 2}^* d_1$ there are no $\rightarrow_2$ transitions. By the same argument also in the path $c \rightarrow_{1\vee 2}^* d_2$ there are no $\rightarrow_2$ transitions. $\Box$

8 Pure Strategies: Payoff Equivalence

We now move on to a study of the elimination of payoff equivalent strategies. This binary relation on the strategies, PE, entails the corresponding reduction relation $\Rightarrow_{PE}$ on the games. Let us recall the definition. Given a game $G := (S_1, \ldots, S_n, p_1, \ldots, p_n)$ and its restriction $G' := (S'_1, \ldots, S'_n, p_1, \ldots, p_n)$

- $G \Rightarrow_{PE} G'$ iff $G \neq G'$ and for all $i \in [1..n]$

$$\text{each } s_i \in S_i \setminus S'_i \text{ is payoff equivalent in } G \text{ to some } s'_i \in S'_i.$$

Note that $\Rightarrow_{PE}$ is not weakly confluent and it does not satisfy the unique normal form property. Indeed, given two payoff equivalent strategies r and s, the removal of r and the removal of s yields two different games. But these games are obviously equivalent in the sense that a renaming of their strategies makes them identical. To study the effect of the removal of the payoff equivalent strategies we shall therefore consider the following ***equivalence relation*** $\sim$ between two games, $G := (S_1, \ldots, S_n, p_1, \ldots, p_n)$ and $G' := (S'_1, \ldots, S'_n, p'_1, \ldots, p'_n)$:

$G' \sim G''$ iff for all $i \in [1..n]$ there exists a 1-1 and onto mapping $f_i : S_i \rightarrow S'_i$ such that for all $i \in [1..n]$ and $s_i \in S_i$, $p_i(s_1, \ldots, s_n) = p'_i(f_1(s_1), \ldots, f_n(s_n))$.

In what follows we shall consider various (also mixed) reduction relations $\Rightarrow_R$ on games in presence of the $\sim$ equivalence relation on the games. In each case it will be straightforward to see that $\Rightarrow_R$ is $\sim$-bisimilar. Intuitively, the $\sim$-bisimilarity of $\Rightarrow_R$ simply means that R does not depend on the strategy names.

Note that if a (mixed) reduction relation R is hereditary, then to prove that $\Rightarrow_R$ is $\sim$-bisimilar it is sufficient on the account of the One-at-a-time Elimination Theorems 3.2 and 5.2 to check that $\Rightarrow_{1,R}$ is $\sim$-bisimilar.

Instead of saying that a reduction relation $\Rightarrow_R$ on the set of all restrictions of a game H satisfies the $\sim$-unique normal form property, we shall simply say that $\Rightarrow_R$ ***is*** $\boldsymbol{\sim}$***-UN***.

To reason about the $\Rightarrow_{PE}$ reduction relation we shall focus on the relation $\Rightarrow_{1,PE}$ concerned with the removal of a single strategy payoff equivalent strategy. The following simple observation holds.

Lemma 8.1 (Weak Confluence) *Consider a game H. The $\Rightarrow_{1,PE}$ relation on the set of all restrictions of a game H is $\sim$-weakly confluent.*

Proof. Suppose $G \Rightarrow_{1,PE} G'$ and $G \Rightarrow_{1,PE} G''$. Let r and s be the strategies eliminated in the first, respectively second, transition. If r and s are payoff equivalent in G, then $G' \sim G''$. Otherwise, by the hereditarity of PE, $G' \Rightarrow_{1,PE} G' \cap G''$ and $G'' \Rightarrow_{1,PE} G' \cap G''$. □

This brings us to the following result that we shall need in the sequel.

Theorem 8.2 (Payoff Equivalence Elimination)

(i) The $\Rightarrow_{1,PE}$ *relation is* $\sim$*-UN.*

(ii) The $\Rightarrow_{PE}$ *relation is* $\sim$*-UN.*

Proof.
(i) We just proved that $\Rightarrow_{1,PE}$ is $\sim$-weakly confluent. Also, this reduction relation is clearly $\sim$-bisimilar. So the conclusion follows by the $\sim$-Newman's Lemma 7.1.
(ii) First note that PE is hereditary, so by the One-at-a-time Elimination Theorem 3.2 $\Rightarrow_{PE}$ satisfies the one-at-a-time property, that is,

$$\Rightarrow^+_{1,PE} = \Rightarrow^+_{PE}.$$

It suffices now to apply the $\sim$-Unique Normal Form Note 7.2. □

Informally, the process of iterated elimination of payoff equivalent strategies yields a unique outcome up to the introduced equivalence relation $\sim$ on the games. This outcome can also be achieved in one step, by replacing each maximal set of at least two mutually payoff equivalent strategies by one representative. The resulting game is called in Myerson [1991] a ***purely reduced*** game. Of course, the above result is completely expected. Still, we find that a concise formal justification of it is in order.

9 Mixed Strategies: Randomized Redundance

The notion of payoff equivalent strategies generalizes in the obvious way to the mixed strategies. We denote by PEM the corresponding mixed dominance relation. So for a strategy s_i and a mixed strategy m'_i of player i s_i PEM m'_i if

$$p_j(s_i, s_{-i}) = p_j(m'_i, s_{-i})$$

for all $j \in [1..n]$ and all $s_{-i} \in S_{-i}$.

As explained in Section 5 *PEM* entails the reduction relation $\Rightarrow_{PEM}$ on games. Recall that a strategy s_i of player i is called ***randomized redundant to*** a mixed strategy m_i if it is payoff equivalent to m_i and $s_i \notin \mathit{support}(m_i)$. Note that for a game $(S_1, \ldots, S_n, p_1, \ldots, p_n)$ and its restriction $G' := (S'_1, \ldots, S'_n, p_1, \ldots, p_n)$ we have

- $G \Rightarrow_{PEM} G'$ when $G \neq G'$ and for all $i \in [1..n]$

  each $s'_i \in S_i \setminus S'_i$ is randomized redundant in G to some $m'_i \in M'_i$.

As in the case of payoff equivalence it is more convenient to focus on the removal of a single strategy, so on the reduction relation $\Rightarrow_{1,PEM}$. The following counterpart of the Weak Confluence Lemma 8.1 holds.

Lemma 9.1 (Weak Confluence) *Consider a game H. The $\Rightarrow_{1,PEM}$ relation on the set of all restrictions of a game H is $\sim$-weakly confluent.*

Proof. Suppose $G \Rightarrow_{1,PEM} G'$ and $G \Rightarrow_{1,PEM} G''$. Let r and t be the strategies eliminated in the first, respectively second, transition. If r and t are payoff equivalent, then, as in the proof of the Weak Confluence Lemma 8.1, $G' \sim G''$.

Otherwise for some $\alpha \in [0,1)$ and $\beta \in [0,1)$ r is payoff equivalent to a mixed strategy $\alpha\, t + (1-\alpha)m_1$ with $r, t \notin \mathit{support}(m_1)$ and t is payoff equivalent to a mixed strategy $\beta\, r + (1-\beta)m_2$ with $r, t \notin \mathit{support}(m_2)$. So r is payoff equivalent to $\alpha\,\beta\, r + \alpha(1-\beta)m_2 + (1-\alpha)m_1$, and hence to

$$m' := (\alpha(1-\beta)m_2 + (1-\alpha)m_1)/(1-\alpha\,\beta).$$

Since $t \notin \mathit{support}(m')$, m' is a mixed strategy in G''. So by the hereditarity of *PEM* r is payoff equivalent to m' in G''. Further, since $r, t \notin \mathit{support}(m')$, m' is a mixed strategy in $G' \cap G''$. So we showed that $G'' \Rightarrow_{1,PEM} G' \cap G''$. By symmetry $G' \Rightarrow_{1,PEM} G' \cap G''$. □

As in the case of the $\Rightarrow_{PE}$ relation we can now conclude.

Theorem 9.2 (Redundance Elimination)
(i) The $\Rightarrow_{1,PEM}$ relation is $\sim$-UN.

(ii) The $\Rightarrow_{PEM}$ relation is $\sim$-UN. □

So the process of iterated elimination of randomized redundant strategies yields a unique up to $\sim$ outcome. The result is called in Myerson [1991] a ***fully reduced*** game.

10 Combining Two Dominance Relations

Given two dominance relation R, Q we now consider the combined dominance relation $R \cup Q$. Such a combination is meaningful to study when Q is such that the $\Rightarrow_Q$ reduction relation is $\sim$-UN. An example is the payoff equivalence PE relation discussed in Section 8.

Given two dominance relations R and Q we would like now to identify conditions that allow us to conclude that the $\Rightarrow_{R \cup Q}$ reduction relation is $\sim$-UN. To this end we introduce the following concept. We say that R is ***closed under*** Q if in all games G for all strategies r, s, t

- $r \ R \ s$ and $s \ Q \ t$ implies $r \ R \ t$,
- $r \ Q \ s$ and $s \ R \ t$ implies $r \ R \ t$,

i.e., if in all games $R \circ Q \subseteq R$ and $Q \circ R \subseteq R$.

Here is a result that we shall use in the sequel.

Theorem 10.1 (Combination) *Consider two dominance relations R and Q such that*

- *$\Rightarrow_R$ and $\Rightarrow_Q$ are $\sim$-bisimilar,*
- *R is a strict partial order,*
- *R is closed under Q,*
- *$\Rightarrow_{1,Q}$ is $\sim$-UN,*
- *$R \cup Q$ is hereditary.*

Then the $\Rightarrow_{R \cup Q}$ relation is $\sim$-UN.

Notice that we do not insist here that R is hereditary. In fact, in one of the uses of the above result the dominance relation R will not be hereditary.

Proof. Since $R \cup Q$ is hereditary, by the One-at-a-time Elimination Theorem 3.2 and the $\sim$-Unique Normal Form Note 7.2 it suffices to prove that $\Rightarrow_{1,R \cup Q}$ is $\sim$-UN. But by assumption both $\Rightarrow_{1,R}$ and $\Rightarrow_{1,Q}$ are $\sim$-bisimilar, so $\Rightarrow_{1,R \cup Q}$ is $\sim$-bisimilar, as well. So on the account of the $\sim$-Newman's Lemma 7.1 the fact that $\Rightarrow_{1,R \cup Q}$ is $\sim$-UN is established once we show that $\Rightarrow_{1,R \cup Q}$ is $\sim$-weakly confluent.

So suppose that $G \Rightarrow_{1,R \cup Q} G'$ and $G \Rightarrow_{1,R \cup Q} G''$. Let r and s be the strategies eliminated in the first, respectively second, transition. By the fourth assumption $\Rightarrow_{1,Q}$ is is $\sim$-weakly confluent, so we only need to consider a situation when $G \Rightarrow_{1,R} G'$.

We can assume that $G' \neq G''$. Then r is in G'' and s is in G'. By definition $r\ R\ t$ holds in G for some strategy t of G' and $s\ R \cup Q\ u$ holds in G for some strategy u of G''. To show that $G'' \Rightarrow_{1,R\cup Q} G' \cap G''$ we consider two cases.

Case 1. t is in G'', i.e., $s \neq t$.

Then, by the hereditarity of $R \cup Q$, $r\ R \cup Q\ t$ holds in G''.

Case 2. t is not in G'', i.e., $s = t$.

Then $r\ R\ s$ holds in G. If $s\ R\ u$ holds in G, then, by the transitivity of R also $r\ R\ u$ holds in G.

If $s\,Q\ u$ holds in G, then by the fact that R is closed under Q $r\,R\,u$ holds in G, as well. Further, $r \neq u$ by the irreflexivity of R, so u is in G'. Hence, by Case 1, $r\ R \cup Q\ u$ holds in G''.

This proves that $G'' \Rightarrow_{1,R\cup Q} G' \cap G''$. To show that $G' \Rightarrow_{1,R\cup Q} G' \cap G''$ we again consider two cases.

Case 1. u is in G', i.e., $u \neq r$.

Then, by the hereditarity of $R \cup Q$, $s\ R \cup Q\ u$ holds in G'. Also u is in G''.

Case 2. u is not in G', i.e., $u = r$.

Then $s\ R \cup Q\ r$ holds in G. If $s\ R\ r$ holds in G, then, by the transitivity of R, $s\ R\ t$ holds in G.

If $s\,Q\ r$ holds in G, then by the fact that R is closed under Q $s\,R\,t$ holds in G, as well. But s and t are strategies of G', so by the hereditarity of R $s\,R\,t$ holds in G'. This shows $G' \rightarrow_R G' \cap G''$.

By the Equivalence Lemma 3.1 the relations $\rightarrow_R$ and $\Rightarrow_R$ coincide, so some strategy t' of $G' \cap G''$ exists such that $s\ R\ t'$, and a fortiori $s\ R \cup Q\ t'$, holds in G'.

This proves that $G' \Rightarrow_{1,R\cup Q} G' \cap G''$. □

This result is a generalization of the Elimination Theorem 3.4. Indeed, it suffices to use instead of $\sim$ the identity relation on games, and use as Q the identity dominance relation (according to which a strategy is only dominated by itself). Then the assumptions of the above theorem reduce to those of the Elimination Theorem 3.4.

As a simple application of this result consider the combination of the strict dominance and the payoff equivalence. The strict dominance relation is hereditary and so is PE, and a union of two hereditary dominance relations is hereditary. Further, strict dominance is a strict partial order and is easily seen to be closed under the payoff equivalence. So the following direct consequence of the Payoff Equivalence Elimination Theorem 8.2(i) and of the above result holds.

Theorem 10.2 (Combined Strict Elimination) *The $\Rightarrow_{S \cup PE}$ relation is $\sim$-UN.* □

In other words, the combined iterated elimination of strategies in which at each step we remove some strictly dominated strategies and some payoff equivalent strategies yields a unique up to the equivalence relation $\sim$ outcome.

11 Combining Nice Weak Dominance with Payoff Equivalence

In this section we show another application of the Combination Theorem 10.1 concerned with a modification of the weak dominance. We denote by $\Rightarrow_W$ the reduction relation on games corresponding to weak dominance. As mentioned earlier, $\Rightarrow_W$ does not satisfy the unique normal form property. An example relevant for us will be provided in a moment.

We studied already one modification of weak dominance in Section 4 by considering inherent weak dominance, a notion due to Börgers [1990]. Another approach was pursued in Marx and Swinkels [1997] (see also Marx and Swinkels [2000]) who studied the notion of nice weak dominance, introduced in Subsection 1.1 and denoted by NW. However, the $\Rightarrow_{NW}$ reduction relation, just as $\Rightarrow_W$, does not satisfy the unique normal form property. To see this consider the following game:

	L	R
T	$2,1$	$2,1$
B	$2,1$	$1,0$

Clearly, all pairs of strategies are compatible, so weak dominance and nice weak dominance coincide here. This game can be reduced by means of the $\Rightarrow_{NW}$ relation both to

	L	R
T	$2,1$	$2,1$

and to

	L
T	2, 1
B	2, 1

In each case we reached a $\Rightarrow_{NW}$-normal form. So the $\Rightarrow_{NW}$ relation (and consequently the $\Rightarrow_W$ relation) is not weakly confluent and does not satisfy the unique normal form property. Note also that the strategy L (nicely) weakly dominates R in the original game but not in the first first restriction. This shows that neither weak dominance nor nice weak dominance is hereditary.

A solution consists of combining nice weak dominance with the payoff equivalence and seeking conditions under which nice weak dominance and weak dominance coincide. This is the approach taken in Marx and Swinkels [1997] who proved that the $\Rightarrow_{NW}$-normal forms of a game are the same up to the removal of the payoff equivalent strategies and a renaming of strategies.[3] They also observed that for the games $(S_1, \ldots, S_n, p_1, \ldots, p_n)$ that satisfy the already mentioned in the Introduction ***transference of decisionmaker indifference*** (TDI) condition:

$$\begin{array}{l} \text{for all } i, j \in [1..n],\ s'_i, s''_i \in S_i \text{ and } s_{-i} \in S_{-i} \\ p_i(s'_i, s_{-i}) = p_i(s''_i, s_{-i}) \text{ implies } p_j(s'_i, s_{-i}) = p_j(s''_i, s_{-i}), \end{array} \tag{1}$$

nice weak dominance and weak dominance coincide on all restrictions. To see the latter note that the compatibility is hereditary and the TDI condition simply amounts to a statement that all pairs of strategies s'_i and s''_i are compatible. So for the games that satisfy the TDI condition the $\Rightarrow_W$-normal forms of a game are the same up to the removal of the payoff equivalent strategies and a renaming of strategies.

Marx and Swinkels [1997] also provided a number of natural examples of games that satisfy this condition. We now present conceptually simpler proofs of their results by following the methodology used throughout the paper. In Section 13 we shall deal with the case of the nice weak dominance by mixed strategies.

The following lemma summarizes the crucial properties of nice weak dominance. They are 'crucial' in the sense that they allow us to directly apply the already discussed Combination Theorem 10.1 to nice weak dominance and payoff equivalence.

Lemma 11.1 (Nice Weak Dominance)

(i) NW is a strict partial order.

[3] Also an addition of payoff equivalent strategies is allowed. Our proof shows this is not needed.

(ii) *NW is closed under PE.*

(iii) $NW \cup PE$ *is hereditary.*

Proof. (i) First, note that the relation NW is clearly irreflexive. To prove transitivity consider a game $(S_1, \ldots, S_n, p_1, \ldots, p_n)$ and suppose that $s''_i \ NW \ s'_i$ and $s'_i \ NW \ s^*_i$.

Then clearly s''_i is weakly dominated by s^*_i. To prove that s''_i and s^*_i are compatible suppose that for some $s_{-i} \in S_{-i}$

$$p_i(s''_i, s_{-i}) = p_i(s^*_i, s_{-i}).$$

Then by the weak dominance

$$p_i(s''_i, s_{-i}) = p_i(s'_i, s_{-i}) = p_i(s^*_i, s_{-i}).$$

Hence by the compatibility of s''_i and s'_i and the compatibility of s'_i and s^*_i for all $j \in [1..n]$

$$p_j(s''_i, s_{-i}) = p_j(s'_i, s_{-i}) = p_j(s^*_i, s_{-i}).$$

(ii) The proofs of the relevant two properties of NW are analogous to the proof of (i) and are omitted.

(iii) Let $G' := (S'_1, \ldots, S'_n, p_1, \ldots, p_n)$ be a restriction of $G := (S_1, \ldots, S_n, p_1, \ldots, p_n)$. Suppose $s'_i, s''_i \in S'_i$ are such that $s'_i \ NW \cup PE \ s''_i$ in G. Then s''_i and s'_i are compatible in G and hence in G'. Moreover

$$p_i(s''_i, s^*_{-i}) \geq p_i(s'_i, s^*_{-i})$$

for all $s^*_{-i} \in S'_{-i}$. If for some $s^*_{-i} \in S'_{-i}$

$$p_i(s''_i, s^*_{-i}) > p_i(s'_i, s^*_{-i}),$$

then s''_i weakly dominates s'_i in G' and consequently s''_i nicely weakly dominates s'_i in G'. Otherwise

$$p_i(s''_i, s^*_{-i}) = p_i(s'_i, s^*_{-i})$$

for all $s^*_{-i} \in S'_{-i}$, so, by the compatibility of s''_i and s'_i in G', s''_i and s'_i are payoff equivalent in G'.

So we showed that $s''_i \ NW \cup PE \ s'_i$ in G'. □

Nice weak dominance clearly satisfies the IIIA condition of Section 4 and by item (i) above it is a strict partial order. So using $R := NW$ in the Inherent Elimination Theorem 4.4 and the One-at-a-time Elimination Theorem 4.5 we get the following result.

Theorem 11.2 (Inherent Nice Weak Elimination)

(i) The $\Rightarrow_{inh-NW}$ relation is UN.

(ii) The $\Rightarrow_{inh-NW}$ relation satisfies the one-at-a-time property. □

Further, the above lemma in conjunction with the Payoff Equivalence Elimination Theorem 8.2(i) means that for $R := NW$ and $Q := PE$ all assumptions of the Combination Theorem 10.1 are satisfied. So we get the following conclusion.

Theorem 11.3 (Nice Weak Elimination) *The $\Rightarrow_{NW \cup PE}$ relation is $\sim$-UN.* □

Also, for games that satisfy the TDI condition (1) the $\Rightarrow_{NW \cup PE}$ and $\Rightarrow_{W \cup PE}$ relations coincide on all restrictions, so the following conclusion follows.

Corollary 11.4 (Weak Elimination) *Consider a game H that satisfies the TDI condition (1). Then the $\Rightarrow_{W \cup PE}$ relation is $\sim$-UN.* □

To establish another form of order independence involving nice weak dominance we shall rely on the following observation that refers to the crucial concept of left commutativity.

Note 11.5 (Left Commutativity) *$\Rightarrow_{PE}$ left commutes with $\Rightarrow_{NW}$.*

Proof. By the One-at-a-time Elimination Theorem 3.2 the reduction relation $\Rightarrow_{PE}$ satisfies the one-at-a-time property, i.e.,

$$\Rightarrow^{+}_{1,PE} = \Rightarrow^{+}_{PE}.$$

So by the Left Commutativity Note 7.3 it suffices to show that $\Rightarrow_{1,PE}$ left commutes with $\Rightarrow_{NW}$. Suppose $G \Rightarrow_{1,PE} G' \Rightarrow_{NW} G''$. In the proof below we repeatedly use the fact that if a strategy r_j is nicely weakly dominated in G' by a strategy t_j, then so it is in G.

Let s_i be the strategy deleted in the first transition. If all strategies that are payoff equivalent to s_i are removed in the second transition, then by the Nice Weak Dominance Lemma 11.1*(ii)* NW is closed under PE which implies $G \Rightarrow_{NW} G''$. Consequently $G \Rightarrow_{NW} G'' \Rightarrow^{\epsilon}_{1,PE} G''$.

Otherwise, by the fact that payoff equivalence is hereditary, we have $G \Rightarrow_{NW} G_1 \Rightarrow_{1,PE} G''$, where G_1 is obtained from G'' by adding s_i to the set of strategies of player i. □

As an aside, note that the same proof shows that $\Rightarrow_{PE}$ left commutes with $\Rightarrow_W$ and with $\Rightarrow_S$. The relevant property is that both W and S are closed under PE.

We reached now the already mentioned result of Marx and Swinkels [1997].

Theorem 11.6 (Structured Nice Weak Elimination) *Suppose that $G \Rightarrow^*_{NW} G'$ and $G \Rightarrow^*_{NW} G''$, where both G' and G'' are closed under the $\Rightarrow_{NW}$ reduction (i.e., are $\Rightarrow_{NW}$-normal forms).*

Then for some $\sim$-equivalent games H' and H'' closed under the $\Rightarrow_{NW \cup PE}$ reduction we have $G' \Rightarrow^*_{PE} H'$ and $G'' \Rightarrow^*_{PE} H''$.

Proof. Since PE is hereditary, each step $H_1 \Rightarrow_{NW \cup PE} H_2$ can be rewritten as $H_1 \Rightarrow_{NW} H_3 \Rightarrow_{PE} H_2$ for some game H_3. So by the Nice Weak Elimination Theorem 11.3 the $\rightarrow_{1 \vee 2}$ relation, where $\rightarrow_1 := \Rightarrow_{NW}$ and $\rightarrow_2 := \Rightarrow_{PE}$, is $\sim$-UN.

It suffices now to use the Left Commutativity Note 11.5 and the Normal Form Lemma 7.4. □

As explained at the end of Section 8 the reductions from G' to H' and from G'' to H'' can be achieved in just one step.

Corollary 11.7 (Structured Weak Elimination) *Consider a game G that satisfies the TDI condition (1). Suppose that $G \Rightarrow^*_W G'$ and $G \Rightarrow^*_W G''$, where both G' and G'' are closed under the $\Rightarrow_W$ reduction.*

Then for some $\sim$-equivalent games H' and H'' we have $G' \Rightarrow^*_{PE} H'$ and $G'' \Rightarrow^*_{PE} H''$. □

Recently, Østerdal [2004] provided an alternative proof of this corollary.

In the Weak Elimination Corollary 11.4 we can weaken the assumption that the initial game H satisfies the TDI condition. Indeed, it suffices to ensure that each time an $\Rightarrow_W$ reduction can take place, it is in fact an $\Rightarrow_{NW}$ reduction. This is guaranteed if the following condition TDI$^+$ is satisfied, given an initial game H:

> for all restrictions $G := (S_1, \ldots, S_n, p_1, \ldots, p_n)$ of H,
> for all $i \in [1..n]$ and $r_i, t_i \in S_i$
> if t_i weakly dominates r_i in G, then r_i and t_i are compatible in G.

An alternative, suggested by Marx and Swinkels [1997] in the context of nice weak dominance by mixed strategies, is to use the following condition

TDI^{++}, where, given a game $(S_1, \ldots, S_n, p_1, \ldots, p_n)$, a strategy s'_i ***very weakly dominates*** a strategy s''_i if

$$p_i(s'_i, s_{-i}) \geq p_i(s''_i, s_{-i})$$

for all $s_{-i} \in S_{-i}$:

for all restrictions $G := (S_1, \ldots, S_n, p_1, \ldots, p_n)$ of H,
for all $i \in [1..n]$ and $r_i, t_i \in S_i$ if t_i very weakly dominates r_i in G, then either t_i weakly dominates r_i in G or r_i and t_i are payoff equivalent in G.

Indeed, it suffices to show that under the TDI^{++} condition all assumptions of the Combination Theorem 10.1 are satisfied by the weak dominance relation W. First, note that W is a strict partial order and is clearly closed under the payoff equivalence.

Denote now the very weak dominance relation by VW. Note that

- $W \subseteq VW$ (i.e., weak dominance implies very weak dominance),
- VW is hereditary.

Additionally, by the TDI^{++} assumption,

- $VW \subseteq W \cup PE$

holds in all restrictions of the initial game H.

This implies under the TDI^{++} assumption that $W \cup PE$ is hereditary since PE is hereditary. By the Combination Theorem 10.1 we conclude then that the $\Rightarrow_{W \cup PE}$ reduction relation is $\sim$-UN.

The same considerations apply to the Structured Weak Elimination Corollary 11.7. However, to be able to use the TDI^{++} condition we need in addition to prove that $\Rightarrow_{PE}$ left commutes with $\Rightarrow_W$. The proof is the same as that of the Left Commutativity Note 11.5.

12 Combining Two Mixed Dominance Relations

We now return to the mixed dominance relations and study a combination $R \cup Q$ of two such relations R and Q. In the applications Q will be the randomized redundance relation *PEM* studied in Section 9.

We say that a combined mixed dominance relation R is ***closed under*** Q if in all games G for all strategies r, s and all mixed strategies m_1, m_2

- $r\ R\ m_1$ and $s\ Q\ m_2$ implies $r\ R\ m_1[s/m_2]$,
- $r\ Q\ m_1$ and $s\ R\ m_2$ implies $r\ R\ m_1[s/m_2]$.

The following counterpart of the Combination Theorem 10.1 holds.

Theorem 12.1 (Combination) *Consider two mixed dominance relations R and Q such that*

- $\Rightarrow_R$ *and* $\Rightarrow_Q$ *are $\sim$-bisimilar,*
- *R is regular,*
- *R is closed under the randomized redundance,*
- $\Rightarrow_{1,Q}$ *is $\sim$-UN,*
- *$R \cup Q$ is hereditary.*

Then the $\Rightarrow_{R \cup Q}$ relation is $\sim$-UN.

Proof. We proceed as in the proof of the Combination Theorem 10.1.

Since $R \cup Q$ is hereditary, by the One-at-a-time Elimination Theorem 3.2 and the $\sim$-Unique Normal Form Note 7.2 it suffices to prove that $\Rightarrow_{1,R \cup Q}$ satisfies the $\sim$-unique normal form. In turn, by the $\sim$-Newman's Lemma 7.1 this is established once we show that $\Rightarrow_{1,R \cup Q}$ is $\sim$-weakly confluent. Indeed, as before $\Rightarrow_{1,R \cup Q}$ is $\sim$-bisimilar.

So suppose that $G \Rightarrow_{1,R \cup Q} G'$ and $G \Rightarrow_{1,R \cup Q} G''$. Let r and s be the strategies eliminated in the first, respectively second, transition. By the fourth assumption $\Rightarrow_{1,Q}$ is $\sim$-weakly confluent, so we only need to consider a situation when $G \Rightarrow_{1,R} G'$.

We can assume that $G' \neq G''$. Then r is in G'' and s is in G'. By definition $r\ R\ m_1$ holds in G for some mixed strategy m_1 of G' and $s\ R \cup Q\ m_2$ holds in G for some mixed strategy m_2 of G''. To show that $G'' \Rightarrow_{1,R \cup Q} G' \cap G''$ we consider two cases.

Case 1. $s \notin support(m_1)$.

Then m_1 is a mixed strategy G'', so $r\ R \cup Q\ m_1$ holds in G'' by the hereditarity of $R \cup Q$.

Case 2. $s \in support(m_1)$.

If $s\ R\ m_2$ holds in G, then, by the regularity of R, $r\ R\ m_1[s/m_2]$ holds in G.

If $sQ\ m_2$ holds in G, then by the fact that R is closed under Q $r R\, m_1[s/m_2]$ holds in G, as well. By assumption m_2 is a mixed strategy of G'', so $s \notin$ *support*(m_2) and consequently $s \notin$ *support*$(m_1[s/m_2])$. So by the first clause of the regularity condition for some mixed strategy m_3 with $r, s \notin$ *support*(m_3) we have $r\ R\ m_3$. Hence, by Case 1, $r\ R \cup Q\ m_1$ holds in G''.

This proves that $G'' \Rightarrow_{1,R\cup Q} G' \cap G''$. To show that $G' \Rightarrow_{1,R\cup Q} G' \cap G''$ we again consider two cases.

Case 1. $r \notin$ *support*(m_2).

Then m_2 is a mixed strategy G', so $s\ R \cup Q\ m_2$ holds in G' by the hereditarity of $R \cup Q$.

Case 2. $r \in$ *support*(m_2).

Recall that $s\ R \cup Q\ m_2$ holds in G. If $s\ R\ m_2$ holds in G, then, by the regularity of R, $s\ R\ m_2[r/m_1]$ holds in G. If $s\ Q\ m_2$ holds in G, then by the fact that R is closed under Q $s\ R\ m_2[r/m_1]$ holds in G, as well. By assumption m_1 is a mixed strategy of G', so $r \notin$ *support*(m_1) and consequently $r \notin$ *support*$(m_2[r/m_1])$. So $m_2[r/m_1]$ is a mixed strategy of G'. This shows $G' \rightarrow_R G' \cap G''$.

By the Equivalence Lemma 5.1 the relations $\rightarrow_R$ and $\Rightarrow_R$ coincide, so some mixed strategy m_3 of $G' \cap G''$ exists such that $s\ R\ m_3$ and a fortiori $s\ R \cup Q\ m_3$, holds in G'.

This proves that $G' \Rightarrow_{1,R\cup Q} G' \cap G''$. □

This result can be directly applied to the combination of the elimination by strict dominance by mixed strategies and by the randomized redundance. Indeed, we already noticed that both mixed dominance relations are hereditary, so their union is, as well. Also, we already saw that strict dominance by means of mixed strategies is regular and it is easy to see it is closed under the randomized redundance. So by the Redundance Elimination Theorem 9.2(i) and the above result we can draw the following conclusions.

Theorem 12.2 (Combined Mixed Strict Elimination) *The* $\Rightarrow_{SM\cup PDM}$ *relation is* $\sim$*-UN.* □

13 Combining Nice Weak Dominance with Randomized Redundance

Finally, we provide a proof of another result of Marx and Swinkels [1997] that deals with the nice weak dominance by mixed strategies. This concept is obtained by generalizing in the obvious way the definition of nice weak dominance to the case when the dominating strategy is mixed.

Recall from Section 6 that given a game G we write s''_i WM m'_i when the strategy s''_i is weakly dominated in G by the mixed strategy m'_i. We also write s''_i NWM m'_i when the strategy s''_i is nicely weakly dominated in G by the mixed strategy m'_i, that is when s''_i WM m'_i and s''_i and m'_i are compatible.

As in Section 11 we summarize first the relevant properties of the nice weak mixed dominance relation.

Lemma 13.1 (Nice Mixed Weak Dominance)

(i) NWM is regular.

(ii) NWM is closed under PEM.

(iii) $NWM \cup PEM$ is hereditary.

Proof. Fix a game $(S_1, \ldots, S_n, p_1, \ldots, p_n)$.
(i) Suppose that for some $\alpha \in (0, 1]$ and some strategy s and a mixed strategy m of player i

$$s \ NWM \ (1 - \alpha)s + \alpha\, m$$

holds. By definition for all $j \in [1..n]$ and all $s_{-i} \in S_{-i}$

$$p_j((1 - \alpha)s + \alpha\, m, s_{-i}) = (1 - \alpha)p_j(s, s_{-i}) + \alpha\, p_j(m, s_{-i}),$$

so for all $op \in \{=, <, \leq\}$

$$p_j(s, s_{-i})\ op\ p_j((1 - \alpha)s + \alpha\, m, s_{-i}) \text{ iff } p_j(s, s_{-i})\ op\ p_j(m, s_{-i}).$$

This implies s NWM m.

Next, consider the strategies t_1 and t_2 and mixed strategies m_1 and m_2 of player i. For some $\alpha \in [0, 1]$ and a mixed strategy m we have $m_1 = \alpha\, t_2 + (1 - \alpha)m$. By definition for all $j \in [1..n]$ and all $s_{-i} \in S_{-i}$

$$p_j(m_1, s_{-i}) = \alpha\, p_j(t_2, s_{-i}) + (1 - \alpha)p_j(m, s_{-i}) \tag{2}$$

and

$$p_j(m_1[t_2/m_2], s_{-i}) = \alpha\, p_j(m_2, s_{-i}) + (1 - \alpha)p_j(m, s_{-i}). \tag{3}$$

It is now easy to check that t_1 WM m_1 and t_2 WM m_2 implies t_1 WM $m_1[t_2/m_2]$.

Suppose now that t_1 WNM m_1 and t_2 WNM m_2. We prove that t_1 and $m_1[t_2/m_2]$ are compatible. So suppose that for some $i \in [1..n]$ and $s_{-i} \in S_{-i}$

$$p_i(t_1, s_{-i}) = p_i(m_1[t_2/m_2], s_{-i}).$$

Then by (2) and (3) and the fact that t_1 *WM* m_1 and t_2 *WM* m_2

$$p_i(t_1, s_{-i}) = p_i(m_1, s_{-i}) \text{ and } p_i(t_2, s_{-i}) = p_i(m_2, s_{-i}).$$

So by the compatibility of t_1 and m_1 and of t_2 and m_2 for all $j \in [1..n]$

$$p_j(t_1, s_{-i}) = p_i(m_1, s_{-i}) \text{ and } p_j(t_2, s_{-i}) = p_i(m_2, s_{-i}),$$

so again by (2) and (3)

$$p_j(t_1, s_{-i}) = p_j(m_1[t_2/m_2], s_{-i}).$$

(ii) The proofs of the relevant two properties of *NWM* are analogous to the proof of (i) and are omitted.

(iii) Analogous to the proof of the Nice Weak Dominance Lemma 11.1(iii) and omitted. □

We can now apply to nice weak mixed dominance the Inherent Mixed Elimination Theorem 6.3. This way we obtain the following result.

Theorem 13.2 (Inherent Nice Weak Mixed Elimination)
(i) The $\Rightarrow_{inh-NWM}$ *relation is UN.*

(ii) The $\Rightarrow_{inh-NWM}$ *relation satisfies the one-at-a-time property.* □

Further, on the account of the Redundance Elimination Theorem 9.2(i) for $R :=$ *NWM* and $Q :=$ *PEM* all assumptions of the Combination Theorem 12.1 are satisfied. We can then draw the following conclusion.

Theorem 13.3 (Nice Weak Mixed Elimination) *The* $\Rightarrow_{NWM \cup PEM}$ *relation is* $\sim$*-UN.* □

To draw a similar conclusion for the weak dominance by mixed strategies, as in Section 11 we provide three alternative conditions. The first one, TDIM, is the direct counterpart of the TDI condition (1):

$$\begin{array}{l}\text{for all } i, j \in [1..n],\ r_i \in S_i,\ m_i \in M_i \text{ and } s_{-i} \in S_{-i}\\ p_i(r_i, s_{-i}) = p_i(m_i, s_{-i}) \text{ implies } p_j(r_i, s_{-i}) = p_j(m_i, s_{-i})\end{array}$$

Equivalently, for all $i \in [1..n]$, $r_i \in S_i$ and $m_i \in M_i$, r_i and m_i are compatible.
Indeed, the compatibility as a mixed dominance relation is hereditary, so the TDIM condition implies that nice weak dominance and weak dominance, both by mixed strategies, coincide on all restrictions.

The second one, TDIM$^+$, is the counterpart of the TDI$^+$ condition of Section 11. Given an initial game H we postulate that

for all restrictions $G := (S_1, \ldots, S_n, p_1, \ldots, p_n)$ of H,
for all $i \in [1..n]$, $r_i \in S_i$ and $m_i \in M_i$
if m_i weakly dominates r_i in G, then r_i and m_i are compatible in G.

Then each time an $\Rightarrow_{WM}$ reduction can take place, it is in fact an $\Rightarrow_{NWM}$ reduction. The last alternative, TDI*, was proposed in Marx and Swinkels [1997]. It refers to the notion of the very weak dominance introduced in Section 11, now used as a mixed dominance relation:

for all restrictions $G := (S_1, \ldots, S_n, p_1, \ldots, p_n)$ of H,
for all $i \in [1..n]$, $r_i \in S_i$ and $m_i \in M_i$
if m_i very weakly dominates r_i in G, then
either m_i weakly dominates r_i in G or r_i and m_i are payoff equivalent in G.

Then the following result holds.

Theorem 13.4 (Weak Mixed Elimination) *Consider a game H that satisfies the TDI* condition. Then the $\Rightarrow_{WM \cup PEM}$ relation is $\sim$-UN.*

Proof. We proceed as in Section 11. Denote the very weak mixed dominance relation by VWM. Note that

- $WM \subseteq VWM$,
- VWM is hereditary.

Additionally, by the TDI* assumption,

- $VWM \subseteq WM \cup PEM$

holds in all restrictions of the initial game H.

So under the TDI* assumption $WM \cup PEM$ is hereditary since PEM is hereditary. By the Combination Theorem 12.1 we conclude that the $\Rightarrow_{WM \cup PEM}$ relation is $\sim$-UN.

To establish another form of order independence involving nice mixed weak dominance we need the following observation.

Note 13.5 (Left Commutativity)
(i) $\Rightarrow_{PEM}$ *left commutes with* $\Rightarrow_{NWM}$.

(ii) $\Rightarrow_{PEM}$ *left commutes with* $\Rightarrow_{WM}$.

Proof. (*i*) By the Nice Mixed Weak Dominance Lemma 13.1(*ii*) NWM is closed under the randomized redundance. The rest of the proof is now analogous to the proof of the Left Commutativity Note 11.5 and is omitted.

(*ii*) By the same argument as in (*i*). □

As in Section 11 we can now draw the following results due to Marx and Swinkels [1997].

Theorem 13.6 (Structured Nice Weak Mixed Elimination) *Suppose that* $G \Rightarrow^*_{NWM} G'$ *and* $G \Rightarrow^*_{NWM} G''$, *where both* G' *and* G'' *are closed under the* $\Rightarrow_{NWM}$ *reduction.*

Then for some $\sim$*-equivalent games* H' *and* H'' *closed under the* $\Rightarrow_{NWM \cup PEM}$ *reduction we have* $G' \Rightarrow^*_{PEM} H'$ *and* $G'' \Rightarrow^*_{PEM} H''$. □

Corollary 13.7 (Structured Weak Mixed Elimination) *Consider a game* G *that satisfies the TDI* condition. Suppose that* $G \Rightarrow^*_{WM} G'$ *and* $G \Rightarrow^*_{WM} G''$, *where both* G' *and* G'' *are closed under the* $\Rightarrow_{WM}$ *reduction.*

Then for some $\sim$*-equivalent games* H' *and* H'' *closed under the* $\Rightarrow_{WM \cup PEM}$ *reduction we have* $G' \Rightarrow^*_{PEM} H'$ *and* $G'' \Rightarrow^*_{PEM} H''$. □

14 Conclusions

In this paper we presented uniform proofs of order independence for various strategy elimination procedures. The main ingredients of our approach were reliance on Newman's Lemma and related results on the abstract reduction systems, and an analysis of the structural properties of the dominance relations. This exposition allowed us to clarify which structural properties account for the order independence of the entailed reduction relations on the games.

In Figure 2 below we summarize the order independence results discussed in this article. We use here the already introduced abbreviations, so:

- S denotes strict dominance,
- W denotes weak dominance,
- NW denotes nice weak dominance,
- PE denotes payoff equivalence.

Further, RM stands for the 'mixed strategy' version of the dominance relation R and *inh-R* stands for the 'inherent' version of the (mixed) dominance relation R discussed in Sections 4 and 6.

Recall also that UN stands for the uniqueness of the normal form, i.e., for the order independence and $\sim$-UN is its 'up to the game equivalence' version. All the results refer to the order independence of the $\Rightarrow_R$ reduction relation on games, introduced in Section 3.

Dominance Notion	Property	Proved in Section	Result originally due to
S	UN	3	Gilboa, Kalai and Zemel [1990], Stegeman [1990]
$inh - W$	UN	4	Börgers [1990]
$inh - NW$	UN	11	
SM	UN	5	Osborne and Rubinstein [1994]
$inh - WM$	UN	6	(Börgers [1990]: equal to SM)
$inh - NWM$	UN	13	
PE	$\sim$-UN	8	
$S \cup PE$	$\sim$-UN	10	
$NW \cup PE$	$\sim$-UN	11	Marx and Swinkels [1997]
PEM	$\sim$-UN	9	
$SM \cup PEM$	$\sim$-UN	12	
$NWM \cup PEM$	$\sim$-UN	13	Marx and Swinkels [1997]

Figure 2: Summary of the order independence results

The reduction relations on games that we studied are naturally related. For example we have $\Rightarrow_S \subseteq \Rightarrow_{SM}$, with the strict inclusion for some games. However, the respective results about these reduction relations are not related. For example, the fact that $\Rightarrow_S$ is UN not a special case of the fact that $\Rightarrow_{SM}$ is UN.

Indeed, given two abstract reduction systems $(A, \rightarrow_1)$ and $(A, \rightarrow_2)$ such that $\rightarrow_1 \subseteq \rightarrow_2$ the uniqueness of a normal form with respect to $\rightarrow_2$ does not imply the uniqueness of a normal form with respect to $\rightarrow_1$. Indeed, just take $\rightarrow_1 := \{(a, b), (a, c)\}$ and $\rightarrow_2 := \rightarrow_1 \cup \{(b, d), (c, d)\}$. This example also shows that weak confluence of $\rightarrow_2$ does not imply weak confluence of $\rightarrow_1$. So the weak confluence of $\Rightarrow_S$ is not a consequence of the weak confluence of $\Rightarrow_{SM}$. The same remarks apply to other pairs of dominance relations.

The provided proofs of the order independence results break down for infinite games. The reason is that the crucial assumption of Newman's Lemma, namely that that no infinite $\rightarrow$ sequences exist, does not hold then anymore. Moreover, for infinite games the Equivalence Lemma 3.1 does not hold. Still, it would be interesting to try to establish the main result of Dufwenberg and Stegeman [2002] using the abstract reduction systems techniques.

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