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Similarity solution for capillary redistribution of two phases in a porous medium with a single discontinuity

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In this paper we consider one-dimensional capillary redistribution of two immiscible and incompressible fluids in a porous medium with a single discontinuity. We study a special time-dependent solution, a similarity solution, which is found when the initial saturation is discontinuous at the same point as the permeability and porosity, and is constant elsewhere. The similarity solution can be used to validate numerical algorithms describing two-phase flow in porous media with discontinuous heterogeneities. We discuss the construction of the similarity solution, in which we pay special attention to the interface conditions at the discontinuity, both for media with positive and zero entry pressure. Moreover, we discuss some qualitative properties of the solution, and outline a numerical procedure to determine its graph. Examples are given for the Brooks–Corey and Van Genuchten model. We also consider similarity solutions for unsaturated water flow, which is a limit case of two-phase flow for negligible nonwetting phase viscosity.

Key words: two-phase flow, capillary pressure, entry pressure, discontinuity, similarity solution.

1 INTRODUCTION

Numerical models are effective tools to study two-phase flow in heterogeneous porous media. These models need to be verified and validated, however. For the purpose of validation of the underlying mathematical model, laboratory experiments and field tests are indispensable. The verification of the numerical model is often established by comparing for specific test problems the numerical solutions with solutions that are obtained by different methods. Ideally, one would like to use explicit solutions for this purpose. However, for nonlinear problems these are available in specific cases only. When explicit solutions are not known, one could try to reduce the problem to a simpler form to find special solutions (e.g. travelling waves, similarity solutions) of which analytical properties can be derived. This approach is chosen in this paper.

A well-known test problem is the Buckley–Leverett problem.[5] The solution of this problem describes the displacement of a nonwetting by a wetting phase in a homogeneous porous medium in the absence of capillarity. When capillarity is included, explicit solutions are not available, except for particular nonlinearities. Examples are given by Philip,[13] who solved a nonlinear diffusion problem related to capillary suction, and Fokas and Yortsos,[8] who solved a convection–diffusion equation occurring in two-phase flow. When the nonlinearities are more general, one can still use analytical techniques to obtain qualitative information about the behaviour of the solution. The graph of the solution, however, has to be determined using a numerical method. Examples of this approach are found in [6,11,15,17,20]. All these papers are dealing with homogeneous porous media.

To our knowledge, not many analytical studies are known for porous media containing heterogeneities. Yortsos and Chang[22] have obtained steady-state solutions for a heterogeneous medium, in which regions of constant permeability are connected by linear transitions. Van Duijn *et al.*[18] derived steady-state solutions for discontinuous heterogeneities. No time-dependent analytical solutions for two-phase flow through porous media with heterogeneities were found in the literature.

In this paper, we present a method to construct a time-dependent solution of self-similar form describing the one-dimensional redistribution of two immiscible and incompressible phases in a heterogeneous porous medium. The redistribution of the phases is caused by capillary forces. The porous medium consists of two homogeneous media of infinite extent which are joined at the origin, so that the permeability and porosity have a jump discontinuity there, and are constant elsewhere. We will show that this problem possesses a similarity solution if one medium is initially saturated by the wetting phase, and the other by the nonwetting phase.

The interface conditions at the point where the permeability and porosity are discontinuous play a crucial role in the construction of the solution. The two conditions that need to be imposed have been derived by Van Duijn *et al.*[18] One condition is that the flux must be continuous across the interface. The other, which is called the extended pressure condition, is a nonlinear relation between the wetting phase saturation at the left- and right-hand side of the discontinuity. It strongly depends on the qualitative behaviour of the capillary pressure.

One aspect in particular plays an important role: the entry pressure. The entry pressure, also known as the displacement pressure or threshold pressure, is the minimum capillary pressure that is needed for a nonwetting fluid to enter a medium that is initially saturated by wetting fluid. A positive entry pressure can have a relevant effect on the fluid flow behaviour. Kueper *et al.*,[9] for instance, have observed experimentally that variation in the entry pressure due to heterogeneities significantly affects the path of migration of a heavy nonwetting fluid in a heterogeneous porous medium saturated with wetting fluid.

When the entry pressure is positive, it may happen that the capillary pressure is not continuous across an interface between two media. Nonetheless, we shall show that the interface conditions still lead to a unique similarity solution. The similarity solution presented here can be used to verify if heterogeneities are correctly treated in numerical models that include entry pressures.

The diffusion problem discussed here resembles, in many respects, the one-dimensional hysteresis problem studied by Philip.[14] In that paper he considers the redistribution of water in an unsaturated soil with different capillary pressure curves on the left- and right-hand side of the origin: a drying curve on one side, a wetting curve on the other side. By a so-called flux-concentration method, Philip obtains approximate solutions for this problem. The solutions in his case always have continuous capillary pressure, since the drying and wetting curve form a closed loop (the hysteresis loop) with zero entry pressure. In this work we give the procedure to obtain solutions without approximations, outline a numerical method to approximate the exact solution, and allow the solutions to have discontinuous capillary pressure.

This paper is organised as follows. In Section 2 we present the mathematical model describing the redistribution of two immiscible phases in a porous medium. Further, we consider the interface conditions needed at a discontinuity in the permeability or porosity.

In Section 3 we use a similarity transformation to transform the partial differential equation into an ordinary differential equation. For this latter equation we shall explain how the solution can be constructed. We give a criterion to determine whether the solution has discontinuous capillary pressure or not. This can be checked before the actual construction of the solution. The technical details of the mathematical justification are presented elsewhere.[7] Furthermore, we provide a numerical method and discuss the qualitative behaviour of the solution.

In Section 4 we give two illustrative examples. We consider similarity solutions for two different models of the capillary-hydraulic properties of the porous medium: the Brooks–Corey model[4] and the Van Genuchten model.[21] Since a Brooks–Corey type of porous medium has a positive entry pressure, solutions may occur with discontinuous capillary pressure. We show for which permeability and porosity contrasts such solutions are found and provide an example.

In Section 5 we discuss for a limit case, i.e. for unsaturated water flow, what interface condition concerning the pressure is appropriate at a discontinuity if the capillary pressure curve has a positive entry pressure. We show that continuity of the capillary pressure leads to unphysical solutions in this case. A conclusion is given in Section 6.

2 MATHEMATICAL MODEL

In this section we give the mathematical formulation of the one-dimensional horizontal redistribution of two immiscible and incompressible phases in a saturated and heterogeneous porous medium. The phases are characterised by their reduced saturations: S_w (saturation of the wetting fluid) and S_n (saturation of the nonwetting fluid), where

$$S_w + S_n = 1 \text{ with } 0 \le S_w,\ S_n \le 1 \tag{1}$$

We denote the wetting phase saturation by $s = s(x,t)$ in the sequel, where x is the spatial coordinate and t is time. Combination of the flow and continuity equations, as in [18], yields for the wetting phase saturation the nonlinear diffusion equation

$$\phi\frac{\partial s}{\partial t} + \frac{\partial}{\partial x}\left(\frac{k}{\mu_w}\bar{\lambda}(s)\frac{\partial p_c}{\partial x}\right) = 0 \tag{2}$$

where ϕ, k and p_c denote the porosity, absolute permeability and capillary pressure, and where

$$\bar{\lambda}(s) = \frac{k_{rw}(s)k_{rn}(s)}{Mk_{rw}(s) + k_{rn}(s)}, \text{ with } M = \frac{\mu_n}{\mu_w} \tag{3}$$

Here, k_{rw} and k_{rn} are the relative permeabilities of the wetting and nonwetting phases, and μ_w and μ_n the respective viscosities; M is the mobility ratio. The capillary pressure is given by the Leverett-relationship[10]

$$p_c = p_c(x,s) = \sigma\sqrt{\frac{\phi(x)}{k(x)}}J(s) \tag{4}$$

where σ is the interfacial tension and J the Leverett function. In this work, hysteresis in the capillary pressure and the relative permeabilities is neglected.

The nonlinear functions k_{rw}, k_{rn} and J have the usual properties:

1. $k_{rw}(s)$ is strictly increasing with $k_{rw}(0) = 0$;
2. $k_{rn}(s)$ is strictly decreasing with $k_{rn}(1) = 0$;
3. $\lim_{s\downarrow 0} J(s) = \infty$, $\mathrm{d}J/\mathrm{d}s < 0$ and $J(1) \geq 0$;
4. $k_{rw}(s)k_{rn}(s)\mathrm{d}J(s)/\mathrm{d}s$ is bounded.

We assumed the latter condition to prevent that the diffusivity $-\bar{\lambda}(s)\mathrm{d}J(s)/\mathrm{d}s$ blows up near $s = 0$ or $s = 1$. This condition is satisfied by most functions k_{rw}, k_{rn} and J found in the literature, e.g. Brooks–Corey and Van Genuchten functions.[4,21] The case of hyperdiffusivity is thus excluded. The entry pressure of a medium at a given point is given by the value of the capillary pressure at $s = 1$. Hence $J(1) > 0$ indicates positive entry pressure.

In order that similarity solutions can be constructed, we restrict ourselves to a porous medium of which the permeability and porosity change abruptly at some point, say $x = 0$, and are constant elsewhere. The permeability and porosity are thus given by

$$k = k(x) = \begin{cases} k_l, & x < 0, \\ k_r, & x > 0, \end{cases} \tag{5}$$

and

$$\phi = \phi(x) = \begin{cases} \phi_l, & x < 0, \\ \phi_r, & x > 0, \end{cases} \tag{6}$$

where subscripts l and r denote left- and right-hand side values.

To make equation (2) dimensionless and to eliminate the parameters μ_w and σ, we introduce reference quantities L (length) and k^* (permeability), and define dimensionless variables according to

$$x := \frac{x}{L},\ t := \frac{t\sigma\sqrt{k^*}}{\mu_w L^2} \tag{7}$$

As a result we obtain

$$\frac{\partial s}{\partial t} = h_l \frac{\partial}{\partial x}\left(D(s)\frac{\partial s}{\partial x}\right) \text{ for } x < 0,\ t > 0 \tag{8}$$

$$\frac{\partial s}{\partial t} = h_r \frac{\partial}{\partial x}\left(D(s)\frac{\partial s}{\partial x}\right) \text{ for } x > 0,\ t > 0 \tag{9}$$

where

$$D(s) = -\frac{\mathrm{d}J(s)}{\mathrm{d}s}\bar{\lambda}(s) \tag{10}$$

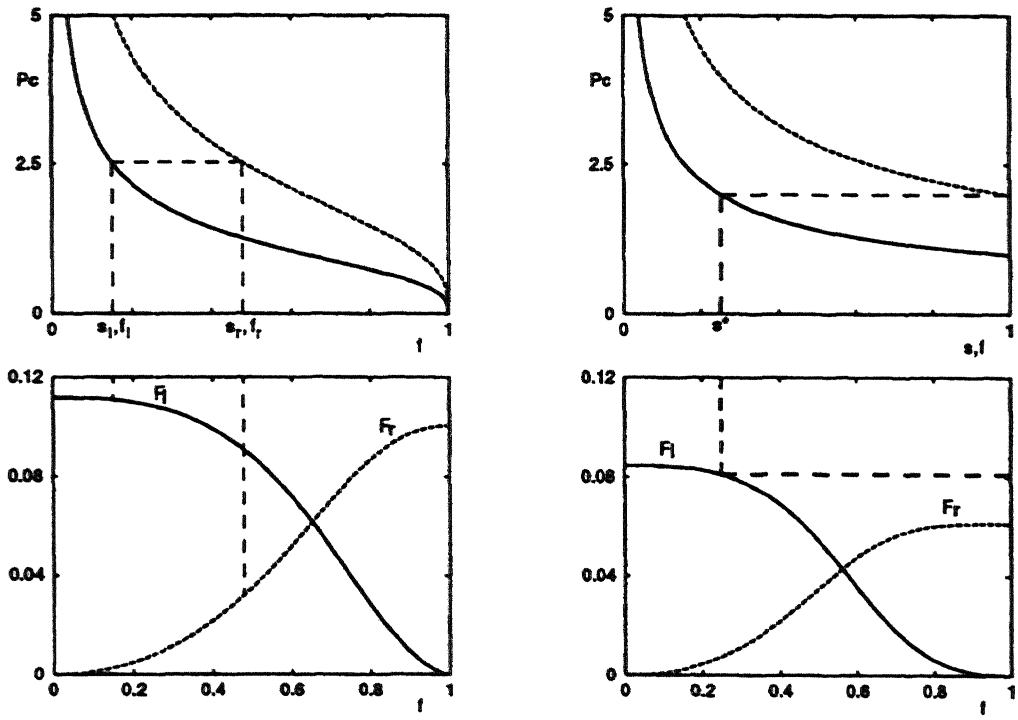

Fig. 1. Capillary pressure (upper figures) and flux functions (lower figures). Solid curves correspond to the left, dashed curves to the right of $\eta = 0$. Left figures are obtained for the Van Genuchten model, right figures for the Brooks-Corey model. Data for the computations are given in Table 1 in Section 4.

and where we introduced for convenience

$$h_l = \left(\frac{k_l}{k^*\phi_l}\right)^{1/2} \text{ and } h_r = \left(\frac{k_r}{k^*\phi_r}\right)^{1/2} \tag{11}$$

The conditions imposed on k_{rw}, k_{rn} and J imply $D(s) > 0$ on $(0,1)$ with $D(0) \geq 0$ and $D(1) \geq 0$.

At $x = 0$ where k and ϕ have a discontinuity, the equations do not hold. At this point we need to impose two interface conditions for all $t > 0$. The first condition is continuity of the flux,

$$\lim_{x\uparrow 0}\left(-\phi_l h_l D(s)\frac{\partial s}{\partial x}\right) = \lim_{x\downarrow 0}\left(-\phi_r h_r D(s)\frac{\partial s}{\partial x}\right) \tag{12}$$

The second interface condition is an extension of a continuity condition for the capillary pressure. This extended pressure condition is derived by Van Duijn *et al.*[18] using continuity of capillary pressure and saturation in a narrow transition zone. It is described as follows.

The media to the left and right of $x = 0$ have different values of $(k/\phi)^{1/2}$. As a result, they have different capillary pressure curves, which follows directly from the Leverett-relationship (4). If, for example, $h_l > h_r$, i.e. coarse material to the left and fine material to the right of $x = 0$, then the capillary pressure curve corresponding to the fine material lies above the curve corresponding to the coarse material (cf. upper two pictures of Fig. 1). We distinguish capillary pressure curves with zero entry pressure, as in Fig. 1 (left), and with positive entry pressure, as in Fig. 1 (right).

If the entry pressure is zero, then to every saturation on one side of the interface, there corresponds a saturation on the other side so that the capillary pressure is continuous [cf. Fig. 1 (left)]. In this case, the second interface condition is simply continuity of capillary pressure, which is expressed by

$$\frac{J(s_r)}{h_r} = \frac{J(s_l)}{h_l} \tag{13}$$

where s_r and s_l denote the right and left limit value of s at $x=0$. Condition (13) is used in the analysis by Philip.[14]

If, however, the entry pressure is positive, we see from Fig. 1 (right) that there is a threshold saturation s^* on the side corresponding to the lower curve, above which the continuity of capillary pressure cannot be established. The threshold saturation s^* is determined by (in case $h_l > h_r$)

$$\frac{J(s^*)}{h_l} = \frac{J(1)}{h_r} \tag{14}$$

If the wetting phase saturation on the side of the interface corresponding to the lower curve is greater than s^*, then the saturation on the other side must be equal to one; the capillary pressure across the interface is then discontinuous. The extended pressure condition is thus given by (in case $h_l > h_r$)

$$\begin{aligned} \frac{J(s_r)}{h_r} &= \frac{J(s_l)}{h_l} && \text{if } s_l \leq s^* \\ s_r &= 1 && \text{if } s^* < s_l \leq 1 \end{aligned} \tag{15}$$

If $h_l \leq h_r$ then the second interface condition is given by equation (15) with the subscripts l and r reversed; the threshold saturation s^* follows then from equation (14) with h_l and h_r reversed.

In the derivation of the extended pressure condition, the Leverett function is assumed to be identical on both sides of the discontinuity. Equally, the condition could have been formulated for different Leverett functions on each side, like, e.g. a drainage and imbibition curve.

Two typical models of relative permeability and Leverett functions are the Brooks–Corey model:[4]

$$\begin{aligned} J(u) &= u^{-1/\lambda} \\ k_{rw}(u) &= u^{3+2/\lambda} \\ k_{rn}(u) &= (1-u)^2(1-u^{1+2/\lambda}) \end{aligned} \tag{16}$$

where $\lambda > 0$, and the Van Genuchten model:[21]

$$\begin{aligned} J(u) &= (u^{-1/m} - 1)^{1-m} \\ k_{rw}(u) &= u^{1/2}(1-(1-u^{1/m})^m)^2 \\ k_{rn}(u) &= (1-u)^{1/2}(1-u^{1/m})^{2m} \end{aligned} \tag{17}$$

where $0 < m < 1$. The latter model is only partly attributable to Van Genuchten; the expression for k_{rn} has been derived by Parker *et al.*[12]. The essential difference between the two models is that Brooks–Corey has a positive entry pressure, whereas Van Genuchten has zero entry pressure. We shall discuss the similarity solutions corresponding to these models in Section 4.

3 SIMILARITY SOLUTIONS

In this section we consider a special solution of equations (8) and (9): a similarity solution. This solution is found for a particular initial condition, which has a discontinuity at $x = 0$ and is constant elsewhere. The original initial value problem can then be transformed into a boundary value problem consisting of ordinary differential equations. We give here the construction of the solution, show that the interface conditions indeed lead to a unique solution, and outline the procedure to determine the similarity solution numerically. Some qualitative properties of the similarity solution are discussed at the end of this section.

We study equations (8) and (9) subject to the initial condition

$$s(x,0) = \begin{cases} 1 & \text{if } x < 0, \\ 0 & \text{if } x > 0 \end{cases} \tag{18}$$

The resulting problem admits solutions of self-similar form. If we set

$$s(x,t) = f(\eta), \text{ with } \eta = \frac{x}{\sqrt{t}} \tag{19}$$

we obtain for f the ordinary differential equations

$$\tfrac{1}{2}\eta f' + h_l(D(f)f')' = 0 \text{ for } \eta < 0 \tag{20}$$

$$\tfrac{1}{2}\eta f' + h_r(D(f)f')' = 0 \text{ for } \eta > 0 \tag{21}$$

Here the primes denote differentiation with respect to η. The initial condition for s yields the boundary conditions

$$f(-\infty)=1 \text{ and } f(\infty)=0 \tag{22}$$

At $\eta = 0$, the solution has to satisfy the flux continuity condition

$$\lim_{\eta\uparrow 0}(-\phi_l h_l D(f)f') = \lim_{\eta\downarrow 0}(-\phi_r h_r D(f)f') \tag{23}$$

and the extended pressure condition (15) with s_r and s_l replaced by $f_r = \lim_{\eta\downarrow 0} f(\eta)$ and $f_l = \lim_{\eta\uparrow 0} f(\eta)$.

3.1 Construction of the solution

To construct the similarity solution, we first solve equations (20), (21) and (22), and then match the corresponding solutions at $\eta = 0$ so that the interface conditions (15) and (23) are satisfied. Thus, we start with the subproblems

$$\begin{aligned} &\tfrac{1}{2}\eta f' + h_r(D(f)f')' = 0, \quad 0<\eta<\infty \\ &f(0)=f_r \quad , \quad f(\infty)=0 \end{aligned} \tag{24}$$

and

$$\begin{aligned} &\tfrac{1}{2}\eta f' + h_l(D(f)f')' = 0, \quad -\infty<\eta<0, \\ &f(-\infty)=1 \quad , \quad f(0)=f_l \end{aligned} \tag{25}$$

where $0 \le f_l, f_r \le 1$ have to be determined from the interface conditions.

It is well-known (e.g. [1,2,19]) that problem (24) has a unique solution $f_+ = f_+(\eta)$ for every $f_r \in [0,1]$. If $f_r = 0$ then $f_+(\eta) = 0$ for all $\eta \ge 0$; if $f_r > 0$ then there is a positive $a_r \le \infty$ such that

$$f_+(\eta)\begin{cases} >0 & \text{for } 0<\eta<a_r \\ =0 & \text{for } \eta \ge a_r \end{cases} \tag{26}$$

and $f'_+(\eta) < 0$ for η between zero and a_r. The behaviour of the diffusion coefficient near $f = 0$ determines whether $a_r = \infty$ or $a_r < \infty$. The precise condition is given in Section 3.2.

Similarly, problem (25) has a unique solution $f_- = f_-(\eta)$ for every $f_l \in [0,1]$. If $f_l = 1$ then $f_-(\eta) = 1$ for all $\eta \le 0$; if $f_l < 1$ then there is a negative $a_l \ge -\infty$ such that

$$f_-(\eta)\begin{cases} =1 & \text{for } \eta \le a_l \\ <1 & \text{for } a_l<\eta<0 \end{cases} \tag{27}$$

and $f'_-(\eta) < 0$ for η between a_l and zero. Here, the behaviour of the diffusion coefficient near $f = 1$ determines whether $a_l = -\infty$ or $a_l > -\infty$. For the precise conditions were refer to Section 3.2.

To apply the interface conditions we need to know the fluxes at $\eta = 0$. Let

$$F_r = -\phi_r h_r D(f_r) f'_+(0) \tag{28}$$

and

$$F_l = -\phi_l h_l D(f_l) f'_-(0) \tag{29}$$

For every value of $f_r \in [0,1]$, which determines the solution f_+ of problem (24), a unique F_r results. We denote this dependence by writing $F_r = F_r(f_r)$. This function is continuous and strictly increasing in $f_r \in [0,1]$ with $F_r(0)=0$. An analytic proof of these statements is given in ref. [19], a computational result is shown in Fig. 1, where the flux function F_r is given for the Brooks–Corey and Van Genuchten model.

In a similar fashion, F_l can be considered as a function of f_l. This function, which is continuous and strictly decreasing in $f_l \in [0,1]$ with $F_l(1) = 0$, is also shown in Fig. 1.

Having discussed these preliminary results, we can now outline the matching procedure.

3.1.1 Existence of a unique pair (f_l, f_r) matching the interface conditions

We consider in detail the case $h_l > h_r$. In Fig. 1 the graphs of the capillary pressure and fluxes are shown, both as functions of the saturation at each side of the origin. Note that here the lower capillary pressure curves correspond to the left-hand side of the origin, the upper curves to the right-hand side. The fluxes were obtained by numerically solving transformed versions of problems (24) and (25) for different f_l and f_r. The details of the transformation and computation are given in Section 3.1.2. We treat the cases with and without entry pressure separately.

3.1.1.1 Zero entry pressure. If the entry pressure is zero, then the saturations at the origin have to satisfy condition (13), reflecting continuity of the capillary pressure. Since the capillary pressure functions are strictly decreasing, it follows from continuity of the capillary pressure that the right saturation depends monotonically on the left saturation [cf. Fig. 1 (left)]: for instance, when we increase the left saturation f_l, then the right saturation f_r increases as well. Furthermore, for increasing f_l, the left flux F_l decreases while the corresponding right flux F_r increases.

Now, using continuity and monotonicity of the graphs in Fig. 1 (left), we find, for f_l increasing from zero, a unique pair (f_l, f_r) such that both pressure and flux are equal. The continuity of the fluxes combined with $F_l(0) > F_r(0)$ and $F_l(1) < F_r(1)$ yields the existence of such a pair. The motonicity of the fluxes implies the uniqueness.

3.1.1.2 Positive entry pressure. If the entry pressure is positive, as in Fig. 1 (right), then the situation is different in the sense that now the saturations at the discontinuity are related to each other through the extended pressure condition (15). Increasing the left saturation f_l, we see that the right saturation increases only if $f_l \le s^*$, but is constant ($f_r = 1$) if $f_l > s^*$. Moreover, for increasing f_l, the left flux F_l decreases while the correspondiing right flux F_r increases only if $f_l \le s^*$, but is constant ($F_r = F_r(1)$) if $f_l > s^*$.

So, if $F_l(s^*) > F_r(1)$, then f_l must be greater than s^* in order to have continuity of the flux. In that case, $f_r = 1$ and $F_r = F_r(1)$. Since $F_l(f_l)$ is a strictly decreasing function of f_l with $F_l(1) = 0$, it follows that there is a unique f_l such that continuity of the flux is satisfied. Note that the capillary pressure is discontinuous in this case.

If $F_l(s^*) \leq F_r(1)$, then it is necessary that $f_l \leq s^*$ in order to have continuity of flux. Consequently, the capillary pressure is continuous, and again as in the case of zero entry pressure, there is a unique pair (f_l, f_r), such that the interface conditions are satisfied.

Remark: The case $h_l \leq h_r$ can be treated in a similar manner. In that case too, the pair (f_l, f_r) is uniquely determined. However, now the capillary pressure is always continuous, since f_l must be less than one in order to have a positive flux F_l, and therefore $f_r < s^*$. Note that in this case $f_l > f_r$.

3.1.2 Computation of (f_l, f_r), the flux functions and the solution

To obtain a solution, we first determine the pair (f_l, f_r) that satisfies the interface conditions, and then use these values to solve problems (24) and (25). It is not necessary to solve these problems separately: their solutions follow directly from the method that will be used to compute the fluxes F_l and F_r.

We first explain how to obtain the pair (f_l, f_r). For the time being, let us assume that the functions $F_l(f_l)$ and $F_r(f_r)$ are known: further on we discuss how they can be determined numerically. We distinguish between the cases $h_l > h_r$ (capillary pressure possibly discontinuous) and $h_l \leq h_r$ (capillary pressure continuous).

If $h_l > h_r$ we first have to check whether the capillary pressure is continuous. For that purpose we need the values of $F_l(s^*)$ and $F_r(1)$. If $F_l(s^*) > F_r(1)$, then the capillary pressure is discontinuous, and hence $f_r = 1$. In that case we have to find the root $f_l \in (s^*, 1)$ of

$$F_l(f_l) = F_r(1) \tag{30}$$

If $F_l(s^*) \leq F_r(1)$, then the capillary pressure is continuous, and hence $f_l \leq s^*$. In that case, determine f_r as a function of f_l using continuity of capillary pressure. Then find the root $f_l \leq s^*$ that solves the equation

$$F_l(f_l) = F_r(f_r(f_l)) \tag{31}$$

If $h_l \leq h_r$, then the capillary pressure is continuous and we proceed as above: determine f_r as a function of f_l using continuity of capillary pressure, and find the root $f_l \in (0,1)$ that solves equation (31). To find the root of equations (30) or (31) we use the bisection method.

Crucial in the construction are the flux functions $F_l(f_l)$ and $F_r(f_r)$. Of course it is not necessary to determine the entire graphs of F_l and F_r. The functions F_l and F_r only have to be evaluated at the iteration points resulting from the algorithm that is used to find the root of equations (30) or (31). We discuss below how $F_r(f_r)$, with $0 \leq f_r \leq 1$, can be solved numerically. The function $F_l(f_l)$ is found in an entirely analogous way. Therefore, those details are omitted.

To determine the right flux $F_r(f_r)$ we need to solve problem (24) and compute the flux at $\eta = 0$. The complication here is the boundary condition $f(\infty) = 0$ which is not always easy to verify. Fortunately, there is a more direct way to obtain the flux-saturation relation at $\eta = 0$. The idea is to transform equation (21) into a differential equation for the flux with the saturation as independent variable.[2] Since f_+ is strictly decreasing on $(0, a_r)$ we can invert

$$f_+ = f_+(\eta) \text{ for } 0 \leq \eta \leq a_r \tag{32}$$

to obtain

$$\eta = \sigma_+(f) \text{ for } 0 \leq f \leq f_r \tag{33}$$

where σ_+ is the inverse of f_+ with $\sigma_+(0) = a_r$ and $\sigma_+(f_r) = 0$. Next, consider the scaled flux (up to the porosity) as a function of saturation, i.e.

$$y(f) := -h_r D(f) f'_+(\sigma_+(f)) \text{ for } 0 \leq f \leq f_r \tag{34}$$

Note that $y(f) > 0$ for $0 < f < f_r$, because f_+ is monotonically decreasing. Using equation (21), one easily verifies that

$$\frac{dy}{df}(f) = \tfrac{1}{2}\sigma_+(f) \text{ for } 0 < f < f_r \tag{35}$$

and

$$y\frac{d^2y}{df^2} = -\tfrac{1}{2}h_r D(f) \text{ for } 0 < f < f_r \tag{36}$$

Since the flux vanishes whenever the saturation vanishes we also have $y(0) = 0$. Thus, for given $f_r \in [0,1]$, we want to solve the boundary value problem

$$\begin{aligned} & y\frac{d^2y}{df^2} = -\tfrac{1}{2}h_r D(f) \quad \text{for } 0 < f < f_r, \\ & \frac{dy}{df}(f_r) = 0 \quad , \quad y(0) = 0, \end{aligned} \tag{37}$$

such that $y > 0$ on $(0, f_r)$. Having established the solution $y = y(f)$, we know the flux at $\eta = 0$ through the relation

$$F_r(f_r) = \phi_r y(f_r) \tag{38}$$

Problem (37) is solved by a shooting technique.[3] In its specific application to problem (37), we replace the boundary condition $y(0) = 0$ by $y(f_r) = y_r$, where y_r is the shooting parameter. The objective is to find y_r such that the boundary condition $y(0) = 0$ is satisfied. We use the bisection method to obtain convergence to the required value of y_r.

The method to compute $y(f)$ can be conveniently employed to determine the solution f_+ of problem (24). In the shooting procedure, the second-order differential equation in y is rewritten into a system of first-order differential equations in the dependent variables y and dy/df. From equation (35) we have

$$\eta = \sigma_+(f) = 2\frac{dy}{df}(f) \text{ for } 0 < f < f_r \tag{39}$$

Hence, using equation (39), the algorithm directly gives for every value of f the corresponding value of η.

3.2 Structure of the solution

From what we learned so far we deduce that the self-similar form of the solution, $s(x,t)=f(\eta)$ with $\eta = x/\sqrt{t}$, has a discontinuity at the origin $x = 0$ ($\eta = 0$), where the initial saturation as well as the properties of the porous medium are discontinuous. Moreover, the self-similar solution $f(\eta)$ has the structure

$$f(\eta)\begin{cases} =1 & \text{for } \eta \le a_l \\ \in (0,1) & \text{for } a_l < \eta < a_r,\ \eta \neq 0 \\ =0 & \text{for } \eta \ge a_r \end{cases} \tag{40}$$

with $f(0^-) = f_l$ and $f(0^+) = f_r$; it is strictly decreasing for η between a_l and zero, and zero and a_r. Let us assume for the moment that a_l and a_r are finite. They characterise moving or free boundaries in the x,t-plane given by

$$x_l(t) = a_l\sqrt{t} \text{ and } x_r(t) = a_r\sqrt{t}, \text{ for } t \ge 0 \tag{41}$$

This implies that the saturation as a function of position and time satisfies for $t > 0$,

$$s(x,t)\begin{cases} =1 & \text{for } x \le x_l(t), \\ \in (0,1) & \text{for } x_l(t) < x < x_r(t),\ x \neq 0, \\ =0 & \text{for } x \ge x_r(t) \end{cases} \tag{42}$$

Moreover, we have for all $t > 0$,

$$\lim_{x\uparrow 0} s(x,t) = f_l \text{ and } \lim_{x\downarrow 0} s(x,t) = f_r \tag{43}$$

In the mathematics literature, precise results are known concerning the finiteness of a_l and a_r, and concerning the behaviour of the similarity solution near these points, e.g. [1,2] or [17]. These results show a relation with the behaviour of the diffusion coefficient near $f = 0$ and near $f = 1$. The conditions for the occurrence of free boundaries are:

$$a_r < \infty \text{ if and only if } \int_0^1 \frac{D(s)}{s}\mathrm{d}s < \infty \tag{44}$$

and

$$a_l > -\infty \text{ if and only if } \int_0^1 \frac{D(s)}{1-s}\mathrm{d}s < \infty \tag{45}$$

When free boundaries exist, we can easily find them from the solution in the flux-saturation plane. From equation (39) we have

$$a_r = 2\frac{\mathrm{d}y}{\mathrm{d}f}(0) \text{ and } a_l = 2\frac{\mathrm{d}y}{\mathrm{d}f}(1) \tag{46}$$

We remark that volume (or mass) conservation is ensured by the continuity of the flux. Namely, integrating equation (20) from a_l to zero and equation (21) from zero to a_r, and using equation (23) yields

$$\phi_l \int_{a_l}^{0} (1 - f(\eta))\mathrm{d}\eta = \phi_r \int_0^{a_r} f(\eta)\mathrm{d}\eta \tag{47}$$

The volume of nonwetting phase to the left of $\eta = 0$ is thus equal to the volume of wetting phase to the right of it. Equation (47) is also valid when a_l or a_r are not finite.

In the case that the capillary pressure curve has a positive entry pressure we can determine when the capillary pressure at $\eta = 0$ is discontinuous, and how it depends on the medium parameters.

Recall that the capillary pressure is discontinuous if $F_l(s^*) > F_r(1)$. The flux function $F_r(f_r)$ is given by equation (38). Notice that if y is scaled by $\sqrt{h_r}$ then the scaled y satisfies problem (37), but with $h_r = 1$. Thus, the flux function F_r can be written as $F_r(f_r) = \phi_r\sqrt{h_r}\hat{F}_r(f_r)$, where $\hat{F}_r$ is the flux function corresponding to $\phi_r = h_r = 1$. Similarly, we have $F_l(f_l) = \phi_l\sqrt{h_l}\hat{F}_l(f_l)$, where $\hat{F}_l$ is the flux function corresponding to $\phi_l = h_l = 1$. Let α denote the ratio of h_r and h_l, and β the ratio of ϕ_r and ϕ_l, i.e.

$$\alpha = \frac{h_r}{h_l} \text{ and } \beta = \frac{\phi_r}{\phi_l} \tag{48}$$

The capillary pressure is thus discontinuous at $\eta = 0$ if and only if

$$\beta < \frac{\hat{F}_l(s^*(\alpha))}{\hat{F}_r(1)\sqrt{\alpha}} = \beta_{cr}(\alpha) \tag{49}$$

where $s^*(\alpha)$ is given by equation (14), and β_{cr} is the critical value of β. Note that β_{cr} is a decreasing function of α with a vertical asymptote at $\alpha = 0$, and $\beta_{cr}(1) = 0$. Thus, the capillary pressure is discontinuous at $\eta = 0$ for given porosity ratio β if α is sufficiently small, i.e. if the permeability ratio k_r/k_l is sufficiently small.

4 EXAMPLES: BROOKS–COREY AND VAN GENUCHTEN MODEL

In this section we consider capillary diffusion for two different sets of relative permeability and Leverett functions which are frequently used in hydrology: the Brooks–Corey model and Van Genuchten model. These models are given at the end of Section 2. For both models we look at the pressure condition, determine if free boundaries occur, and discuss the similarity solutions. For the Brooks–Corey model we also show when the capillary pressure at $x = 0$ is discontinuous, depending on the permeability and porosity ratio.

The extended pressure condition for a Brooks–Corey curve leads to the following relation between s_l and s_r when $h_l > h_r$:

$$s_r = \begin{cases} s_l/s^* & \text{if } 0 \le s_l \le s^*, \\ 1 & \text{if } s^* \le s_l \le 1 \end{cases} \tag{50}$$

with

$$s^* = \left(\frac{h_r}{h_l}\right)^\lambda \tag{51}$$

To obtain the interface condition for $h_l < h_r$, one has to interchange s_l and s_r, as well as h_l and h_r in equations (50) and (51). The graphs of the relations between s_r and s_l for the Brooks–Corey and Van Genuchten model are given in Fig. 2.

The graphs of the diffusion functions for each model are

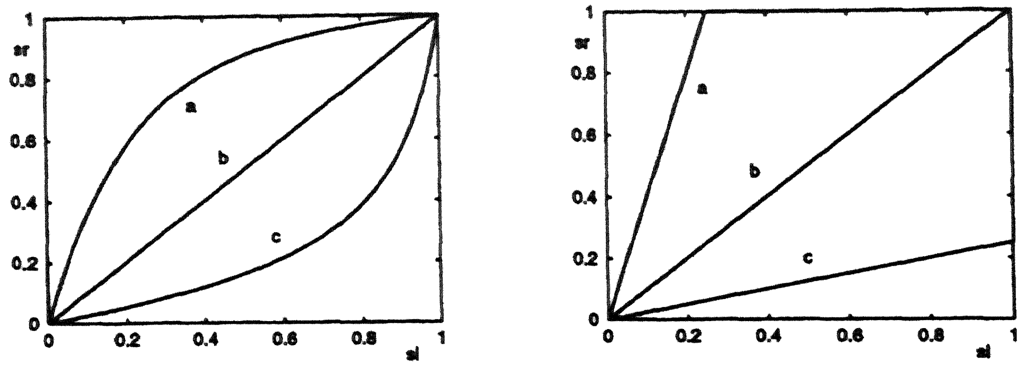

Fig. 2. Second interface condition for the Van Genuchten model (left) and Brooks-Corey model (right), with $m = 2/3$ and $\lambda = 2$, for different ratios of h_r/h_l: (a) 0.5, (b) 1, (c) 2.

given in Fig. 3. Note that in both cases $D(0) = D(1) = 0$. Hence, free boundaries may occur. We determine for both media if the similarity solution has free boundaries by examining the integrability conditions for the diffusion function in equations (44) and (45). The diffusion function (10) for the Brooks–Corey model has the asymptotic behaviour:

$$D(s) = O(s^{2+1/\lambda}) \text{ as } s \downarrow 0 \tag{52}$$

$$D(s) = O((1-s)^3) \text{ as } s \uparrow 1 \tag{53}$$

Therefore, the integrability conditions in equations (45) and (44) are satisfied for all $\lambda > 0$, independent of the mobility ratio M. Hence, we have free boundaries as $s \uparrow 1$ and $s \downarrow 0$ for all $\lambda > 0$. The diffusion function for the Van Genuchten model yields:

$$D(s) = O(s^{(2+m)/2m}) \text{ as } s \downarrow 0 \tag{54}$$

$$D(s) = O((1-s)^{m+1/2}) \text{ as } s \uparrow 1 \tag{55}$$

Again the integrability conditions in equations (45) and (44) are satisfied, now for all relevant values of $m(0 < m < 1)$, independent of M. Hence, also for the Van Genuchten model both free boundaries occur for all m.

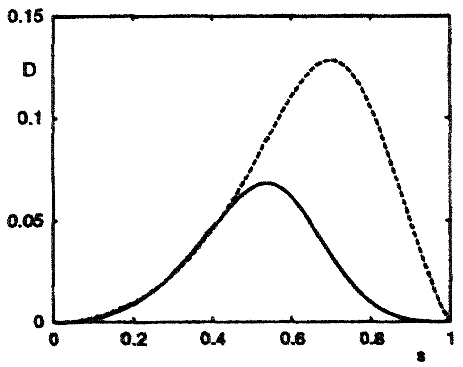

Fig. 3. Diffusion functions for Brooks-Corey (solid) and Van Genuchten (dashed)model, with $\lambda = 2$, $m = 2/3$ and $M = 1$.

In Figs 4 and 5, solutions for the Brooks–Corey and Van Genuchten model are shown as curves in the flux-saturation plane and as similarity profiles $f = f(\eta)$. The data are given in Table 1. The parameter m is chosen such that the capillary pressure curve of the Brooks–Corey and Van Genuchten model have identical asymptotic behaviour as s tends to zero.

Note that for this data set the capillary pressure for the solution corresponding to the Brooks–Corey model is discontinuous at the origin ($f_l > s^* = 0.25$). Furthermore, we observe that the nonwetting front ($\eta = a_l$) for the solution corresponding to the Van Genuchten model is much further to the left than the nonwetting front for the solution corresponding to the Brooks–Corey model. This is due to the diffusion near $f = 1$, which is much larger for the Van Genuchten model than for the Brooks–Corey model because of the steeper slope of the J-curve near $f = 1$ (cf. Figs 1, and 3). Equal saturation gradients lead then for the Van Genuchten model to a higher diffusive flux. Therefore, in a Van Genuchten medium the nonwetting phase enters the region originally occupied by the wetting phase easier than in a Brooks–Corey medium.

The time-dependent behaviour of the Brooks–Corey similarity solution is shown in Fig. 6. Observe that the values at the origin are fixed and that the behaviour near the free boundaries remains unchanged, except for the $\sqrt{t}$ scaling. We remark that for negative fixed x the saturation $s(x,t)$ tends to $f_l \in (0,1)$ as $t \rightarrow \infty$, while for positive fixed x it tends to $f_r = 1$.

Finally, we investigate for the Brooks–Corey model which porosity and permeability ratios yield a discontinuous capillary pressure at $x = 0$. In Section 3 we derived that the capillary pressure is discontinuous if and only if $(\phi_r/\phi_l) < \beta_{cr}(h_r/h_l)$, where the function β_{cr} is given by equation (49). In Fig. 7 we have plotted the curve β_{cr} for $\lambda = 2$ and $M = 1$, and we indicated which region corresponds to a discontinuous capillary pressure. We observe that the point (0.5, 1), which corresponds to the data in Table 1, is indeed located in the region corresponding to a discontinuous capillary pressure.

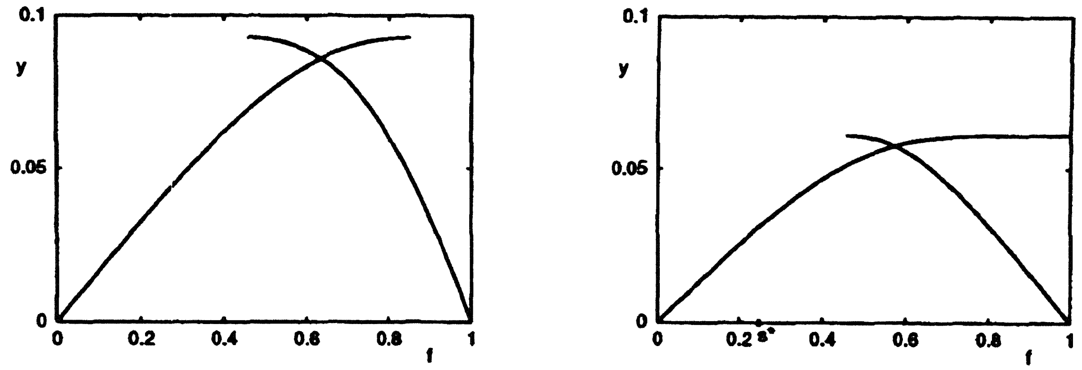

Fig. 4. Solutions in the flux-saturation plane for the Van Genuchten (left) and Brooks-Corey model (right).

5 DISCUSSION: PRESSURE CONDITION FOR UNSATURATED WATER FLOW

In this section we consider similarity solutions for horizontal unsaturated water flow in the case that the capillary pressure curve has a positive entry pressure. The equation governing unsaturated water flow is a limit case of the two-phase model presented in Section 2 for vanishing mobility ratio. For unsaturated water flow, similarity solutions can also be constructed when continuity of the capillary pressure is used as interface condition. We show that use of continuity of the capillary pressure as interface condition, as opposed to the extended pressure condition (15), leads to unphysical similarity solutions.

In unsaturated air–water flow, the pressure of air is assumed to be constant; water is the wetting and air is the nonwetting phase. The governing equation can be obtained from the two-phase model by letting $M \rightarrow 0$ in equation (3). It is thus given by equations (2) and (4) with $\bar{\lambda}(s)$ replaced by $k_{rw}(s)$. The capillary pressure is equal to $p_c = -p$, where p denotes the water pressure. Using equation (4), the water saturation can be expressed as a function of the water pressure, i.e. $s = S(p)$. Thus, equation (2) can be written as

$$\phi \frac{\partial}{\partial t}(S(p)) = \frac{\partial}{\partial x}\left(\frac{k}{\mu_w} k_{rw}(S(p)) \frac{\partial p}{\partial x}\right) \tag{56}$$

which is the pressure formulation of Richards' equation.[16] This formulation admits a varying water pressure when the water saturation is equal to one. If $s < 1$ then equation (56) is equivalent to equations (2) and (4) with $\bar{\lambda} = k_{rw}$. The interface condition for the capillary pressure needed at a discontinuity, however, cannot be derived directly by using a regularisation procedure as in [18].

When we use the extended pressure condition (15), in the sense that $-p(0^+) = \sigma(\phi_r/k_r)^{1/2} J(1)$ if $S(p(0^-)) > s^*$, we can construct a similarity solution of equation (56) in the same way as for the two-phase model. It can be shown that the similarity solution of the full two-phase model converges for vanishing mobility ratio to the solution of equation (56) with extended pressure condition. This is illustrated in Fig. 8.

For the pressure formulation (56) it is also possible to construct a similarity solution using continuity of the capillary pressure as interface condition instead of the extended pressure condition (15). It can be shown that this leads to a unique similarity solution as well. An example is given in Fig. 9, where we have plotted the graphs of the similarity solutions of equation (56) corresponding to the two different interface conditions. We observe that the solution obtained for continuous capillary pressure is equal to one in a region to the right of $\eta = 0$. The flux is constant in this region and the capillary pressure increases linearly. The saturation becomes less than one at the point where the capillary pressure exceeds the entry pressure. Further, we observe that the solution obtained for the extended pressure condition has a discontinuous capillary pressure at $\eta = 0$ in this case.

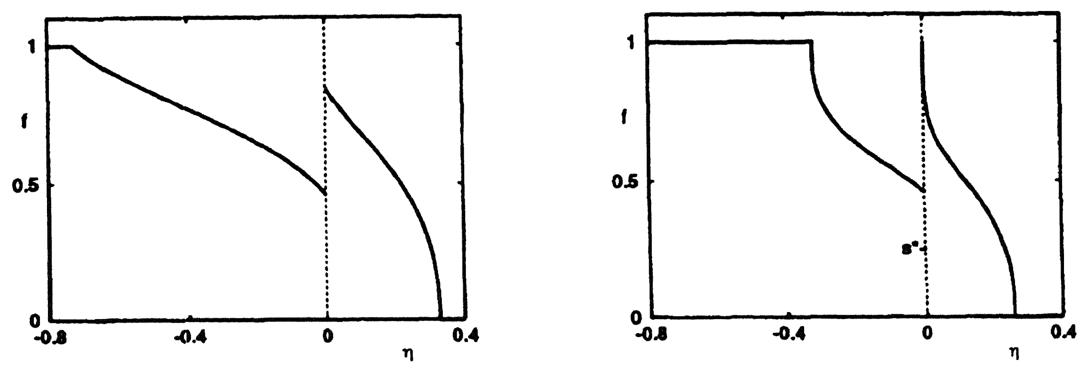

Fig. 5. Similarity profiles $f = f(\eta)$ for the Van Genuchten (left) and Brooks-Corey model (right).

Table 1. Data set of parameters

Parameter	Value	Parameter	Value
h_r	0.5	M	1
h_l	1	λ	2
ϕ_r/ϕ_l	1	m	2/3

The solution with continuous capillary pressure cannot be physically correct, however, because air entering on the left side has to pass through a region fully saturated by water. This problem does not occur when the extended pressure condition is used. Moreover, the similarity solution of equation (56) corresponding to the extended pressure condition is identical to the similarity solution of the two-phase model for vanishing mobility ratio. Thus, if the capillary pressure curve has a positive entry pressure, then the extended pressure condition is also in unsaturated flow problems an appropriate interface condition, whereas continuity of the capillary pressure is not.

6 CONCLUSION

We presented a method to construct a similarity solution of a capillary redistribution problem for a porous medium with a single jump discontinuity in permeability and porosity. Special attention was paid to the extended pressure condition, the interface condition corresponding to capillary pressure curves with a positive entry pressure. For both zero and positive entry pressure we showed that the interface conditions lead to a unique similarity solution. We indicated when the capillary pressure is discontinuous at the interface and how it depends on permeability and porosity contrasts.

We considered the similarity solutions for the Brooks–Corey model (positive entry pressure) and the Van Genuchten model (zero entry pressure). We showed for both models that free boundaries always occur. For the Brooks–Corey model we gave an example of a solution with discontinuous capillary pressure.

We discussed the pressure condition for unsaturated water flow when the entry pressure is positive. We demonstrated that continuity of capillary pressure may lead to unphysical solutions in this case. The extended pressure condition turned out to be appropriate for unsaturated water flow as well.

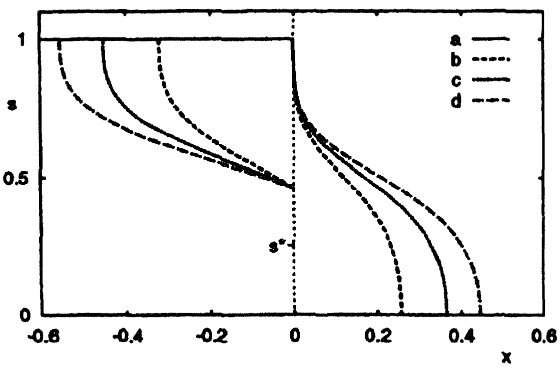

Fig. 6. Similarity solution for the Brooks-Corey model at (a) $t = 0$, (b) $t = 1$, (c) $t = 2$, and (d) $t = 3$.

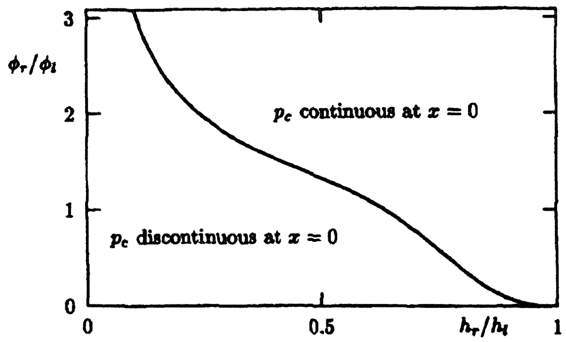

Fig. 7. Graph of the critical porosity ratio β_{cr} for the Brooks-Corey model with $\lambda = 2$ and $M = 1$.

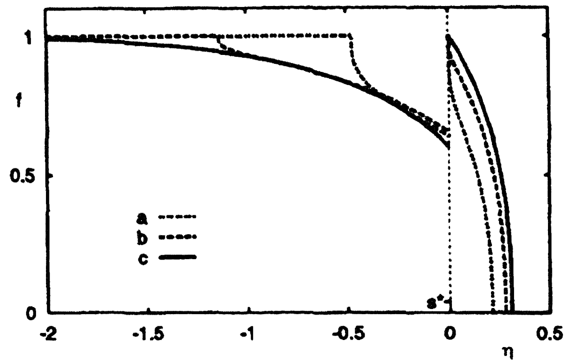

Fig. 8. Similarity solution of the two-phase model for vanishing mobility ratio: (a) $M = 0.1$, (b) $M = 0.001$, (c) similarity solution of (56) with extended pressure condition (Brooks-Corey model, $h_r = 0.2$, other data from Table 1).

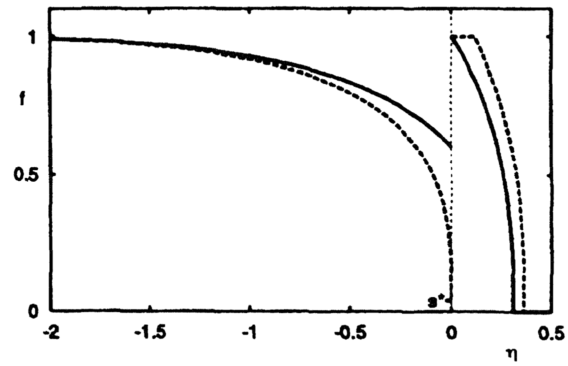

Fig. 9. Similarity solution of Richards' equation using continuity of capillary pressure (dashed) or the extended pressure condition (solid) for $h_r/h_l = 0.2$ (Brooks-Corey model, other data given in Table 1).

REFERENCES

1. Atkinson, F. V. & Peletier, L. A. Similarity profiles of flows through porous media. *Arch. Rat. Mech. Anal.*, 1971, **42**(5) 369–379.
2. Atkinson, F. V. & Peletier, L. A. Similarity solutions of the nonlinear diffusion equation. *Arch. Rat. Mech. Anal.*, 1974, **54**(4) 373–392.
3. Bailey, P. B., Shampine, L. F. and Waltman, P. E., *Nonlinear Two Point Boundary Value Problems*. Academic Press, 1968.
4. Brooks, R. H. & Corey, A. T. Properties of porous media affecting fluid flow. *J. Irrig. Drain. Div. Am. Soc. Civ. Eng.*, 1966, **92**(IR2) 61–88.
5. Buckley, S. E. & Leverett, M. C. Mechanism of fluid displacement in sands. *Trans. Am. Inst. Min. Metall. Pet. Eng.*, 1942, **146** 107–116.
6. Chen, Z.-X., Bodvarsson, G. S., Witherspoon, P. A. & Yortsos, Y. C. An integral equation formulation for the unconfined flow of groundwater with variable inlet conditions. *Transport in Porous Media*, 1995, **18** 15–36.
7. de Neef, M. J., Two-phase flow in heterogeneous porous media. Ph.D. thesis, Delft University of Technology, 1998. In preparation.
8. Fokas, A. S. & Yortsos, Y. C. On the exactly solvable equation $S_t = [(\beta S + \lambda)^{-2} S_x]_x + \alpha(\beta S + \lambda)^{-2} S_x$ occurring in two-phase flow in porous media. *SIAM J. Appl. Math.*, 1982, **42**(2) 318–332.
9. Kueper, B. H., Abbott. W. & Farquhar, G. Experimental observations of multiphase flow in heterogeneous porous media. *J. Contam. Hydrol.*, 1989, **5** 83–95.
10. Leverett, M. C. Capillary behavior in porous solids. *Trans. AIME, Petr. Eng. Div.*, 1941, **142** 152–169.
11. McWhorter, D. B. & Sunada, D. K. Exact integral solutions for two-phase flow. *Water Resour. Res.*, 1990, **26**(3) 399–413.
12. Parker, J. C., Lenhard, R. J. & Kuppusamy, T. A parametric model for constitutive properties governing multi-phase flow in porous media. *Water Resour. Res.*, 1987, **23**(4) 618–624.
13. Philip, J. R. A very general class of exact solutions in concentration dependent diffusion. *Nature*, 1960, **185** 233.
14. Philip, J. R. Horizontal redistribution with capillary hysteresis. *Water Resour. Res.*, 1991, **27**(7) 1459–1469.
15. Philip, J. R. Theory of infiltration. *Adv. Hydrosci.*, 1996, **5** 215–296.
16. Richards, L. A. Capillary conduction of liquids through porous mediums. *Physics*, 1931, **1** 318–333.
17. van Duijn, C. J. & Floris, F. J.T. Mathematical analysis of the influence of power-law fluid rheology on a capillary diffusion zone. *J. Pet. Sc. Eng.*, 1992, **7** 215–237.
18. van Duijn, C. J., Molenaar, J. & de Neef, M. F. The effect of capillary forces on immiscible two-phase flow in heterogeneous porous media. *Transport in Porous Media*, 1995, **21** 71–93.
19. van Duijn, C. J. & Peletier, L. A. A class of similarity solutions of the nonlinear diffusion equation. *Nonlinear Analysis, Theory, Methods and Applications*, 1977, **1**(3) 223–233.
20. van Duijn, C. J. & Peletier, L. A. A boundary-layer problem in fresh-salt groundwater flow. *Q. Jl. Mech. Appl. Math.*, 1992, **45** 1–23.
21. van Genuchten, M.Th. A closed-form equation for predicting the hydraulic conductivity of unsaturated soils. *Soil Sci. Soc. Am. J.*, 1980, **44** 892–898.
22. Yortsos, Y. C. and Jincai Chang, Capillary effects in steady-state flow in heterogeneous cores. *Transport in Porous Media*, 1990, **5**, 399–420.