Parametrization problems for spaces of linear dynamical input-output systems

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1. Introduction

This note is concerned with a fairly persvasive problem in modeling and identification. Namely the general problem: What is a "good" parametrization for a given model class? Where "good" of course has to be specified and may depend on other factors than just the model class in question. In some of its aspects it is a very old problem and has been with us ever since it was noted that there are several competing cartographic projections which can be used to map the earth and that none of them is perfect (or best) for all purposes.

I shall try to address this question in the context of modeling by means of linear dynamical input-output systems of a priori known state-space dimension (MacMillan degree). That is, we shall assume that our input-output observations are to be modeled by means of a system

$$\dot{x} = Ax + Bu, \ \ y = Cx, \ \ x \in \mathbf{R}^n, u \in \mathbf{R}^m, y \in \mathbf{R}^p \quad (\Sigma) \tag{1.1}$$

where A,B,C are constant (unknown) matrices of the appropriate sizes, and where it is assumed that (1.1) is completely reachable (cr) and completely observable (co). (For algebraic criteria for these two conditions c.f. below). A system like (1.1) induces an input-output map V_Σ, which, assuming that the machine (i.e. the system, or the model) starts at $x=0$ at time $t=0$, is given by

$$V_\Sigma : u(\cdot) \mapsto y(\cdot), \ \ y(t) = \int_0^t Ce^{(t-\tau)A} Bu(\tau) d\tau \tag{1.2}$$

The only data we have available are input-output data. So all that is knowable (identifiable) about (1.1) is the information about $\Sigma = (A,B,C)$ which is encoded in V_Σ. However V_Σ does not determine (A,B,C) uniquely, i.e. the map $(A,B,C) \mapsto V_\Sigma$ is not injective on the space $L^{co,cr}_{m,n,p}$ of all *cr* and *co* matrix triples (A,B,C) of the indicated dimensions. Indeed let $S \in Gl_n(\mathbf{R})$, i.e. S is an invertible real $n \times n$ matrix. Consider

$$\Sigma^S = (A,B,C)^S = (SAS^{-1}, SB, CS^{-1}) \tag{1.3}$$

It is totally elementary to observe that $V_{\Sigma^s} = V_\Sigma$. The transformation (1.3) corresponds to a base change $x' = Sx$ in state space. It is also a fact that this is the only redundancy in the description $(A,B,C) = \Sigma$ with respect to V_Σ. (I.e. if $\Sigma, \Sigma' \in L^{co,cr}_{m,n,p}$, and $V_\Sigma = V_{\Sigma}'$, then $\exists S \in Gl_n(\mathbf{R})$ such that $\Sigma' = \Sigma^S$). The relation

$$\Sigma \sim \Sigma' \Leftrightarrow \exists S \in Gl_n(\mathbf{R}) \text{ such that } \Sigma' = \Sigma^S \tag{1.4}$$

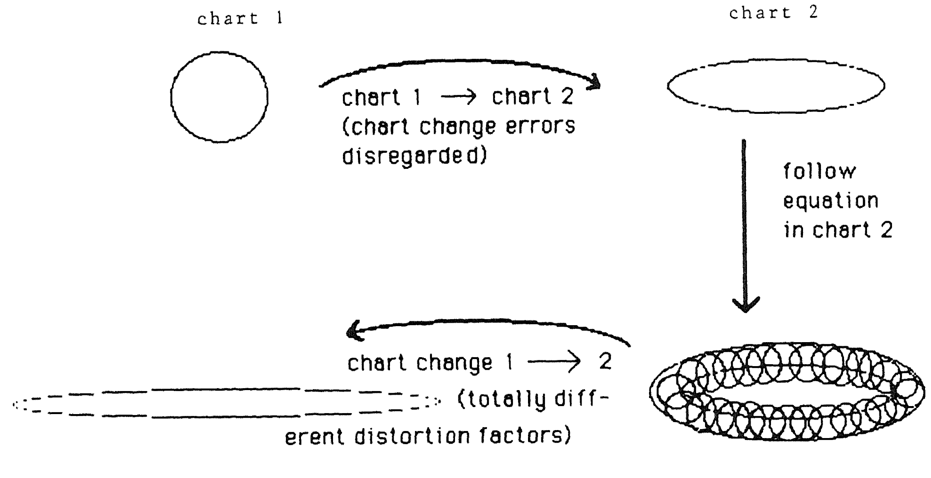
is of course an equivalence relation. The import of the remark above is thus that all we can identify on the basis of input-output data is the equivalence class of a system under this equivalence relation. Or, in other words, what can be identified is a point of the quotient space

$$M^{co,cr}_{m,n,p} = L^{co,cr}_{m,n,p} / \sim \; = L^{co,cr}_{m,n,p} / Gl_n(\mathbf{R}) \tag{1.5}$$

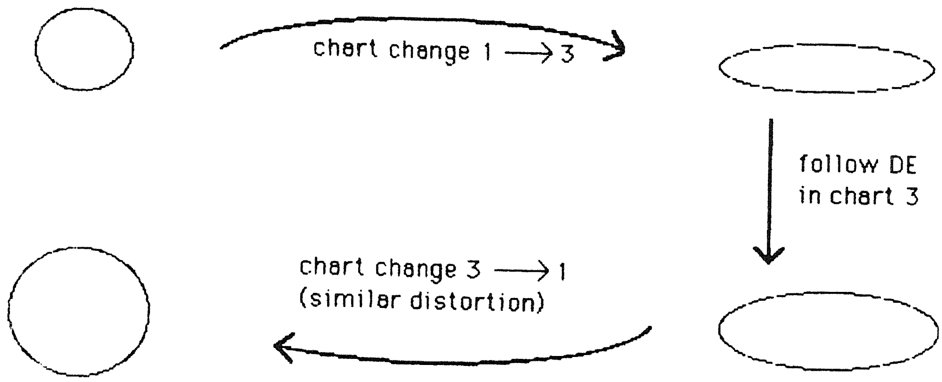
It now turns out that $M^{co,cr}_{m,n,p}$ is in fact quite nice. It is a differentiable manifold (of dimension $mn+np$), c.f. below. That is, it is locally like $\mathbf{R}^{mn+np}$ and can be described by $nm+np$ coordinates ((coordinate) charts) locally, together with correspondence rules, to yield an atlas, very much like an atlas of the world. Cf. below for an explicit atlas of this kind. It also turns out that if $m>1$ and $p>1$ it is not possible to make do with one particular chart. (Similarly it is not possible to have one global coordinate system for the whole earth (the sphere S^2) giving a unique correspondence (continuous both ways) between a part $U \subset \mathbf{R}^2$ and S^2).

Now imagine that we are engaged in a recursive identification procedure. So at time t we have ("best") estimates $\hat{A}_t, \hat{B}_t, \hat{C}_t$) for A, B, C (and $\hat{x}_t$ for the state x at time t). New information comes in and we want to update our estimates. $(\hat{A}_t, \hat{B}_t, \hat{C}_t)$ determines a point in $M^{co,cr}_{m,n,p}$ and we are looking for an optimal nearby point representing our updated estimate. This can be done using a coordinate chart valid at $(\hat{A}_t, \hat{B}_t, \hat{C}_t)$, calculating the relevant numerical coordinates, and calculating the updated versions of these coordinates according to some criterium function as expressed in these same coordinates. Proceeding in this gives a sequence of points $m_t, m_{t+1}, m_{t+2}, \ldots..$ in $M^{co,cr}_{m,n,p}$ (represented by, say, $(\hat{A}_t, \hat{B}_t, \hat{C}_t)$, $(\hat{A}_{t+1}, \hat{B}_{t+1}, \hat{C}_{t+1}), \ldots.$) and there may come a time when it becomes necessary to switch to another chart, because, say, m_{t+k} is no longer in the domain where the chart we are using is defined, or, in any case, is getting too near the "edge" of this chart to make these chart coordinates very reliable. Think again of using an ordinary street atlas, say, and changing charts when needed. In this framework one can make the general parametrization problem more precise; for instance as follows. Given a differential equation (or class of them), what are good atlases and good switching rules between coordinate charts in order to be able to follow this differential equation well numerically.

To illustrate the point consider the following situation



We start in chart 1 with a point known up to a small uncertainty as indicated. At this point chart 2 is also applicable. Changing coordinates at this point changes the uncertainty circle into an ellipse. (Uncertainty less in y-direction, more in x-direction). Following the equation in chart 2 introduces some additional uncertainty fattening up the ellipse (and even if it did not the difficulties would remain). It now becomes necessary to transfer back to chart 1 again. But now at this point in space the distortion factors may have changed totally. (In the picture a transformation $2 \mapsto 1$ at the first point compresses in the x-direction and magnifies in the y-direction; at the second point it magnifies in the x-direction and compresses in the y-direction). The result is a very elongated ellipse of uncertainty in chart 1 coordinates. Suppose we could also have worked with the coordinates of a chart 3 which as it happened had the following chart change distortion behaviour.



Obviously in this case having chart 3 available was advantageous even though the whole manifold could perhaps have been described in terms of charts 1 and 2 only. (It is by the way very easy to construct examples where this happens).

As described the good-atlases-and-parametrizations-problem seems particularly relevant in the case of recursive identification procedures. The problem however does not go away in the non-recursive case. There remains selecting a best (or good) chart from the several which may be available (and discarding one which turns out to be unsuitable in favour of a new one). And even if one could make do with one chart (on the basis of prior (structural) information concerning the class of models (e.g. in case $p = 1$ or $m = 1$ this is always possible) this may not be a particularly good one to use for a given problem. (Think of using a map of the earth covering all except the North-pole with in fact the region of interest very near the North-pole but not including it). Algorithms for identification based on overlapping coordinate charts, i.e. atlases, have in fact been developed, c.f. [2,7].

Related to the fact that as a rule it is impossible to use one chart to describe all of $M^{co,cr}_{m,n,p}$ is the fact that it is impossible to select a complete distinguishable class of models in $L^{co,cr}_{m,n,p}$ which is continous with respect to the data. (Nonexistence of continuous canonical forms [3,5]). All this means the following for a class $C \subset L^{co,cr}_{m,n,p}$

(i) Complete: for every input-output operator V (of the type coming from a Σ as in (1.1)) there is in fact a $\Sigma \in C$ such that $V_\Sigma = V$.

(ii) Distinguishable: $\Sigma_1, \Sigma_2 \in C$ and $\Sigma_1 \neq \Sigma_2 \Rightarrow V_{\Sigma_1} \neq V_{\Sigma_2}$.

(iii) Continuous: Let $V \mapsto \Sigma$ be the map determined by (i). Then this map is continuous.

So, roughly speaking it is not possible to select in a nice way one representant of each equivalence class of systems so as to remove the (statistical) indeterminacy of identifying A,B,C on the basis of input-output data alone.

This note which contains material presented at a most stimulating conference in the Pfalz academy in Lambrecht last March, is meant as an introduction to the problem and as an opportunity to introduce to the more applied community the sometimes advantageous possibility (and occasionally the necessity) of using several coordinate charts, and whole atlases. I hope and plan to write a much fuller version in the future. It is a pleasure to thank the organizer of the conference, Prof. H. Neunzert, for bringing this unusual group of scientists together.

2. Description of the spaces of all linear systems of a given degree.

As in § 1 above, let $L_{m,n,p}$ be the space of all triples (A,B,C) of matrices of sizes $n \times n, n \times m$ and $p \times n$ respectively. The triple (A,B,C) (in fact the pair (A,B)) is called completely reachable if the $(n+1)m \times n$ reachability matrix

$$R(A,B) = (B\,|AB\,|A^2B\,|\,\cdots\,|A^nB) = R(A,B,C) = R(\Sigma) \tag{2.1}$$

has rank n. Dually the triple (A,B,C) (in fact the pair (A,C)) is called completely observable if the $n \times (n+1)p$ observability matrix

$$Q(A,C) = \begin{bmatrix} C \\ CA \\ \cdot \\ \cdot \\ \cdot \\ CA^n \end{bmatrix} = Q(A,B,C) = Q(\Sigma) \tag{2.2}$$

has rank n. The spaces of *cr*, resp. *co*, resp. *cr* and *co* triples are denoted $L^{cr}_{m,n,p}$, $L^{co}_{m,n,p}$, $L^{co,cr}_{m,n,p}$. All three are open dense subspaces of $L_{m,n,p}$ (in the natural topology).

The group of invertible $n \times n$ real matrices $Gl_n(\mathbf{R})$ acts on $L_{m,n,p}$ by the formula given in (1.3) above. The subspaces of *cr*, *co*, *cr* and *co* systems are stable under this action. Indeed

$$R((A,B,C)^S) = R(SAS^{-1}, SB, CS^{-1}) = SR(A,B,C) \tag{2.3}$$

so that $rk\ R(\Sigma)=n$ iff $rk\ R(\Sigma^S)=n$. And $Q((A,B,C)^S) = S^{-1}Q(A,B,C)$.

The quotient spaces of $L_{m,n,p}$, $L^{co}_{m,n,p}$ and $L^{co,cr}_{m,n,p}$ by this action of $Gl_n(\mathbf{R})$ are denoted $M^{cr}_{m,n,p} = L^{cr}_{m,n,p} / Gl_n(\mathbf{R}), M^{co}_{m,n,p} = L^{co}_{m,n,p} / Gl_n(\mathbf{R}), M^{co,cr}_{m,n,p} = L^{co,cr}_{m,n,p} / Gl_n(\mathbf{R})$. All these quotient spaces are non-compact, smooth manifolds of dimension $mn + np$.

Below in this section we shall give one detailed description of $M^{cr}_{m,n,p}$ in terms of (coordinate) charts and gluing (= chart correspondence) rules, i.e. in terms of an atlas. To do this we need a few definitions. Consider an array $J_{m,n}$ of $n \times (n+1)m$ dots as indicated below

$$J_{m,n} = \{(i,j)\colon i \in \{0,...,n\},\ j \in \{1,...,m\}\} \tag{2.4}$$

$$J_{5,7} = \begin{array}{ccccc} \cdot & \cdot & \cdot & \cdot & \cdot \\ \cdot & \cdot & \cdot & \cdot & \cdot \\ \cdot & \cdot & \cdot & \cdot & \cdot \\ \cdot & \cdot & \cdot & \cdot & \cdot \\ \cdot & \cdot & \cdot & \cdot & \cdot \\ \cdot & \cdot & \cdot & \cdot & \cdot \\ \cdot & \cdot & \cdot & \cdot & \cdot \\ \cdot & \cdot & \cdot & \cdot & \cdot \end{array} \tag{2.5}$$

The first row of $J_{m,n}$ represents the columns of the matrix B, the second one the columns of AB, etc. Thus $(i,j) \in J_{m,n}$ represents the vector $A^i b_j$ if $B = (b_1, \cdots, b_m)$. A subset α of size n of $J_{m,n}$ is called a *nice selection* if $(i,j) \in \alpha$, $i \geq 1 \Rightarrow (i-1,j) \in \alpha$. Pictorially, if α is depicted as a set of crosses in the array as visualized by (2.5), this means that if a cross appears anywhere then in the column above if there are only crosses. Thus e.g. the left subset of $J_{5,7}$ in (2.6) below is nice, the middle one is not.

$$\begin{array}{ccccc} x & \cdot & x & x & \cdot \\ x & \cdot & \cdot & x & \cdot \\ \cdot & \cdot & \cdot & x & \cdot \\ \cdot & \cdot & \cdot & x & \cdot \\ \cdot & \cdot & \cdot & \cdot & \cdot \\ \cdot & \cdot & \cdot & \cdot & \cdot \\ \cdot & \cdot & \cdot & \cdot & \cdot \\ \cdot & \cdot & \cdot & \cdot & \cdot \end{array} \qquad \begin{array}{ccccc} \cdot & x & \cdot & x & x \\ x & \cdot & \cdot & \cdot & x \\ x & \cdot & \cdot & x & \cdot \\ \cdot & \cdot & \cdot & x & \cdot \\ \cdot & \cdot & \cdot & \cdot & \cdot \\ \cdot & \cdot & \cdot & \cdot & \cdot \\ \cdot & \cdot & \cdot & \cdot & \cdot \\ \cdot & \cdot & \cdot & \cdot & \cdot \end{array} \qquad \begin{array}{ccccc} x & * & x & x & * \\ x & \cdot & * & x & \cdot \\ * & \cdot & \cdot & x & \cdot \\ \cdot & \cdot & \cdot & x & \cdot \\ \cdot & \cdot & \cdot & * & \cdot \\ \cdot & \cdot & \cdot & \cdot & \cdot \\ \cdot & \cdot & \cdot & \cdot & \cdot \\ \cdot & \cdot & \cdot & \cdot & \cdot \end{array} \tag{2.6}$$

For a nice selection α and $j \in 1,...,m$ let $s(\alpha,j)$ be the element $(k,j) \in J_{n,m}$ determined by $(k,j) \notin \alpha$ and $(i,j) \in \alpha$ for $i \leq k-1$. This one is called the j-th successor index. In (2.6) above the successor indices of the nice selection on the left are indicated by * in the rightmost diagram. Given an $n \times (n+1)m$ matrix R and a subset α of $J_{m,n}$ let R_α denote the matrix obtained from R by removing all columns whose index is not in α.

LEMMA 2.7 *Let* $(A,B,C) \in L^{cr}_{m,n,p}$, *then there is a nice selection* α *such that the* $n \times n$ *matrix* $R(A,B,C)_\alpha$ *is invertible.*

This follows from the special structure of $R(A,B,C)$ given that $R(A,B,C)$ has rank n because (A,B,C) is *cr*.

Let $L_\alpha = \{(A,B,C) \in L^{cr}_{m,n,p}\colon R(A,B)_\alpha \text{ is invertible}\}$. Note that $\Sigma^S \in L_\alpha$ if $\Sigma \in L_\alpha$, for all $S \in Gl_n(\mathbf{R})$. Then by the lemma above

$$\bigcup_{\alpha \text{ nice}} L_\alpha = L^{cr}_{m,n,p} \tag{2.8}$$

LEMMA 2.9 *Let* $\Sigma \in L_\alpha$, α *a nice selection. Then there is precisely one* $S \in Gl_n(\mathbf{R})$ *such that* $R(\Sigma^S)_\alpha = I_n$, *the* $n \times n$ *identity matrix (and* $\Sigma^S \in L_\alpha$ *of course).*

This follows immediately from the observation that

$$R(\Sigma^S)_\alpha = S(R(\Sigma)_\alpha) \quad \text{all } \alpha \subset J_{m,n} \tag{2.10}$$

LEMMA 2.11 *Let α be a nice selection. Let $x=(y_1,...,y_m,z)$ be an element of $\mathbf{R}^{mn+np}$ written as a sequence of m n-vectors $y_1,....,y_m$ and a $p\times n$ matrix z. Then there is precisely one $\Sigma_\alpha(x) = (A_\alpha(x),B_\alpha(x),C_\alpha(x)) \in L_\alpha \subset L^{cr}_{m,n,p}$ such that*

$$R(\Sigma_\alpha(x))_\alpha = I_n,\ R(\Sigma_\alpha(x))_{s(\alpha,j)} = y_j, C_\alpha(x)=z. \tag{2.12}$$

The matrices $B_\alpha(x),A_\alpha(x)$ are very easy to write down explicitly. They always consist of columnvectors which are either equal to one of the standard basis vectors of $\mathbf{R}^n$ or to one of the vectors y_j. Indeed in the case of the example of the nice selection α of (2.6) above we have, writing $e_1,....,e_7$ for the standard basis of $\mathbf{R}^7$:

$$B = (e_1,y_2,e_2,e_3,y_5), A = (e_4,y_3,e_5,y_1,e_6,e_7,y_4)$$

$$\begin{array}{ccccc} e_1 & y_2 & e_2 & e_3 & y_5 \\ e_4 & & y_3 & e_5 & \\ y_1 & & & e_6 & \\ & & & e_7 & \\ & & & y_4 & \end{array}$$

I.e. label the crosses $e_1,...,e_7$, write in y_j for the successor spots * and read of B and A directly from the resulting pattern remembering that the first row represents the columns of B, the second one the columns of AB, etc.. From these three lemmas there follows immediately the following description of $M^{cr}_{m,n,p}$ in terms of local coordinate charts and correspondence rules between these charts.

2.13 Description of the manifold $M^{cr}_{m,n,p}$

The manifold $M^{co,cr}_{m,n,p}$ is the union of open neighborhoods $V_\alpha, \alpha \subset J_{m,n}$ running through all nice selections. Each V_α is diffeomorphic to $\mathbf{R}^{mn+np}$ via a coordinate chart $\phi_\alpha : V_\alpha \xrightarrow{\sim} \mathbf{R}^{nm+np}$. Let $x \in \mathbf{R}^{mn+np} = \psi_\alpha(V_\alpha)$, $x' \in \mathbf{R}^{mn+np} = \psi_\beta(V_\beta)$. Then x and x' correspond to the same element of $M^{co,cr}_{m,n,p}$ (i.e. $\psi_\beta(\psi_\alpha^{-1}(x))=x'$) iff

$$R(\Sigma_\beta(x')) = (R(\Sigma_\alpha(x))_\beta)^{-1} R(\Sigma_\alpha(x)),\quad z'=zR(\Sigma_\alpha(x))_\beta \tag{2.14}$$

where as above $x=(y_1,...,y_m,z), x'=(y_1',....,y_m',z')$. Note that if $x \in \psi_\alpha(V_\alpha)=\mathbf{R}^{mn+np}$ are the α-coordinates of $P \in V_\alpha \subset M^{cr}_{m,n,p}$, then the β-coordinates of P are defined iff $R(\Sigma_\alpha(x))_\beta$ is invertible, a condition which is purely in terms of the α-coordinates of P. Note also that because $y_j' = R(\Sigma_\beta(x')_{s(\beta j)}$ the β-coordinates of P are then given in terms of explicit rational expressions in the α-coordinates.

Thus (abstractly)

$$M^{cr}_{m,n,p} = \frac{\amalg}{\alpha\ nice} V_\alpha'/\sim$$

where $V_\alpha'=\mathbf{R}^{mn+np}$ for each α and $x \sim x'$, $x \in V_\alpha'$, $x' \in V_\beta'$ iff (2.14) holds.

The manifold $M^{co,cr}_{m,n,p}$ is an open submanifold of $M^{cr}_{m,n,p}$ obtained by gluing together in exactly the same way the open subsets $V^{co}_\alpha \subset V_\alpha$ defined by

$$V^{co}_\alpha = \{x \in V_\alpha = \mathbf{R}^{mn+np}: \Sigma_\alpha(x) \textit{ is co}\} \tag{2.15}$$

Note that this is an explicit (polynomial) condition in terms of the coordinates of x. For more details and proofs of the above cf. [3,5].

3. $M^{co,cr}_{m,n,p}$ as an imbedded manifold.

It is perhaps more customary to view a manifold like S^2, the sphere, as imbedded in some euclidean space like $\mathbf{R}^3$ and to view the distortions involved in taking local coordinates as measuring the differences between the geometry of the charts and the (true) geometry of the imbedded manifold (with its notions of distance etc. coming from the ambient euclidean space). As it happens the space $M^{co,cr}_{m,n,p}$ does come with a natural imbedding into a euclidean space. This and the relation of this imbedding with various atlases for $M^{co,cr}_{m,n,p}$ is the topic of this section.

Let $\mathcal{H}$ be the space of all sequences of $p \times m$ matrices $H_0, H_1, \ldots, H_{2n}$ with the normal Euclidean topology. Define a map

$$v: L_{m,n,p} \to \mathcal{H},\ (A,B,C) \mapsto (CB, CAB, \ldots, CA^{2n}B) \tag{3.1}$$

It is elementary to observe that $v(\Sigma) = v(\Sigma^S)$ for all $S \in Gl_n(\mathbf{R})$ so that v induces a quotient map also denoted v which can be restricted to $M^{co,cr}_{m,n,p}$

$$v: M^{co,cr}_{m,n,p} \to \mathcal{H} \tag{3.2}$$

THEOREM 3.3 (Kalman). *The map (3.2) is an injection. The image of (3.2) consists precisely of all sequences of matrices $H_0, H_1, \ldots, H_{2n}$ such that*

$$rk \begin{bmatrix} H_0 & H_1 & \cdots & H_{n-1} \\ H_1 & H_2 & \cdots & H_n \\ \cdot & & & \cdot \\ \cdot & & & \cdot \\ \cdot & & & \cdot \\ H_{n-1} & H_n & \cdots & H_{2n-2} \end{bmatrix} = n = rk \begin{bmatrix} H_0 & H_1 & \cdots & H_n \\ H_1 & H_2 & \cdots & H_{n+1} \\ \cdot & & & \cdot \\ \cdot & & & \cdot \\ \cdot & & & \cdot \\ H_n & H_{n+1} & \cdots & H_{2n} \end{bmatrix}. \tag{3.4}$$

In fact the map (3.2) is an imbedding of the differentiable manifold $M^{co,cr}_{m,n,p}$ into $\mathcal{H} = \mathbf{R}^{mp(2n+1)}$. It is worth noting that the matrices occuring in $v(\Sigma)$ are directly related to the input-output operator V_Σ associated to Σ. Indeed if $y(t) = V\Sigma u(t)$ and $Y(s), U(s)$ denote the Laplace transforms of $y(t), u(t)$, then

$$Y(s) = T_\Sigma(s) U(s) \tag{3.5}$$

with

$$T_\Sigma(s) = C(sI - A)^{-1}B = CBs^{-1} + CABs^{-2} + CA^2Bs^{-3} + \cdots \tag{3.6}$$

the socalled transferfunction of Σ.

It follows that if $H_0, H_1, \ldots, H_{2n}$ is a sequence of $p \times m$ matrices such that condition (3.4) is fulfilled then there must be a $\Sigma = (A,B,C) \in L^{cr,co}_{m,n,p}$ such that $H_i = CA^iB$. An algorithm for finding such an A,B,C is called a realization algorithm. And (clearly) such algorithms are not unrelated to the matter of finding coordinate charts for $M^{co,cr}_{m,n,p}$. Here is one ([6]). First observe that if $H_i = CA^iB$ then for the Hankel matrices of $(H_0, \ldots, H_{2n})$ and (A,B,C) we have

$$\begin{bmatrix} H_0 & H_1 & \cdots & H_n \\ H_1 & H_2 & \cdots & H_{n+1} \\ \cdot & & & \cdot \\ \cdot & & & \cdot \\ \cdot & & & \cdot \\ H_n & H_{n+1} & \cdots & H_{2n} \end{bmatrix} = \begin{bmatrix} CB & CAB & \cdots & CA^nB \\ CAB & CA^2B & \cdots & CA^{n+1}B \\ \cdot & & & \cdot \\ \cdot & & & \cdot \\ \cdot & & & \cdot \\ CA^nB & & \cdots & CA^{2n}B \end{bmatrix}$$

$$= Q(A,B,C)R(A,B,C) =: H(A,B,C)$$

Now because (A,B,C) is cr there is a nice selection α_c of the columns of $R(A,B,C)$ such that $R(A,B,C)_{\alpha_c}$ is invertible. Similary there is a nice selection α_R of the rows of $Q(A,B,C)$ such that $Q(A,B,C)_{\alpha_R}$ is invertible. Now observe that

$$H(A,B,C)_{\alpha_R,\alpha_c} = (Q(A,B,C)_{\alpha_R})(R(A,B,C)\alpha_c) \tag{3.7}$$

where of course H_{α_R,α_c} means the matrix obtained from H by retaining only those columns whose index is in α_R and only those rows whose index is in α_c. The first step of the realization algorithm is hence to find a nice α_R and α_c such that $S = H_{\alpha_R,\alpha_c}$ is invertible. These given (3.4) exist. We now also know that among all the (A,B,C) with this given Hankel matrix there is precisely one with $R_{\alpha_c} = I_n$. This is the one we are going to construct. Then of course $Q_{\alpha_R} = S$ which is now known. Also $H^{\alpha_R} = Q_{\alpha_R} R$ so that we know $R(A,B) = Q_{\alpha_R}^{-1} H_{\alpha_R}$ from which A and B can be recovered (Lemma 2.11). In fact A and B consist of column vectors which are either standard basis vectors or the vectors labelled by the succes indices $s(\alpha_c, j)$ of $Q_{\alpha_R} H_{\alpha_R}$. Finally if $\underline{p}$ denotes the lables of the first p rows of H we have $C = H_{\underline{p},\alpha_c}$.

This particular realization algorithm is clearly much related to the coordinate charts described in § 2 above.

The reader may wonder what the role is of the two rank conditions (3.4) in this algorithm. The first condition in fact ensures that there are nice α_R and α_c such that H_{α_R,α_c} is invertible. The second one sees to it that the construction in fact yields an A,B,C such that $H_i = CA^i B$ for all i.

4. Can the distortions involved in the coordinate changes be kept under control?

As a start and for the purpose of this note. I shall interpret this question as follows. Consider $M_{m,n,p}^{co,cr}$ as imbedded in $\mathcal{H}$. Consider a set of coordinate charts $M = M_{m,n,p}^{co,cr} \supset U_a \xrightarrow{\phi_a} \mathbf{R}^{mn+np}$. Give $M_{m,n,p}^{co,cr}$ the (Riemanian) metric induced by the imbedding. (This is not the only natural metric on M, $cf.$[4] for another important one). Is it true that one can find an atlas $(U_a,\phi_a)\alpha$ such that for all $P \in M$ there is a good chart in that for a certain predetermined ϵ the Jacobian of ϕ_a at P and its inverse are both at least ϵ away from the subset of singular matrices in the space of all square matrices of size $dimM \times dimM$? This would for example be the case of we could find a finite atlas $(U_\alpha,\phi_a)_\alpha$ (i.e. one with finitely many charts) such that for each a there is a compact set $D_a \subset \phi_a(U_a)$ such that for each $P \in M$ there is an a such that $\phi_a(P) \in D_a$. This, however, would imply that M is compact (as image of $\coprod_a D_a$) which is never the case.

The question is open but is obviously of great relevance for accurate numerical (recursive) identification problems.

The following observation of Bosgra and VAN DER WEIDEN [1] is probably going to be of importance here. Consider again the realization algorithm described in § 3 above. Because of the Hankel structure of H there are indentical ones among the entries of H which are actually used in constructing (A,B,C). It turns out that in fact precisely $nm+np$ entries of the matrices $H_0, \ldots, H_{2n}$ are used. This means that to each pair of nice selections (α_R,α_c) there is associated a subset of size $nm+np$ of the $mp(2n+1)$ coordinates of $\mathcal{H}$ such that projection onto these $nm+np$ coordinates is in fact a local coordinate chart. And of course the coordinate neighborhoods thus obtained cover all of M. This certainly does not yet give a positive answer to the question asked above but it is a positive indicator in that it is so particularly simple to indicate for a particular $P \in M$ which subsets of the coordinates of $\mathcal{K}$ of the type determined by a pair of nice selections (α_R, α_c) may be used as local coordinate charts around $P \in M$. Of course in itself, abstractly, the fact that for an imbedded manifold dimension r, say $M \subset \mathbf{R}^N$ the r-element set projections $\mathbf{R}^N \to \mathbf{R}^r$ restricted to M may be used as coordinate charts means nothing. Indeed let $P \in M \subset \mathbf{R}^N$. Locally around P the manifold M is then the image of a differentiable map $i: \mathbf{R}^r \to \mathbf{R}^N, 0 \mapsto P$, of rank r near 0. That means that the Jacobian matrix $J(i)(0)$ of i at 0 has rank r and so there is a subset α of size r of N such that $J(i)(0)_\alpha$ is invertible. Let $\pi_\alpha: \mathbf{R}^N \to \mathbf{R}^r$ be the projection corresponding to α. Then $\mathbf{R}^r \to \mathbf{R}^N \to \mathbf{R}^r$ is a diffeomorphism near 0 so that π_α is a good coordinate chart for M near P.

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