Explicit generators for the ring of quasisymmetric functions over the integers

by

Abstract.

MSCS:

Key words and key phrases:

1. The Witt polynomials.

This is well known stuff, see e.g. Chapter 3 of [1], included here for completeness sake and to establish notation.

Let \( \text{Symm} \) be the ring of symmetric functions over the integers in infinitely many variables

\[
\text{Symm} = \mathbb{Z}[e_1, e_2, \cdots] \subset \mathbb{Z}[x_1, x_2, \cdots]
\]  

(1.1)

Here the \( e_i \) are the elementary symmetric functions in the \( x_j \). There is another free polynomial basis of \( \text{Symm} \), that is related to the free polynomial basis \( \{e_1, e_2, \cdots\} \) by the formula

\[
\prod_{i=1}^\infty (1 - a_i t^i) = 1 - e_1 t + e_2 t^2 - e_3 t^3 + \cdots = \prod_{i=1}^\infty (1 - x_i t)
\]  

(1.2)

The free polynomial basis \( \{a_1, a_2, \cdots\} \) generalizes in a natural way to a free polynomial basis over the integers for the ring \( \mathcal{QSymm} \) of quasisymmetric functions. It is of course obvious from (1.2) that \( \{a_1, a_2, \cdots\} \) is a free polynomial basis of \( \text{Symm} \).

Let

\[
w_n(X) = \sum_{d|n} dX_{a/d}^{n/d}
\]  

(1.3)

be the well known Witt polynomials (in a set of commuting variables \( X_1, X_2, \cdots \)). Let

\[
p_n = \sum_i x_i^n \in \text{Symm}
\]  

(1.4)

be the power sums. Then

\[
w_n(a_1, a_2, \cdots, a_n) = p_n
\]  

(1.5)

To see this just apply \(-t \frac{d}{dt}\log\) to the formula (1.2) (the outer parts).
2. The wll-ordering

Let \( \alpha = [a_1, a_2, \ldots, a_m] \), \( a_i \in \mathbb{N} = \{1, 2, \ldots\} \) be a composition. The length of such a composition is \( \text{lg}(\alpha) = m \), and its weight is \( \text{wt}(\alpha) = a_1 + a_2 + \cdots + a_m \). The empty composition \([\ ]\) has length and weight zero. A composition \( \alpha \) defines a monomial quasisymmetric function as follows

\[
\alpha = \sum_{i_1 < i_2 < \cdots < i_n} x_{i_1}^{a_1} x_{i_2}^{a_2} \cdots x_{i_n}^{a_n}
\]  

(2.1)

As a rule we shall not distinguish between a composition and the quasisymmetric function it defines. The empty composition is the unit element in the ring \( \text{QSymm} \) of quasisymmetric functions. The monomial symmetric functions (2.1) form a free Abelian group basis for \( \text{QSymm} \).

We shall use a total ordering on the set of composition called the wll-ordering. This stands for “weight first, than length, and finally lexicographic”. Thus, for instance

\[
[5] >_{wll} [1,1,2] >_{wll} [2,2] >_{wll} [1,3]
\]

3. Substitution (= plethysm)

Given a composition \( \alpha \) and a composition \( \beta \) define a new quasisymmetric function \( \alpha \circ \beta \), “\( \beta \) substituted in \( \alpha \)” as follows. Order the summands of the quasisymmetric function \( \beta \) lexicographically and substitute these in that order for the \( x_1, x_2, \ldots \) in the quasisymmetric function \( \alpha \). The result is a new quasisymmetric function of weight \( \text{wt}(\alpha)\text{wt}(\beta) \).

The transformation

\[
s_\beta: \alpha \mapsto \alpha \circ \beta
\]

(3.1)

is (obviously) a ring endomorphism. The transformation

\[
t_\beta: \beta \mapsto \alpha \circ \beta
\]

is not a ring homomorphism; it is a plethysm. Indeed, the \( t_\beta \) define a \( \lambda \)-ring structure on \( \text{QSymm} \).

3.2. Example. For a composition \( \alpha = [a_1, a_2, \ldots, a_m] \) and a natural number \( n \) let \( n\alpha = [na_1, na_2, \ldots, na_m] \). Then for the power sums \( p_n \)

\[
p_n \circ \alpha = n\alpha
\]

(3.3)

3.4. Example. For two compositions \( \alpha, \beta \) let \( \alpha \ast \beta \) denote their concatenation. Thus, for example, \([1,2] \ast [1,6,4] = [1,2,1,6,4] \), and \([1,2] \ast [1,2,1,2,1,2] \ast [1,2] = [1,2,1,2,1,2,1,2,1,2] \). Let \( e_n \) be the \( n \)-th elementary symmetric function. Then if \( \alpha \) is a Lyndon word

\[\text{It would actually probably be better to write} \ f_\alpha \alpha, \text{for these are the right Frobenius Hopf algebra endomorphisms of} \ \text{QSymm}; \text{they are also the Adams endomorphisms corresponding to the} \ \lambda - \text{ring structure already mentioned.} \]
\[ e_n \circ \alpha = \alpha^n + (\text{wll-smaller}) \quad (3.5) \]

where (wll-smaller) means a sum of monomial quasisymmetric functions that are strictly wllsmaller than \( \alpha^n \).

4. Lyndon-Witt generators

For a composition \( \alpha = [a_1, a_2, \cdots, a_m] \) let \( g(\alpha) = \gcd(a_1, a_2, \cdots, a_m) \) and define

\[ A_\alpha = a_{g(\alpha)} \circ \alpha_{\text{red}} \quad (4.1) \]

where the \( a_i \) are the symmetric functions of section 1 above and

\[ \alpha_{\text{red}} = [g(\alpha)^{-1} a_1, g(\alpha)^{-1} a_2, \cdots, g(\alpha)^{-1} a_m] \quad (4.2) \]

Note that \( A_\alpha \) is homogeneous of weight \( \text{wt}(\alpha) \).

4.3. Lemma. Let \( \alpha \) be a reduced composition, i.e. \( g(\alpha) = 1 \). Then

\[ \sum_{d|n} dA_{n/d} = n\alpha \quad (4.4) \]

Proof. This follows immediately from the definition of the \( A_\beta \) by applying the operation

"substitute \( \alpha^n \) to formula (1.5), using (3.3)."

Note that formula (4.4) again establishes that all \( A_\beta \) are quasisymmetric functions while their

integrality is assured by the definition (4.1).

Let \( LYN \) be the set of Lyndon compositions (Lyndon words).

4.5. Theorem. The set \( \{A_\alpha : \alpha \in LYN\} \) is a set of free polynomial generators for \( QSymm \).

Proof. Let \( R \) be the subring of \( QSymm \) generated by the \( A_\alpha \), \( \alpha \in LYN \). Because the number of proposed homogeneous generators is just right for each weight it will suffice to show that \( R = QSymm \), i.e. that each composition \( \alpha \) is in \( R \).

To start with, let \( \beta \) be a Lyndon composition. Then taking \( a = \beta_{\text{red}} \), and \( n = g(\beta) \) in

formula (4.4) we see that \( \beta \in R \).

We now proceed with induction for the wll-ordering. The case of weight 1 is trivial. For each separate weight the induction starts because of what has just been said because compositions of length 1 are Lyndon.

So let \( \alpha \) be a composition of weight \( \geq 2 \) and length \( \geq 2 \). By the Chen-Fox-Lyndon concatenation factorization theorem

\[ \alpha = \beta_1^{r_1} \ast \beta_2^{r_2} \ast \cdots \ast \beta_k^{r_k}, \quad \beta_i \in LYN, \quad \beta_i >_{\text{lex}} \beta_2 >_{\text{lex}} \cdots >_{\text{lex}} \beta_k \quad (4.6) \]

where, as before, the \( \ast \) denotes concatenation and \( \beta >_{\text{lex}} \beta' \) means that \( \beta \) is lexicographically
strictly larger than $\beta'$.

If $k \geq 2$, take $\alpha' = \beta_1^n$ and for $\alpha''$ the corresponding tail of $\alpha$ so that $\alpha = \alpha' \cdot \alpha''$. Then

$$\alpha' \alpha'' = \alpha' \cdot \alpha'' + (\text{wll - smaller than } \alpha) = \alpha + (\text{wll - smaller than } \alpha)$$

and with induction it follows that $\alpha \in R$. there remains the case that $k = 1$ in the CFL-factorization (4.6). If then also $r = \eta = 1$, $\alpha$ is Lyndon, and hence by what has been said at the start of the proof $\alpha \in R$. The remaining case is that of a word of the form $\gamma = \beta''$, $r \geq 2$ and this case requires a rather different argument.

Let $\mathcal{P}$ be the ideal in $QSymm$ generated by all nontrivial products $\alpha_1 \alpha_2$, $\lg(\alpha_1), \lg(\alpha_2) \geq 1$. Now in formula (4.4) take $\alpha = \beta_{\text{red}}$ and $n = g(\beta)r$ to see that

$$g(\beta)rA_{\beta} = g(\beta)rA_{\beta_{\text{red}}} \equiv g(\beta)r \alpha = r\beta \mod \mathcal{P} \quad (4.7)$$

Now consider the $r$-th Newton relation in $Symm$

$$p_r - e_1 p_{r-1} + \cdots + (-1)^{r-1} e_{r-1} p_1 + (-1)^r r e_r = 0 \quad (4.8)$$

And apply the operation ‘substitute $\beta'$ to it. Using Examples 3.2, 3.4, there results that

$$r\beta \equiv \pm r(\beta'' + (\text{wll - smaller than } \beta''')) \mod \mathcal{P} \quad (4.9)$$

Now combine this with (4.7) and use that $QSymm / \mathcal{P}$ is torsion free to see that (for Lyndon $\beta$)

$$g(\beta)A_{\beta} \equiv \pm \beta'' + (\text{wll - smaller than } \beta''') \quad (4.10)$$

With induction this finishes the proof.

4.11. Remarks.

Note that this proof requires that one has already shown that $QSymm / \mathcal{P}$ is torsion free which follows from the theorem that abstractly (without specifying explicit generators) the ring $QSymm$ is freely polynomially generated, which is proved in [3, 4].

The idea of using Chen-Fox-Lyndon factorization to prove theorems like 4.5 goes back (at least) to [5]. This is also the key technique for proving a $p$-adic version of theorem 4.5, i.e. over $\mathbb{Z}_{(p)}$, (also with explicit generators) in [2, 3]. Which, in turn, suffices to establish the torsion freeness of $QSymm / \mathcal{P}$.

4.12. Corollary

Take another free polynomial basis of $Symm$, like the elementary symmetric functions $e_n$ or the complete symmetric functions $h_n$. Define quasisymmetric functions

$$E_{\alpha} = e_{g(\alpha)} \circ \alpha_{\text{red}}, \quad H_{\alpha} = h_{g(\alpha)} \circ \alpha_{\text{red}} \quad (4.13)$$

Then also $\{E_{\alpha}; \alpha \in LYN\}$ and $\{H_{\alpha}; \alpha \in LYN\}$ are free polynomial bases for $QSymm$. 
References.