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LECTURES ON INTERACTIONS OF CONTROL THEORY AND *K*-THEORY

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and
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THE ABDUS SALAM INTERNATIONAL CENTRE FOR THEORETICAL PHYSICS

LECTURES ON INTERACTIONS OF CONTROL THEORY AND K-THEORY

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Abstract

Perhaps somewhat surprisingly, there are a fair number of relations between questions of control and system theory and algebraic K-theory. In the direction from K-theory to control these are mostly applications of the Quillen-Suslin theorem. In the other direction there are some ideas from control that have applications to the K-theory of endomorphisms. These lectures are an attempt to do a survey of these interrelations.

MIRAMARE – TRIESTE

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Introductory remarks.

Below there are write-ups of four lectures on interactions of algebraic K-theory with control and system theory and related matters (mostly applications of the Quillen-Suslin theorem to control type questions, but also other matters and some insights into the K-theory of endomorphisms and deformations of representations that came from the control world). These lectures were delivered in March at the Abdus Salam ICTP in Trieste, Italy.

This write-up is quite a bit more complete than the handwritten notes that were distributed at the time. However, for full details of the proofs and proper accreditations of all the results the attentive reader is referred to the references. These are collected, for all four lectures, just after the text of lecture 4.

In addition there is an appendix. This appendix mainly consists of a list of references on the topics of these lectures and also quite a bit more "applied K-theory". In particular the reader can there find the material I collected at the time for four more lectures on interactions of K-theory with other areas in mathematics (than control) such as Corona type theorems (in H^∞), Lambda rings and Beta rings and ..., Tiling, Polylogarithms.

Lecture 1. Delay control systems and the Quillen-Suslin theorem.

More details and proofs of the material in this first lecture can be found in the two references [4, 5], and the papers quoted there.

1.1. KRONECKER INDICES OF A CONTROL SYSTEM.

A linear constant-time *control system*, or input system, is a set of differential equations, or difference equations of the form

$$\dot{x} = Ax + Bu,\ \ x(t) \in \mathbf{R}^n,\ \ u(t) \in \mathbf{R}^m \quad (\text{or } \ x \in \mathbf{C}^n,\ \ u \in \mathbf{C}^m \tag{1.1.1}$$

where $\mathbf{R}$ and $\mathbf{C}$ are respectively the real and complex numbers, or

$$x_{i+1} = Ax_i + Bu_i,\ \ x_i \in K^n,\ \ u_i \in K^m \tag{1.1.2}$$

where K is any field (or more generally a ring) and A and B are matrices of the appropriate sizes. The interpretation is that x gives the *state* of the system at a given time, and the u's are *controls* or *inputs*.

One of the simpler typical questions is: Do there exist controls u that steer the initial state $x = 0$ to any desired state (*reachability*). To answer this consider the so-called *reachability matrix*

$$R(A,B) = (B \vdots AB \vdots A^2B \vdots \cdots \vdots A^nB) \tag{1.1.3}$$

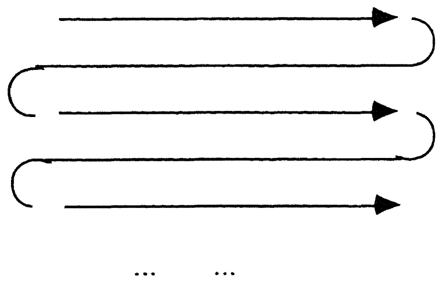
consisting of the $n \times m$ matrix B with next to it placed the $n \times m$ matrix AB and so on. One well known theorem says:

1.1.4. *Theorem.* The pair (A, B) is *completely reachable* (abbreviated '*cr*'), i.e. any state can be reached from the zero state by suitable controls, if and only if the reachability matrix $R(A, B)$ has the maximal rank it can have, viz n.

Now depict the $(n+1)m$ columns of $R(A, B)$ as indicated below

$$\begin{array}{ccccccc}
\circ & \circ & \circ & \circ & \circ & & B \\
\circ & \circ & \circ & \circ & \circ & & AB \\
\circ & \circ & \circ & \circ & \circ & & A^2B \\
 & \cdots & & & \cdots & & \\
\circ & \circ & \circ & \circ & \circ & & A^nB
\end{array}$$

So the first row depicts the columns of B, the next one the ones of AB, and so on. Now go through this collection of vectors row by row from left to right starting with the first row



and put a cross whenever a nonzero vector is found that is not in the space spanned by all previously encountered vectors. This will result in a pattern of crosses, like e.g. the following one

$$\begin{array}{ccccc}
\circ & \times & \circ & \times & \times \\
\circ & \times & \circ & \circ & \times \\
\circ & \circ & \circ & \circ & \times \\
\circ & \circ & \circ & \circ & \circ \\
\circ & \circ & \circ & \circ & \circ \\
\circ & \circ & \circ & \circ & \circ \\
\circ & \circ & \circ & \circ & \circ
\end{array} \tag{1.1.5}$$

(Here $m = 5$, $n = 6$.) It is a relatively easy lemma, the *nice selection lemma*, see loc. cit. that the patterns of crosses that so arise have the following *nice selection* property: "If a cross occurs, then above it there are only crosses" (as is the case in the example.) Thus the pattern is uniquely described by specifying the number of crosses in each column. This is the *Kronecker selection*, $\tilde{\kappa}(A, B)$, defined by the pair (A, B). In example (1.1.5) we have: $\tilde{\kappa}(A, B) = (0,2,0,1,3)$.

It is simple to see that all patterns that satisfy the nice selection property can indeed occur. For instance to obtain example (1.1.5) one can take

$$B=(0,e_1,0,e_2,e_3),\ \ A=(e_4,0,e_5,0,e_6,0)$$

where 0 is the zero vector in $\mathbf{R}^6$ and $e_1,\cdots,e_6$ are the six standard basis vectors. The *Kronecker indices* of the pair (A,B) are the Kronecker selection numbers in decreasing order, where final zeros are often omitted. Thus in the example

$$\kappa_1(A,B)=3,\ \ \kappa_2(A,B)=2,\ \ \kappa_3(A,B)=1$$

If the pair is completely reachable $\kappa(A,B)$ is a partition of n. The Kronecker indices derive their name from the fact that they turn up as invariants of the certain special pencils of matrices under left and right multiplication by invertible matrices. The problem of invariants of pencils of matrices under this equivalence relation was studied by Kronecker in the 19-th century.

1.2. Families of control systems and linear dynamical systems.

A (linear constant-time) input-output system, or linear dynamical system is a set of equations

$$\begin{array}{ll} \dot{x}=Ax+Bu & \quad x_{i+1}=Ax_i+Bu_i \\ y=Cx & \quad y_i=Cx_i \end{array} \tag{1.2.1}$$

i.e. it consists of a control system plus an observation, or output, equation with values in p-dimensional space.

Intuitively, or perhaps naively, a family of systems, parametrized by $s\in S$ would simply be a family of triples of matrices $(A(s),B(s),C(s))_{s\in S}$ depending continuously (algebraically, differentably, analytically, ...) on the parameter s.

This turns out to be not good enough, even to analyze the properties of systems that, in fact, can be represented in this way. The right global definition is as follows.

A family of linear dynamcial systems over a topological space S consists of a vectorbundle E over S together with three vectorbundle homomorphisms

$$\begin{array}{ccc} S\times\mathbf{R}^m & \xrightarrow{B} & E \\ \downarrow & & \downarrow \\ S & = & S \end{array},\quad \begin{array}{ccc} E & \xrightarrow{A} & E \\ \downarrow & & \downarrow \\ S & = & S \end{array},\quad \begin{array}{ccc} E & \xrightarrow{C} & S\times\mathbf{R}^p \\ \downarrow & & \downarrow \\ S & = & S \end{array}$$

Even over a point this is not quite the same as a triple of matrices (A,B,C). It is exactly the same as a triple of vectorspace homomorphisms

$$B:\mathbf{R}^m\to V,\ \ A:V\to V,\ \ C:V\to C$$

where V is an n-dimensional vectorspace. Choosing a basis in V and taking the canonical bases in $\mathbf{R}^m$ and $\mathbf{R}^p$ this becomes a triple of matrices.

Thus a system over a point really is a triple of matrices up to *state-space base change*, i.e. up to the equivalence relation (in the real case)

$$(A,B,C) \sim (SAS^{-1}, SB, CS^{-1}), \quad S \in GL_n(\mathbf{R})$$

(This is just right, because the "state space is hidden"; what one can see is inputs (or controls) and outputs (or observations)).

One interesting question immediately arises: in what ways can the Kronecker indices of a family of systems vary within a family. This matter will be settled in lecture 4.

1.3. Systems with delays.

Let α_1, α_2 be two (incommensurate) positive real numbers. As an example of a continuous time delay system take

$$\begin{aligned} \dot{x}_1(t) &= 2x_1(t) + x_2(t-\alpha_1) + x_2(t-2\alpha_2) + u(t) \\ \dot{x}_2(t) &= x_1(t-3\alpha_1-\alpha_2) - u(t-\alpha_2) \\ y(t) &= x_1(t) + 3x_2(t-\alpha_2) \end{aligned} \tag{1.3.1}$$

One might as well take the α_1, α_2 incommensurate because otherwise there is an α such that $\alpha_1 = a\alpha, \alpha_2 = b\alpha$ for some $a, b \in \{1, 2, \cdots\}$, and everything could be written in terms of one single delay.

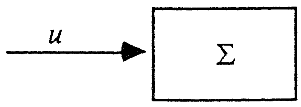
More generally consider delay systems involving r incommensurate delays $\alpha_1, \alpha_2, \cdots, \alpha_r$. Associated to a delay system like (1.3.1) there is a family of systems obtained by treating the α 's as (polynomial) parameters. This is even a family of systems in the naive sense. Equivalently, the delay system can be seen as a system over the polynomial ring $\mathbf{R}[\sigma_1, \sigma_2, \cdots, \sigma_r]$ where the σ 's act on functions of time as the operators $\sigma_i f(t) = f(t-\alpha_i)$. Thus the matrices involved in example (1.3.1) are

$$A(\sigma) = \begin{pmatrix} 2 & \sigma_1 + \sigma_2^2 \\ \sigma_1^3\sigma_2 & 0 \end{pmatrix}, \quad B(\sigma) = \begin{pmatrix} 1 \\ -\sigma_2 \end{pmatrix}, \quad C(\sigma) = (1, 3\sigma_2)$$

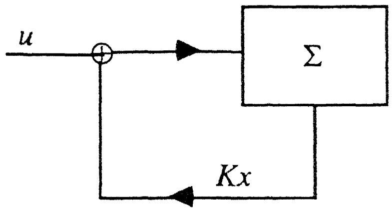
As far as I know there are no relations between solutions of a delay system like (1.3.1) and the associated family of systems. However, the study of the associated family as a family does give results and insights in the properties of the original delay system, as we shall see presently.

1.4. The feedback group.

Consider a control system $\dot{x} = Ax + Bu$ (or a discrete version)



A (linear) state feedback is a mapping $K : \mathbf{R}^n \to \mathbf{R}^m$. It changes the control system $\Sigma = (A, B)$ to a new control system $(A + BK, B)$



$$\dot{x} = (A + BK)x + Bu$$

i.e. a linear function of the state is fed back as input.

This is a standard technique when designing control systems. Think e.g. of the regulator on the (Watt) steam engine. Modern machines such as airplanes tend to have hundreds of such feedback loops.

A natural question is: To what extent can a system be changed by state space feedback, or, more or less equivalently, what properties of the system remain invariant under feedback.

One answer is as follows. Consider the following three kinds of transformations

$$\begin{aligned}&\text{Feedback: } (A,B) \mapsto (A+BK, B),\ K \in \mathbf{R}^{m\times n}\\&\text{State space base change: } (A,B) \mapsto (SAS^{-1}, SB),\ S \in GL_n(\mathbf{R})\\&\text{Input space base change: } (A,B) \mapsto (A, BT),\ T \in GL_m(\mathbf{R})\end{aligned} \tag{1.4.1}$$

Together these transformations generate the so-called *feedback group*.

1.4.2. *Theorem* (R E Kalman, P Brunovsky). The invariants of completely reachable pairs (A,B) under the feedback group are precisely the Kronecker indices.

1.5. THE BLOCK COMPANION CANONICAL FORM.

In order to sketch, in the next section, a theorem on stabilization and related matters for systems with delays we need one more somewhat technical result from linear control theory.

1.5.1. *Theorem.* Consider a family of control systems $(A(\sigma), B(\sigma))$ such that the Kronecker selection is constant as a function of σ. Then there exists a family of upper triangular matrices $T(\sigma)$ and an invertible family of matrices $S(\sigma)$ whose coefficients are polynomial in the coefficients of $(A(\sigma), B(\sigma))$ such that the input and state space transformed system $(S(\sigma)A(\sigma)S(\sigma)^{-1}, S(\sigma)B(\sigma)T(\sigma))$ is in block companion canonical form. In particular, if $A(\sigma)$ and $B(\sigma)$ are polynomial, the base change matrices $T(\sigma)$ and $S(\sigma)$ can be taken to be polynomial.

Just what "block companion canonical form" is, should be clear from the following example. Let the Kronecker selection in question be $\tilde{\kappa} = (2,3,0,1)$, then the corresponding block companion form looks like

$$A=\left(\begin{array}{cc|ccc|c} 0 & 1 & 0 & 0 & 0 & 0 \\ * & * & * & * & * & * \\ \hline 0 & 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 & 0 \\ * & * & * & * & * & * \\ \hline * & * & * & * & * & * \end{array}\right), \qquad B=\left(\begin{array}{cccc} 0 & 0 & 0 & 0 \\ 1 & 0 & 0 & 0 \\ \hline 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ \hline 0 & 0 & 0 & 1 \end{array}\right)$$

where the stars are elements that are not further specified.

1.5.2. *Remark.* As a rule it is not true that a family of systems with constant Kronecker indices can be brought into such a canonical form in a continuous manner.

See [5], section 9, for details about all this.

1.6. Pole placement and coefficient assignability.

A linear dynamical system $\dot{x}=Ax+Bu, y=Cx$ takes an input signal $u(t)$ into an output signal $y(t)$. This depends a bit on the initial state $x(0)$, but is essentially determined by what happens in the case the initial state is zero, $x(0)=0$, in which case it is given by

$$y(t)=\int_0^t Ce^{(t-\tau)A}Bu(\tau)d\tau \tag{1.6.1}$$

The Laplace transformed version of this is

$$Y(s)=T(s)U(s), \quad T(s)=C(sI_n-A)B \tag{1.6.2}$$

$T(s)$ is called the transfer functions, and its poles, i.e. the eigenvalues of A, are the danger spots; these represent frequencies for which the system is unstable. For stable input-output behaviour the poles need to be in the left half plane and a frequently used design criterium is that they be in the sector $\{z \in \mathbf{C} : \mathrm{Re}(z) \le 0, \mathrm{Im}(z) \le \mathrm{Re}(z)\}$. Historically, this is where the Boeing 707 got its name from: $\cos(45^\circ)=\frac{1}{2}\sqrt{2}=0.707\cdots$.

Thus there arise:

(i) The *pole placement problem*: given (A,B), can we place the poles as desired using feedback?

(ii) The *coefficient assignability problem*: given (A,B), is there a linear feedback K such that $A+BK$ has a desired characteristic polynomial?

1.6.3. *Theorem.* The pair (A,B) is coefficient assignable iff it is cr (i.e. iff the reachability matrix $R(A,B)$ has rank n).

1.6.4. *Theorem.* If (A,B) is cr, it is pole assignable and in particular stabilizable by feedback.

These theorems hold over any field. For discrete time systems the role of the Laplace transform is taken over by the z-transform.

The question now arises whether something similar can maybe be done for families of systems and delay systems.

1.6.5. *Theorem* (C.I.Byrnes). Let $\Sigma(\sigma)$ be a polynomial family of cr systems over a field K and suppose that the Kronecker indices $\kappa(\sigma)$ are constant as a function of σ. Then $\Sigma(\sigma)$ is coefficient assignable.

1.6.6. *Corollary*. Let Σ be a delay system such that the associated polynomial family is cr everywhere and has $\kappa(\sigma)$ constant. Then Σ is coefficient assignable as a delay system.

One proof of this theorem makes heavy use of the Quillen -Suslin theorem to the effect that a polynomial vector bundle over a field is trivial. A short outline of the structure of this proof follows.

Look at the subspace of $\overline{K}^n$ spanned by $B(\sigma), A(\sigma)B(\sigma), \cdots, A(\sigma)^{i-1}B(\sigma)$ for all σ and $i = 1, \cdots, n$. Let the dimension be $d_i(\sigma)$. Now the $d_i(\sigma)$ and the $\kappa_i(\sigma)$ are narrowly related. In detail, if

$$e_i = d_i - d_{i-1}, d_0 = 0$$

then

$$\kappa_1 = \#\{i : e_i \geq 1\}, \cdots, \kappa_m = \#\{i : e_i \geq m\}$$

and thus the d_i determine the κ_i and vice versa. Thus if the $\kappa_i(\sigma)$ are constant so are the $d_i(\sigma)$. So $\langle B(\sigma)\rangle$, the space generated by the columns of $B(\sigma)$, has constant rank and that means by the Quillen Suslin theorem that there is a polynomial invertible matrix $T_1(\sigma)$ such that the first $d_1 = e_1$ columns of $B(\sigma)T_1(\sigma)$ are linearly independent for all $\sigma \in \overline{K}^r$. Note that this corresponds to a polynomially dependent base change in input space. Now look at the space $\langle B(\sigma) \vdots A(\sigma)B(\sigma)\rangle$ of dimension d_2 and the quotient vector bundle $\langle B(\sigma) \vdots A(\sigma)B(\sigma)\rangle / \langle B(\sigma)\rangle$. This quotient bundle is also trivial. It follows that there is a further polynomial input base change, involving only the first d_1 vectors of $B(\sigma)$, such that the first e_2 vectors of $A(\sigma)B(\sigma)T_1(\sigma)T_2(\sigma)$ are linearly independent modulo $\langle B(\sigma)\rangle$. Thus after two base changes in input space, both polynomial, things have been arranged so that the Kronecker selection $\kappa(\sigma)$ of the family looks like

$$\begin{array}{l} \underbrace{\times \quad \times \quad \cdots \quad \times}_{e_1} \quad \circ \quad \circ \quad \cdots \quad \circ \\ \underbrace{\times \quad \cdots \quad \times}_{e_2} \quad \circ \quad \circ \quad \circ \quad \cdots \quad \circ \\ \qquad \cdots \qquad\qquad \cdots \end{array}$$

i.e. it is constant as regards the first two rows. Continuing in this way we can see to it that the whole Kronecker selection is constant after a suitable transformation of input space. Using the block canonical form of the previous section it is now a simple matter to find a feedback so that the new A matrix has precisely a desired characteristic polynomial.

Indeed, in the example above BK can be any matrix of the form

$$\begin{pmatrix} 0 & 0 & 0 & 0 & 0 & 0 \\ * & * & * & * & * & * \\ 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \\ * & * & * & * & * & * \\ * & * & * & * & * & * \end{pmatrix}$$

so that the second, fifth, and sixth row of the matrix A can be changed arbitrarily while the other rows remain as before. In particular A can be changed into anything of the form

$$\begin{pmatrix} 0 & 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 \\ * & * & * & * & * & * \end{pmatrix}$$

and for matrices of this form the coefficients of the characteristic polynomial are, up to sign, the entries of the bottom row.

1.7. THE FINE MODULI SPACE $\mathcal{M}^{co,cr}_{m,n,p}$.

Let's start with the dual concept to cr, which is co, *complete observability*. A pair of matrices (A,C) of sizes $n\times n$ and $p\times n$ is co if the observability matrix, $Q(A,C)$ has its maximal rank n, where

$$Q(A,C)=\begin{pmatrix} C \\ AC \\ A^2C \\ \vdots \\ A^nC \end{pmatrix}$$

1.7.1. *Theorem.* There exists a space $\mathcal{M}^{co,cr}_{m,n,p}$ and a continuous universal family Σ^u of systems over it that is co and cr everywhere, such that for any continuous family, Σ, of co and cr systems over a space X there is a unique continuous map $f: X \to \mathcal{M}^{co,cr}_{m,n,p}$ such that the pullback of Σ^u is Σ, $\Sigma = f^!\Sigma^u$.

1.7.2. *Remarks.* There are several analogous theorems. E.g. for $\mathcal{M}^{cr}_{m,n,p}$ and $\mathcal{M}^{co}_{m,n,p}$ and these, as well as $\mathcal{M}^{co,cr}_{m,n,p}$, are all smooth algebraic varieties. A space that should be denoted $\mathcal{M}_{m,n,p}$ probably exists in one sense or another but will be singular.

1.8. Realization theory.

Both in the continuous case and the discrete time case, the input-output, behaviours of a system is given by a transfer function, or, more precisely, by the matrices

$$H_i = CA^iB, \quad i = 0,1,2,\cdots \tag{1.8.1}$$

For instance in the continuous time case

$$T(s) = H_0 s^{-1} + H_1 s^{-2} + H_2 s^{-3} + \cdots$$

Thus the main question of realization theory arises. Given a sequence of $p \times m$ matrices $H_0, H_1, H_2, \cdots$, when do there exist matrices A, B, C (of sizes $n \times n$, $n \times m$, $p \times n$) such that (1.8.1) holds. Note than the integer n also needs to be determined. The answer is as follows

1.8.2. *Theorem* (R E Kalman). A sequence of $p \times m$ matrices $H_0, H_1, H_2, \cdots$ over a field K is realizable if and only if the block Hankel matrix

$$\mathcal{H} = \begin{pmatrix} H_0 & H_1 & H_2 & \cdots \\ H_1 & H_2 & H_3 & \cdots \\ H_2 & H_3 & H_4 & \cdots \\ \cdots & \cdots & \cdots & \cdots \end{pmatrix}$$

has finite rank n (and that n is the size of the matrix A of a minimal size realization).

There is an algorithm for carrying out this realization and it is continuous, even algebraic in the parameters of the sequence $H_0, H_1, H_2, \cdots$ as long as the rank of the block Hankel matrix remains constant, see [4, 5].

A system with delays gives rise to a matrix transfer function of the form

$$T(s, e^{-\alpha_1 s}, \cdots, e^{-\alpha_r s}) \tag{1.8.3}$$

that is rational in $s, e^{-\alpha_1 s}, \cdots, e^{-\alpha_r s}$. In turn this gives a family of matrices

$$H_0(\sigma_1, \cdots, \sigma_r), H_1(\sigma_1, \cdots, \sigma_r), \cdots$$

depending polynomially on $\sigma = (\sigma_1, \cdots, \sigma_r)$. Let

$$n(\sigma) = \mathrm{rk} \begin{pmatrix} H_0(\sigma) & H_1(\sigma) & \cdots \\ H_1(\sigma) & H_2(\sigma) & \cdots \\ \cdots & \cdots & \cdots \end{pmatrix}$$

1.8.4. *Theorem.* Let $n(\sigma)$ be constant $< \infty$ as a function of σ. Then the delay transfer function (1.8.3) is realizable.

The proof goes as follows. For each individual σ theorem 1.8.2 gives a realization. The algorithm is continuous in σ. Thus there results a continuous algebraic map

$$\varphi : \overline{K}^r \to \mathcal{M}_{m,n,p}^{co,cr}$$

Now pull back the universal family by φ. This gives a polynomial family over $\overline{K}^r$. By the Quillen-Suslin theorem the underlying state space vector bundle is trivial. So, choosing a trivialization, there is a polynomial family $(A(\sigma), B(\sigma), C(\sigma))$ that does the job. Now reintepret this family as a delay system.

1.8.5. *Remarks.* Using continuity and genericity arguments, or, alternatively, working over the quotient field $K(\sigma)$ first and then using that the obtained realization is generically polynomial (after a base change), it seems to me that one could show that the theorem remains true without the hypothesis that $n(\sigma)$ be constant (but retaining the necessary condition that $n(\sigma) < \infty$ (uniformly). I know of no explicit statement or proof of such a theorem in the published literature.

Lecture 2. On the K-theory of endomorphisms. Some applications of control.

This second lecture goes totally in the opposite direction of the first. The first was concerned with two applications of the Quillen-Suslin theorem to control. Now we will see some applications of the ideas and methods from the mathematics of control, particularly realization theory, to K-theory.

2.1. The functor $W_0(A)$.

Let A be a commutative ring with unit element. $\mathcal{E}nd(A)$ is the category of pairs (P, f), where P is a finitely generated projective module and f is an endomorphism of P. A morphism in $\mathcal{E}nd(A)$, $(P,f) \xrightarrow{\varphi} (Q,g)$, is a morphism of A-modules $P \xrightarrow{\varphi} Q$ such that $g\varphi = \varphi f$ There is an obvious notion of short exact sequence: a sequence in $\mathcal{E}nd(A)$ is a short exact sequence if and only if the underlying sequence of A-modules is a short exact sequence. Thus we can form the Grothendieck group $K_0(\mathcal{E}nd(A))$. Tensor product and direct sum turn $K_0(\mathcal{E}nd(A))$ into a commutative ring with unit element.

To a projective module P assign the pair $(P, 0)$. This defines an ideal in the ring $K_0(\mathcal{E}nd(A))$, which identifies with $K_0(A)$. The quotient

$$W_0(A) = K_0(\mathcal{E}nd(A)) / K_0(A) \tag{2.1.1}$$

is what we are interested in in this lecture. I like to call $W_0(A)$ the ring of rational Witt vectors over A (for reasons that will become clear in a moment).

2.2. THE (BIG) WITT VECTOR FUNCTOR $W(A)$.

For each commutative ring R with unit element let $W(R)$ be the set

$$W(R) = \{1 + r_1 t + r_2 t^2 + \cdots : r_i \in R\} = 1 + tR[[t]] \tag{2.2.1}$$

Multiplication of power series turns $W(R)$ into an Abelian group. It is also (functorially) a commutative ring with unit element and quite generally one of the most fascinating and important objects in mathematics. See [3] for a detailed treatment of the Witt vectors.

2.3. ALMKVIST'S HOMOMORPHISM.

Let $(P, f) \in \mathcal{End}(A)$ and let Q be a finitely generated projective module such that $P \oplus Q$ is free. Consider the endomorphism $f \oplus 0$ of $P \oplus Q$, and consider $\det(\mathrm{Id} + t(f \oplus 0))$. This is a polynomial in t that does not depend on the choice of Q.

It induces a homomorphism $K_0(\mathcal{End}(A)) \to W(A)$ that is obviously zero on $K_0(A)$ and hence induces a functorial homomorphism

$$c : K_0(\mathcal{End}(A)) / K_0(A) \to W(A) \tag{2.3.1}$$

2.3.2. *Theorem* (Almkvist, [1]). The homomorphism c of (2.3.1) is injective and its image consists precisely of those power series $1 + a_1 t + a_2 t^2 + \cdots$ that are rational in the sense that they can be written as a quotient

$$1 + a_1 t + a_2 t^2 + \cdots = \frac{1 + b_1 t + b_2 t^2 + \cdots}{1 + d_1 t + d_2 t^2 + \cdots}, \quad b_i, d_j \in A$$

2.4. REPRESENTABLE FUNCTORS.

A (covariant) functor $F : \mathcal{C} \to \mathcal{Set}$ is representable if there is a universal example of whatever it is that the functor deals with. This means that there is a pair $(X, u), u \in F(X)$ such that

$$\mathcal{C}(X, Y) \to F(Y), \ (X \xrightarrow{\varphi} Y) \mapsto F(\varphi)(u)$$

is a functorial isomorphism $\mathcal{C}(X, -) \to F(-)$. It is pretty straightforward to adapt this definition to contravariant representability.

Many of the more important objects in mathematics are (or can be) defined by representable functors.

The functor of the big Witt vectors $W : \mathcal{Ring} \to \mathcal{Ab}$ from commutative rings with unit element to Abelian groups is representable by the pair

$$\mathbf{Z}[X_1, X_2, \cdots], \ u = 1 + X_1 t + X_2 t^2 + \cdots \in W(\mathbf{Z}[X])$$

As it turns out the subfunctor $W_0 : \mathcal{Ring} \to \mathcal{Ab}$ of rational Witt vectors is not representable, but "nearly so"; see below.

We met something similar, to be precise a contravariant almost (in a different way) representable functor in lecture 1. There the functor

$$\Sigma(S) = \text{continuous maps } s \mapsto (A(s), B(s), C(s)) \text{ to co and cr systems}$$

was not representable. But a more general object "Families of co and cr systems with possibly nontrivial family of state spaces" was indeed representable (by the fine moduli space $\mathcal{M}^{co,cr}_{m,n,p}$ and the universal family of systems Σ^u over it).

2.5. OPERATIONS.

Given a functor $F : \mathcal{C} \to \mathcal{S}et$ the operations on F are the functorial endomorphisms of F, i.e., so to speak, the natural constructions (operations) one can carry out on elements of $F(C)$. E.g. if F is Abelian group valued, $a \mapsto a^2$ is an operation.

If F is representable the operations correspond naturally to the set of endomorphisms $\mathcal{C}(X, X)$ if (X, u) represents F.

In this lecture we are interested in determining all continuous operations of the functor $W_0 : \mathcal{R}ing \to \mathcal{A}b$ or the functor $W_0 : \mathcal{R}ing \to \mathcal{S}et$ where $W_0(A)$ is given the topology determined by the subgroups $(1 + t^n A[[t]]) \cap W_0(A) \subset W(A)$.

2.6. "REPRESENTING" THE FUNCTOR W_0. PART 1.

Consider the ring $\mathbf{Z}[X] = \mathbf{Z}[X_1, X_2, \cdots]$ of polynomials over the integers in countably many commuting indeterminates, and consider the Hankel matrix

$$\begin{pmatrix} 1 & X_1 & X_2 & X_3 & \cdots \\ X_1 & X_2 & X_3 & X_4 & \cdots \\ X_2 & X_3 & X_4 & X_5 & \cdots \\ X_3 & X_4 & X_5 & X_6 & \cdots \\ \cdots & \cdots & \cdots & \cdots & \cdots \end{pmatrix} \tag{2.6.1}$$

Let J_n be the ideal in $\mathbf{Z}[X]$ generated by all the $n \times n$ minors of the infinite Hankel matrix (2.6.1). Let $V_n = \mathbf{Z}[X]/J_n$ and let $\xi_n = \rho_{n*}(1 + X_1 t + X_2 t^2 + \cdots)$ where $\rho_n : \mathbf{Z}[X] \to V_n$ is the natural projection.

2.6.2. *Remarks and warning.*

(i) ξ_n is not an element of $W_0(V_n)$

(ii) J_n is a prime ideal

(iii) V_n is not Noetherian

(iv) Let $D_n = \rho_n(\tilde{D}_n)$ where $\tilde{D}_n$ is the top left $n \times n$ minor of (2.6.1). Then the localization $(V_n)_{D_n}$ is Noetherian and ξ_n is a rational Witt vector over $(V_n)_{D_n}$. It is, however, still not true that ξ_n over $(V_n)_{D_n}$ is universal for rational Witt vectors of numerator degree

$\leq (n-1)$ and denominator degree $\leq n$.

However, the collective $(V_n, \xi_n)_n$ do, in some sense, represent W_0, see below. To formulate this precisely we need to discuss the Fatou property.

2.7. The Fatou property.

Let R be an integral domain. Then R is said to be *Fatou* if for every power series $a(s) = 1 + a_1 s + a_2 s^2 + \cdots$, $a_i \in R$ for which there exist polynomials $p(s), q(s) \in Q(R)[s]$ such that

$$a(s) = \frac{p(s)}{q(s)} \tag{2.7.1}$$

there also exist polynomials $p'(s), q'(s) \in R[s]$ such that $a(s) = p'(s)/q'(s)$. Here $Q(R)$ is the quotient field of the integral domain R.

2.7.2. *Theorem* (Sontag). Every (commutative) Noetherian integral domain is Fatou.

2.7.3. Open question. Does there exist a Fatourization construction? I.e. does there exist for all commutative integral domains R a Fatou integral domain R_F together with a homomorphism $R \to R_F$ such that every ring homomorphism $R \to S$ with S Fatou factorizes through $R \to R_F$. Guess: probably yes.

2.8. Representing the functor W_0. Part 2.

2.8.1. *Theorem.* For each $a = 1 + a_1 t + a_2 t^2 + \cdots \in W_0(A)$ let $\varphi_a : \mathbf{Z}[X] \to A$ be defined by $X_i \mapsto a_i$. Then $a \mapsto \varphi_a$ is injective and functorial $W_0(A) \to \mathcal{Ring}(\mathbf{Z}[X], A) = W(A)$ and φ_a is continuous with respect to the J-topology on $\mathbf{Z}[X]$ and the discrete topology on A. If A is Fatou, $a \mapsto \varphi_a$ is a bijection between $W_0(A)$ and the continuous ring homomorphisms $\mathbf{Z}[X] \to A$.

Here the J-topology on $\mathbf{Z}[X]$ is that defined by the ideals J_n.

So, in particular, if A is a Noetherian integral domain, then $W_0(A)$ correponds bijectively to the continuous ring homomorphisms $\mathbf{Z}[X] \to A$.

Note that $\psi : \mathbf{Z}[X] \to A$ is continuous if and only if ψ factors through some V_n. It is in this sense that the V_n collectively represent W_0.

2.9. Operations on W_0

2.9.1. *Theorem.* The continuous operations on W_0 correspond precisely to the continuous endomorphisms of $\mathbf{Z}[X]$ for the J-topology (on both source and target).

The only ring structure preserving operations are the Frobenius operations (which are the same as the Adams operations in this case).

2.10. Some comments.

For details see [6].

It may not be obvious in what sense this bit of mathematics relates to control theory and indeed is more or less an application of ideas from control. To this end let me remark that the consideration of the Hankel matrix (2.6.1) and the ideals J_n really come from realization theory (Lecture 1, part 1.8) and that theorem 2.7.2 first appeared in discrete control theory.

Another object of considerable interest is $K_0(\mathrm{End}(P))$, where P is a fixed projective finitely generated A-module, see [9]. The results described above certainly have implications and applications in this setting, but that has not yet been worked out.

Lecture 3. Hilbert 90 with parameters and polynomial rigidity of representations.

3.1. The Hilbert 90 with parameters theorem.

Let K/k be a Galois extension of fields with Galois group $\mathrm{Gal}(K/k) = \Gamma$. Consider the group $GL_n(K[z_1,\cdots,z_m])$ of invertible $n \times n$ polynomial matrices with coefficients in K. The group $\mathrm{Gal}(K/k) = \Gamma$ acts on $GL_n(K[z])$ by acting on the coefficients.

3.1.1. *Theorem.* The cohomology set

$$H^1(\mathrm{Gal}(K/k), GL_n(K[z_1,\cdots,z_m]) = \{e\}$$

i.e. it is trivial.

For a definition of the cohomology sets with distinguished element, $H^1(\Gamma,A)$, see below.

If $n=1, m=0$ theorem 3.1.1 reduces to the 'classical Hilbert 90 theorem' to the effect that $H^1(Gal(K/k), K^*) = \{1\}$, which in turn is a dual cohomological generalization of the Hilbert 90 theorem for cyclic extensions as originally due to Hilbert and to be found in his "Die Theorie der algebraische Zahlkörper".

3.1.2. *Application.* Take $k = \mathbf{R}, K = \mathbf{C}$ to find that if A is a polynomial matrix over the complex numbers such that $A\bar{A} = I_n$, then there exists a polynomial matrix B over $\mathbf{C}$ such that $A = \bar{B}B^{-1}$.

3.2. The cohomology set $H^1(\Gamma, A)$.

Let A be a group, not necessarily commutative, and let Γ be a group acting on A. Usually Γ is finite or profinite and in the latter case the action is usually taken to be continuous where A is given the discrete topology.

The word *action* means that there is a mapping $\Gamma \times A \to A$, $(\gamma, a) \mapsto \gamma(a) \in A$, such that $\gamma(ab) = \gamma(a)\gamma(b)$, $\gamma(e_A) = e_A$, $a, b \in A$, e_A the identity element of A, (i.e. $a \mapsto \gamma(a)$ is a group automorphism of A), and such that $e_\Gamma(a) = a$, $\gamma(\gamma'(a)) = (\gamma\gamma')(a)$, where e_Γ is the identity element of Γ. These latter two conditions are those that make the mapping an action.

If A is commutative, specifying a Γ action on A is the same as giving A the structure of a $\mathbf{Z}[\Gamma]$ module.

A 1-cocycle (of Γ in A) is a map $\Gamma \to A$, $s \mapsto a_s$ such that

$$t(a_s) = a_t^{-1} a_{ts} \tag{3.2.1}$$

There is a very special 1-cocyle, the trivial 1-cocycle defined by $s \mapsto e_A$ for all s. Two 1-cocycles $s \mapsto a_s$, $s \mapsto a_s'$ are equivalent if and only if there is a $b \in A$ such that

$$a_s' = b^{-1} a_s s(b) \tag{3.2.2}$$

The set of equivalence classes is denoted $H^1(\Gamma, A)$. There is a distinguised element in $H^1(\Gamma, A)$, viz. the equivalence class of the trivial 1-cocycle. As a rule this is the only natural structure on $H^1(\Gamma, A)$: it is a set with distinguished element, nothing more.

3.3. The motivating problem. Polynomial rigidity of representations.

The problem here is the following. Let G be a compact group, for instance a finite group. Let

$$\rho(z) : G \to GL_n(k[z_1, \cdots z_m]) \tag{3.3.1}$$

be a homomorphism; i.e. we have a polynomial family of representations. Here k is a field. Is it true that $\rho(z) \sim \rho(0)$ over $k[z] = k[z_1, \cdots, z_m]$, i.e. is there an $S(z) \in GL_n(k[z])$ such that

$$\forall g \in G \quad \rho(z)(g) = S(z)\rho(0)(g)S(z)^{-1} \tag{3.3.2}$$

Such a polynomial family of representations will be called trivial.

3.3.3. *Theorem* (Fagnani-De Concini, [2]). Let k be a splitting field for a finite group G. Then there are no nontrivial polynomial families of representations.

A field k is *splitting* for a finite group G if the group algebra $k[G]$ is isomorphic to a direct sum of matrix algebras $k[G] \simeq \bigoplus_{i=1}^{r} M_{n_i}(k)$, where $M_{n_i}(k)$ is the algebra of $n_i \times n_i$ matrices over k. This, for instance, is always the case if $k = \mathbf{C}$, and more generally if k is algebraically closed of a characteristic that does not divide the order of the group G.

A sketch of a proof of theorem 3.3.3 will be given below in section 3.5.

Also in the case of a compact group and $k = \mathbf{C}$ there are no nontrivial families of representations.

Thus for algebraically closed fields of characteristic zero the matter is settled. What about general fields of characteristic zero. There is a standard technique for dealing with such matters called the philosophy (or technique) of forms. This is much related to the more general technique of 'descent theory' in algebraic geometry.

For the problem at hand the philosophy of forms gave nothing, but it did provide the motivation for studying 'Hilbert 90 with parameters'. There is, of course, also independent interest in this result.

3.4. THE PHILOSOPHY OF FORMS.

Let K be a field extension of a field k. Let T be an (algebraic) object over k. Another object T' over k (of the same type), is called a form of T (more precisely a K/k-form of T) if $T \otimes_k K \simeq T' \otimes_k K$ over K.

The next bit of this theory is that there is a natural map

$$\vartheta : \{\text{Forms of } T \text{ up to isomorphism}\} \to H^1(\mathrm{Gal}(K/k), \mathrm{Aut}(T \otimes_k K))$$

In quite a number of (good) cases this map is an isomorphism or at least injective. See [11, 13] for more details on this and some results like this.

This provided the motivation for looking at $H^1(Gal(K/k), GL_n(k[z_1, \cdots z_n]))$, because, as is easily seen by the Schur Lemma, using the fact that polynomial rigidity holds over characteristic zero algebraically closed fields, $\mathrm{Aut}(\rho \otimes_k K)$ is a direct product of $GL_n(K[z_1, \cdots, z_m])$.

3.5. POLYNOMIAL RIGIDITY OF REPRESENTATIONS OF FINITE GROUPS.

As it turned out the form philosophy gave nothing concerning polynomial rigidity of representations. The situation turned out to be quite simple (in characteristic zero). In that case $k[G]$ is semisimple and hence a direct sum of matrix algebras over division algebras over k:

$$k[G] \simeq \bigoplus_i M_{n_i}(D_i).$$

If no noncommutative D_i turn up the answer is yes, nontrivial families do not exist; otherwise it is no, see [2]. Here is a restatement of one of the principal results (not the most general) and an outline of the proof (following [2]).

3.5.1. *Theorem.* Let k be a splitting field for a finite group and let M (or ρ) be a representation of the group G over $k[z_1, \cdots z_m] = R$. Then there exist a vectorspace W and a representation ρ_0 of G in W such that $M \simeq W \otimes_k R$.

Sketch of the proof as given in [2]. As k is splitting, $k[G] = \bigoplus_i A_i$, $\varphi_i : A_i \xrightarrow{\sim} M_{n_i}(k)$, where the $M_{n_i}(k)$ are matrix algebras. Let e_{st} be the (s,t) elementary matrix and let

$\varphi_i(a_{st}^{(i)}) = e_{st}$. Then one has the orthogonality relations

$$\sum_{i=1}^{n}\sum_{s=1}^{n_i} a_{ss}^{(i)} = 1, \quad a_{su}^{(i)}a_{vt}^{(j)} = \delta_{ij}\delta_{uv}a_{st}^{(i)} \tag{3.5.2}$$

Let $M^{i,s}$ be the submodule $a_{ss}^{(i)}M$ of M. Then

$$M = \bigoplus_{i,s} M^{i,s}$$

and by the Quillen-Suslin theorem each of the $M^{i,s}$ is a free $R = k[z]$ module.

For each i take an R-basis $m_1^{(i)},\cdots,m_{r_i}^{(i)}$ for $M^{i,1}$. Then the elements

$$a_{s1}^{(i)}m_k^{(i)}, \quad i = 1,\cdots,n; s = 1,\cdots,n_i; k = 1,\cdots,r_i \tag{3.5.3}$$

Form a basis for M because $M^{i,1} \simeq M^{i,s}$ the isomorphism being given by multiplication with $a_{s1}^{(i)}$.

Let W be the subspace over k spanned by the elements (3.5.3). Let $g \in G \subset k[G]$

$$g = \sum \gamma_{\mu\nu}^{(j)}a_{\mu\nu}^{(j)}, \quad \gamma_{\mu\nu}^{(j)} \in k$$

Then, using the relations (3.5.2)

$$\begin{aligned}\rho_g a_{s1}^{(i)}m_k^{(i)} &= \sum \gamma_{\mu\nu}^{(j)}a_{\mu\nu}^{(j)}a_{s1}^{(i)}m_k^{(i)} = \sum \delta_{ij}\delta_{\nu s}\gamma_{\mu\nu}^{(j)}a_{\mu 1}^{(i)}m_k^{(i)} \\ &= \sum \gamma_{\mu\nu}^{(i)}a_{\mu 1}^{(i)}m_k^{(i)} \in W\end{aligned}$$

3.6. Hilbert 90 with parameters.

3.6.1. *Theorem.* Let K/k be a Galois extension with Galois group Γ. Then

$$H^1(\Gamma, GL_n(k[z_1,\cdots z_m])) = \{e\}.$$

Outline of the proof. It suffices to prove this for finite Γ. Let $s \mapsto A_s$ be a 1-cocycle. For each $v \in K[z]^n$, let

$$b = \sum_{s\in\Gamma} A_s s(v)$$

For each particular set of values $\lambda \in \bar{k}^n$ of the z's, where $\bar{k}$ is an algebraic closure of k, there is a finite set of v's such that the corresponding b's span $\bar{k}^n$. See [10], p. 159. Because $K[z]$ is Noetherian there exists a (larger) finite set of polynomial vectors $v_1,\cdots,v_r \in K[z]^n$ such that the corresponding b's span $\bar{k}^n$ for every value $\lambda \in \bar{k}^n$ of the z's. Let B be the polynomial matrix formed by the columns $b_1,\cdots,b_r \in K[z]^n$. The rows of B span a vectorbundle over $\mathrm{Spec}(\bar{k}[z])$. The cocyle condition is now used to show that this vectorbundle is defined over k. By the Quillen-Suslin theorem the bundle is trivial over k. This means that there is a unimodular matrix $U \in GL_r(k[z])$ such that the first n columns of B form a unimodular $n \times n$ matrix B'. A straightforward calculation now shows that $t(B') = A_t^{-1}B'$ for all $t \in \Gamma$, which is what is needed. See [8] for details.

3.6.2. *Remarks.* The proof sketched above is basically a polynomial version of the proof of $H^1(\Gamma, GL_n(K)) = \{e\}$ as given in [10]. In [13] there is a completely different proof of $H^1(\Gamma, GL_n(K)) = \{e\}$, based on descent theory. This proof can also be adapted to prove Hilbert 90 with parameters and this proof also uses the Quillen-Suslin theorem essentially.

Here is a sketch of the argument. The first step is the form philosophy in a somewhat different form. Let N be an "algebraic structure" over a ring R and let S/R be a faithfully flat ring extension. Then, [13], p. 136, the R-isomorphisms classes of S/R-forms of N correspond bijectively to $H^1(S/R, \mathrm{Aut}(N))$ where this H^1 is a special cohomology group defined by descent data and $\mathrm{Aut}(N)$ is the automorphism scheme of the structure N.

Further if G is a group scheme that preserves products, $G(A \times B) = G(A) \times G(B)$, and S/R is Galois with Galois group Γ the descent cohomology set $H^1(S/R, G)$ can be rewritten as the Galois cohomology set $H^1(\Gamma, G(S))$, [13], p. 137.

Now apply this to the situation $R = k[z]$, $S = K[z]$ and N the free R-module of rank n over R. By Quillen-Suslin there are no nontrivial forms of N over S, and Hilbert 90 with parameters follows.

Lecture 4. Vectorbundles over the Riemann sphere, representations of the symmetric groups, Kronecker indices of systems, etc., etc.

4.1. THE SPECIALIZATION ORDERING.

There is a certain ordering on partitions of n, that has a habit of cropping up in many quite different parts of mathematics. It has many names: two of them are the *specialization ordering* and the *majorization ordering*. Others are mixing ordering, Snapper ordering; there are quite a few more names.

Let $\kappa = (\kappa_1 \geq \kappa_2 \geq \cdots \geq \kappa_m \geq 0)$ and $\kappa' = (\kappa'_1 \geq \kappa'_2 \geq \cdots \geq \kappa'_m \geq 0)$ be two partitions of n; i.e. $\sum \kappa_i = \sum \kappa'_i = n$.

The specialization ordering on partitions is defined by

$$\kappa \succ \kappa' \quad \Leftrightarrow \quad \kappa_1 \leq \kappa'_1, \kappa_1 + \kappa_2 \leq \kappa'_1 + \kappa'_2, \cdots, \kappa_1 + \cdots + \kappa_m \leq \kappa'_1 + \cdots + \kappa'_m \tag{4.1.1}$$

This ordering turns up with a rather surprising frequency in a lot of different parts of mathematics (and statistics, and chemistry, and physics, and ...). By no means all will be discussed here. For references to many others see [7].

Let's start with a K-theoretic manifestation.

4.2. THE SHATZ THEOREM.

According to a theorem of Grothendieck an algebraic or holomorphic vectorbundle over the Riemann sphere, $\mathbf{P}^1(\mathbf{C})$, splits as a direct sum of line bundles:

$$E \simeq L(\kappa_1) \oplus \cdots \oplus L(\kappa_m), \quad L(i) = L(1)^{\otimes i}$$

where $L(1)$ is the canonical ample line bundle over $\mathbf{P}^1(\mathbf{C})$. Call E positive if $\kappa_i(E) \geq 0$ for all i. Up to isomorphism we can order the line bundle summands so that

$$\kappa_1(E) \geq \kappa_2(E) \geq \cdots \geq \kappa_m(E)$$

and let $\kappa(E) = (\kappa_1(E) \geq \kappa_2(E) \geq \cdots \geq \kappa_m(E))$

4.2.1. *Theorem* (Shatz). Let E_t be a holomorphic family of positive vectorbundles over the Riemann sphere $\mathbf{P}^1(\mathbf{C})$, then for small enough t, $\kappa(E_t) \succ \kappa(E_0)$. And, inversely, if $\kappa \succ \kappa'$ there exist families with $\kappa(E_t) = \kappa$ for $t \neq 0$ and $\kappa(E_0) = \kappa'$.

4.3. THE SNAPPER, LIEBLER-VITALE, LAM, YOUNG THEOREM.

For a partition κ of n, let $S_\kappa = S_{\kappa_1} \times \cdots \times S_{\kappa_m} \subset S_n$ be the corresponding Young subgroup. Let

$$\rho(\kappa) = \mathrm{Ind}_{S_\kappa}^{S_n}(1)$$

where 1 is the trivial representation, and let $[\kappa]$ be the irreducible representation defined by the partition κ.

4.3.1. *Theorem.* $[\kappa]$ occurs in $\rho(\kappa')$ if and only if $\kappa < \kappa'$.

The result can be found hidden somewhere in the works of Young. At one time it was known as the Snapper conjecture. Independent proofs were given of the Snapper conjecture by Lam and Liebler-Vitale.

4.4. DEGENERATION OF SYSTEMS.

Let $L_{m,n} = \{(A,B) \in M^{n\times n}(k) \times M^{n\times m}(k) : (A,B) \text{ is cr}\}$. As we saw in lecture 1, the only invariants of $L_{m,n}$ under the feedback group are the Kronecker indices, i.e. partitions of n. Let

$$\mathcal{O}_\kappa = \{(A,B) : \kappa(A,B) = \kappa\} \qquad (4.4.1)$$

4.4.2. *Theorem* (Byrnes, Hazewinkel, Kalman, Martin).

$$\overline{\mathcal{O}_\kappa} \supset \mathcal{O}_{\kappa'} \iff \kappa \succ \kappa'$$

4.5. GERSTENHABER-HESSELINK THEOREM.

Let $\mathcal{N}_n$ be the space of all $n \times n$ complex nilpotent matrices. Let $GL_n(\mathbf{C})$ act on $\mathcal{N}_n$ by similarity: $A^S = S^{-1}AS$. The orbits are classified by partitions of n (given by the sizes of the Jordan blocks of the Jordan canonical form of a nilpotent matrix). Let Θ_κ be the orbit labelled by κ.

4.5.1. *Theorem.*

$$\overline{\Theta}_\kappa \supset \Theta_{\kappa'} \iff \kappa \prec \kappa'$$

4.6. RUCH-SCHÖNHOFER THEOREM.

Let μ^* be the dual partition of μ, i.e.

$$\mu_i^* = \#\{j : \mu_j \geq i\}. \tag{4.6.1}$$

4.6.2. *Theorem.* $\langle \rho(\kappa), \overline{\rho}(\mu) \rangle = 1$ if and only if $\kappa \succ \mu^*$

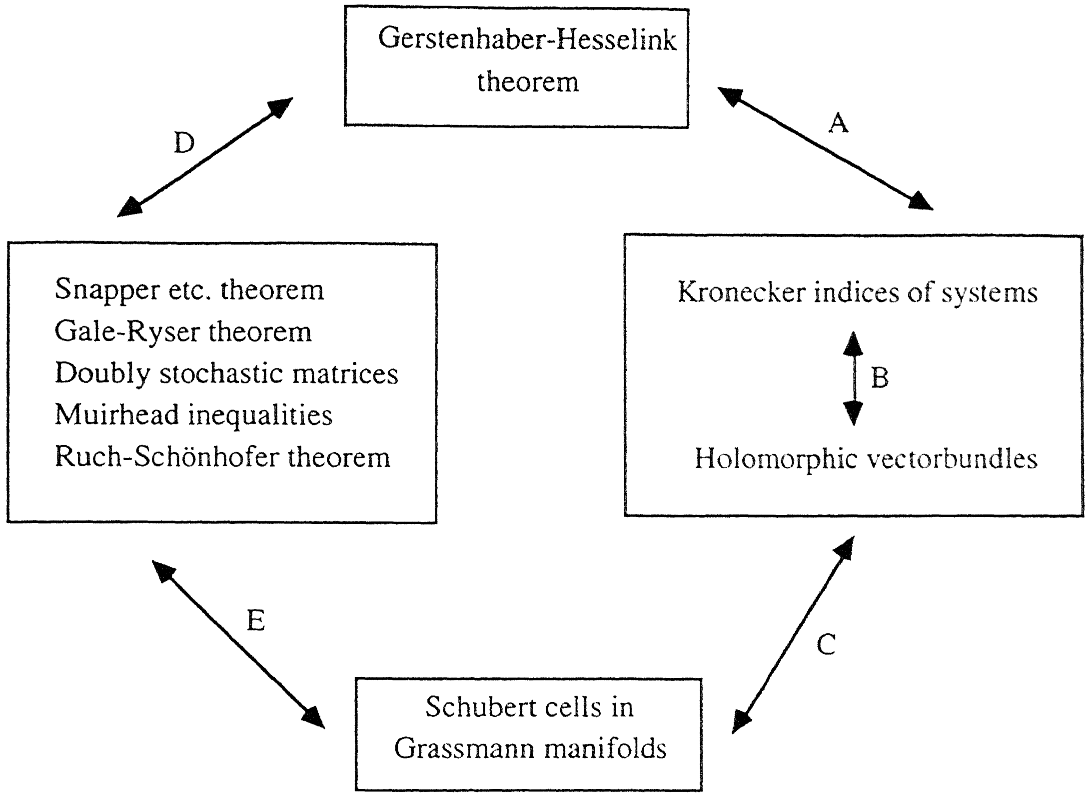
Here $\overline{\rho}(\mu) = \mathrm{Ind}_{S_\mu}^{S_n}(\mathrm{alt})$, alt is the sign representation, and $\langle\ ,\ \rangle$ is the standard inner product on representations that counts how many irreducible representations the two representations involved have in common (with multiplicity).

4.7. OTHER MANIFESTATIONS.

Some other manifestations of the specialization order involve the Muirhead inequalities, the Gale-Ryser theorem on (0,1)-matrices, doubly stochastic matrices, the Ruch theorem on increasing mixing ordering (which is an infinite dimensional generalization of the law of increasing entropy).

4.8. DIAGRAM OF INTERRELATIONS.

All the named manifestations of the specialization ordering are narrowly related. A schematic overview of some of the interrelations is below.



That the first four topics in the leftmost box above are narrowly related is well known. The Ruch-Schönhofer theorem, which came out of theoretical chemistry and was originally overlooked by the mathematicians, also belongs in this box. See [12] for how to connect integral matrices and representations.

Below I shall try to outline the basic ideas and constructions of the various interrelations A-E. For details (also on the interrelations inside the left-most box), see [7].

4.9. INTERRELATION B. THE HERMANN-MARTIN VECTORBUNDLE OF A CONTROL SYSTEM.

Let $Gr_n(\mathbf{C}^{n+m})$ be the Grassmann manifold of n-dimensional subspaces of the $(n+m)$-dimensional complex space $\mathbf{C}^{n+m}$. Define an m-dimensional holomorphic vectorbundle over $Gr_n(\mathbf{C}^{n+m})$ by

$$\xi_m(x) = \mathbf{C}^{n+m}/x \tag{4.9.1}$$

where x stands for both a point of $Gr_n(\mathbf{C}^{n+m})$ and the n-dimensional subspace of $\mathbf{C}^{n+m}$ that is that point (that that point represents).

Now let $\Sigma = (A, B)$ be cr control system. Then

$$\begin{aligned} s &\mapsto [sI_n - A \,\vdots\, B] \\ \infty &\mapsto [I_n \,\vdots\, 0] \end{aligned} \tag{4.9.2}$$

where for a complex $n\times(n+m)$ matrix M the symbol $[M]$ stands for the subspace of $\mathbf{C}^{n+m}$ that is spanned by the rows of the matrix M, defines a holomorphic map

$$\varphi_\Sigma : \mathbf{P}^1(\mathbf{C}) \to Gr_n(\mathbf{C}^{n+m})$$

(The property 'cr' is equivalent to $(\operatorname{rank}(sI_n - A\vdots B) = n$ for all s).)

The so-called *Hermann-Martin vectorbundle* of a control system is now defined as the pullback

$$E(\Sigma) = \varphi_\Sigma^!(\xi_m) \tag{4.9.3}$$

4.9.4. *Theorem.* The Grothendieck numbers of the vectorbundle $E(\Sigma)$ associated to the control system Σ (see 4.2 above) are the same as the contol indices (= Kronecker indices) (see lecture 1 , section 1.1) of the control system Σ.

It follows immediately that the Shatz theorem on families of holomorphic vectorbundles and the degeneration of systems theorem 4.4.1 are really two aspects of the same thing.

4.9.5. *Remark.* The bundle ξ_m is the algebraic geometer's universal vectorbundle; (algebraic topologists more often work with $\eta_n(x) = x$; the reason that algebraic geometers prefer ξ_m is that this bundle has $n+m$ natural holomorphic sections, viz $x \mapsto e_i/x$, $i = 1,\cdots,n+m$, while η_n has no nonzero holomorphic sections.

4.9.6. *Remark.* The definition of the Hermann-Martin vectorbundle given above differs from the original one. The difference amounts precisely to working with the algebraic geometer's vectorbundle instead of the topologist's one. This way theorem 4.9.4 comes out easier; otherwise there would have been an additional duality involved. Also this version fits better with Schubert cells and such, see below.

4.10. On connections D and E.

The best connection here is probably one due to H-P Kraft. Consider the variety $\mathcal{N}_\kappa$ of nilpotent matrices whose Jordan canonical form blocks are given by the partition κ of n. Take its closure $\overline{\mathcal{N}_\kappa}$ and consider also the subvariety C of all diagonal matrices inside the variety of all $n\times n$ matrices. Now take the scheme-theoretic intersection $\overline{\mathcal{N}_\kappa} \cap C$. As a variety this is just one point, the zero matrix; but it carries a nontrivial nilpotent structure sheaf. There is a natural action of the symmetric group on this intersection and the corresponding representation is $\rho(\kappa)$. Moreover the structure ring involved is graded and the top graded part is $[\kappa]$.

The way to link Snapper Liebler-Vitale Lam Young theorem on representations of the symmetric group and Schubert cells is via the construction of a family of representations parametrized by $Gr_n(\mathbf{C}^{n+m})$, see section 11 of [7]. This is a very rich family of representations, rich enough to contain all the degeneration examples needed to get the Snapper, Liebler-Vitale, Lam, Young theorem. Just how rich it is still needs to be sorted out. The way representations change in this family depends on the Schubert cells in $Gr_n(\mathbf{C}^{n+m})$, which is the next topic to discuss. This is connection E.

4.11. SCHUBERT CELLS IN GRASSMANN MANIFOLDS.

Consider again the Grassmann manifold $Gr_n(\mathbf{C}^{n+m})$ of complex n-planes in $(n+m)$-space. Let $\mathcal{A} = (A_1, \cdots, A_n)$ be a sequence of n subspaces of $\mathbf{C}^{n+m}$, $0 \neq A_1 \subset A_2 \subset \cdots \subset A_n$, with each inclusion strict. To each such subsequence $\mathcal{A}$ associate the closed subset

$$SC(\mathcal{A}) = \{x \in Gr_n(\mathbf{C}^{n+m}) : \dim(x \cap A_i \geq i\} \tag{4.11.1}$$

In particular if

$$\gamma = (0 < \gamma_1 < \cdots < \gamma_n \leq n+m)$$

is a stricly increasing sequence of natural numbers less than or equal to $n+m$, then we define

$$SC(\gamma) = SC(\mathbf{C}^{\gamma_1}, \cdots, \mathbf{C}^{\gamma_n})$$

where $\mathbf{C}^r$ is viewed as the subspace of all vectors in $\mathbf{C}^{n+m}$ whose last $n+m-r$ coordinates are zero.

4.12. SCHUBERT CELLS AND CONTROL INDICES. ON CONNECTION C.

Let $\Sigma = (A, B)$ be a control system, $\varphi_\Sigma : \mathbf{P}^1(\mathbf{C}) \to Gr_n(\mathbf{C}^{n+m})$ the corresponding holomorphic map that defines the Hermann-Martin vectorbundle, as in section 4.8 above.

The control indices of the system and the Schubert cells in the Grassmann manifold are narrowly related through the map φ_Σ.

To formulate the theorems which embody this we need the following definition. Let κ be a partition of n. Associate to it a sequence of n integers $\tau(\kappa)$ as follows:

$$\tau(\kappa) = 2, 3, \cdots, \kappa_1 + 1; \kappa_1 + 3, \cdots, \kappa_1 + \kappa_2 + 2; \; \cdots \; ; \kappa_1 + \cdots \kappa_{m-1} + m + 1, \cdots, \kappa_1 + \cdots + \kappa_m + m$$

4.12.1. *Theorem.* Let Σ be a cr contol system and φ_Σ the corresponding map as above. Then there is a Schubert cell $SC(\mathcal{A})$, $\mathcal{A} = (A_1, \cdots, A_n)$ such that $\mathrm{Im}(\varphi_\Sigma) \subset SC(\mathcal{A})$ and $\dim(A_i) = \tau_i(\kappa(\Sigma))$.

And, in a precise sense, the Schubert cell of theorem 4.12.1 is the smallest one possible for this.

4.12.2. *Theorem.* Let $\mathcal{A} = (A_1, \cdots, A_n)$ be such that $\mathrm{Im}(\varphi_\Sigma) \subset SC(\mathcal{A})$, and let $p(i) = j$ if and only if $\kappa_1(\Sigma) + \cdots + \kappa_j(\Sigma) < i \leq \kappa_1(\Sigma) + \cdots \kappa_{j+1}(\Sigma)$. Then $\dim(A_i) \geq i + p(i) = \tau_i(\kappa(\Sigma))$.

The Schubert cell closure relations fit with the majorization ordering as follows:

$$SC(\mu) \subset \overline{SC}(\tau) \iff \mu_i \leq \tau_i, \; i = 1, \cdots, n$$

4.13. On connection A.

This connection first takes the form of a completely dual proof of the Gerstenhaber-Hesselink theorem on the one hand and the degeneration of systems theorem on the other hand. Once the arguments proving these two theorems are in place one can set up an order reversing correspondence as follows.

For a nilpotent matrix N let

$$s(N) = \{(A,B) : (A,B) \text{ is cr, and } N^i A^{i-1} B = 0,\ i = 1,\cdots,n\}$$

and for a cr control system (A,B) let

$$t(A,B) = \{N : N^i A^{i-1} B = 0,\ i = 1,\cdots,n\}$$

Then, using the arguments that go into the two proofs alluded to, one shows that

$$ts(\overline{\mathcal{C}}(\kappa)) = \overline{\mathcal{C}}(\kappa),\quad st(\overline{\mathcal{U}}(\kappa)) = \overline{\mathcal{U}}(\kappa)$$

where $\overline{\mathcal{C}}(\kappa)$ is the closure of the similarity orbit of nilpotent matrices with indices κ and $\overline{\mathcal{U}}(\kappa)$ is the closure of the feedback orbit of control systems with indices κ. Thus t and s set up a bijective order reversing correspondence between closures of orbits in the two cases.

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Appendix. Selected references on applied K-theory in general

The list below pertains to a number of papers on (roughly) applied K-theory; better: interrelations between K-theory and other parts of mathematics. The relations between number theory and algebraic K-theory and algebraic geometry and algebraic K-theory are well known. These have largely been omitted below. The same holds for K-theory and EXT (as in the theory of C*-algebras).

There has been no attempt at completeness; these are simply some of the papers of this type that I know about.

A series of lectures based on this material was given in March 1999, at the Abdus Salam ICTP in Trieste, Italy.

The lectures were planned as follows:

- Lecture 1. An application of the Quillen-Suslin theorem to delay control systems.
- Lecture 2. The K-theory of endomorphisms and control theory.
- Lecture 3. Control systems with symmetry and Hilbert 90 with parameters.
- Lecture 4. Tilings and K_0.

- Reserve lecture 1. $K(\mathbf{PC}^1)$, the majorization ordering, and representations of the symmetric groups.
- Reserve lecture 2. Λ-operations, K-theory, Witt vectors, etc.
- Reserve lecture 3. The stable rank of H^∞ and applications.
- Reserve lecture 4. Polylogarithms.

For the references that are below that have to do with one of these eight topics this is so indicated. Not all references below were used for these lectures.

As it turned out the actual lectures given were Lectures 1-3 and Reserve Lecture 1.

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