ON MATHEMATICAL CONTROL ENGINEERING

Michiel HAZEWINKEL
CWI, Amsterdam
P.O. Box 4079
1009 AB AMSTERDAM

Abstract : Mathematical systems theory nowadays uses many and sophisticated tools from "pure" mathematics. Notably from algebraic and differential topology and from operator theory.

AMS 1980, classification : 93-02

Introduction

It used to be that when applying mathematics in sciences like engineering, chemistry or biology, and most of physics a quite modest mathematical training was sufficient. A fair amount of linear algebra, some differential equation theory, stability theory and transform theory, perhaps a bit of complex functions of one variable and a few special functions and one was well equipped to tackle most problems. Certainly, at best but little use was made of the tools of "pure" mathematics developed in fields like number theory, algebraic geometry, differential and algebraic topology, algebraic K-theory, operator theory and functional analysis, etc.. And these times are not all that long ago.[2)]

Things have changed in the last quarter century or so. Together with a strikingly increasing mathematization of many fields, there has come a shift in the kind of mathematics being used and nowadays there is an awareness that virtually any tool can an occasion be profitably used. It is interesting in this respect to compare the table of contents of, say, the journal of chemical physics of 10 years ago with that of last year. Or, more relevant of my present theme, the program of, say, the 10-th annual CDC (Conference on Decision and Control) and the program of the last one so far: the 22-nd of December 1983 in San Antonio.

One area in which all kinds of mathematics are used and which is turn generates many fascinating mathematical problems is mathematical (electrical) engineering a.k.a. mathematical systems theory.[1)]

In these few pages I will try to illustrate these remarks by means of a few selected examples of problem areas, combined with a few remarks concerning the mathematics used.

1. Systems

The basic object of study is a system, which is a device which accepts inputs or controls, which are functions of time, and which from these produces outputs or observations, also functions of time.

For example we might have a system governed by a finite dimensional system of ordinary differential or difference equations as in (1.1) and (1.2) below.

$$\dot{x} = f(x,u),\ y = h(x),\ x \in \mathbb{R}^m,\ u \in \mathbb{R}^m,\ y \in \mathbb{R}^p \tag{1.1}$$

$$x_{k+1} = f(x_k,u_k),\ y_k = h(x_k),\ x \in \mathbb{R}^n,\ u \in \mathbb{R}^m,\ y \in \mathbb{R}^p. \tag{1.2}$$

Here the u's are the inputs or controls, the y's are the outputs or observations and the x's are the state variables describing the true state of the system. Think for example of the historically important example of an aeroplane. Here the u's represent the controls, the x's the state of the plane in terms of mechanics, i.e. all relevant velocities and positions and the y's represent the observations, i.e. those quantities derived from the x's for which a measuring instrument is present.

Another example would be a (well-stirred) chemical reaction tank or a distillation column, with the x's representing various concentrations and perhaps additional variables like pressures and temperatures, the u's represent inputs, e.g. various reaction chemicals which are fed in and perhaps a heat source and the y's represent monitoring devices and or chemical end products which are drained away.

Equations (1.1) and (1.2) represent, as things go, but a relatively mildly difficult class of systems. That is there can be much more complicated objects which govern the dynamics. E.g. partial differential equations[4], or equations with delays, or integro-differential equations may appear instead of the differential or difference equations in (1.1) and (1.2)[6]. Not that such complications are necessary to make the mathematics interesting. Even in the very simplest case of linear systems, cf. section 3 below, very powerfull mathematics is useful and used.

One way to look at, say, equation (1.1) is to regard it as a mechanical system in the sense of classical mechanics, where now the u's represent external forces and the y's represent those functions of the state which are directly observable. In a way this seems a much better and more accurate set up to study nature than the more traditional one in which external forces are absent and in which one implicitly assumes that all the state variables are directly and precisely observable[3].

Abstractly the control part of (1.1), i.e. disregarding the outputs, is a family of O.D.E's parametrized by $u \in \mathbf{R}^m$ and one could expect that ideas from deformation theory, bifurcation theory, unfolding of singularities etc. could be very useful. This is quite likely true but this approach has not yet been tried out[5] seriously. Part of the reason for this is no doubt the kind of question one asks of a system like (1.1) and (1.2). These tend to be rather different from the classical questions of analysis pertaining e.g. to uniqueness and existence of solutions and properties of these solutions, as we shall see below.

2. Some typical questions from mathematical systems theory.

Consider again a system, like (1.1) for definiteness sake. Often there is a natural initial state $x_0 \in \mathbb{R}^n$ which is an equilibrium (not necessarily stable). From now on I shall assume that we are dealing with such an initialized system.

2.1. Stabilization by state feedback. A first natural question is then: can one find a function $k : \mathbb{R}^n \to \mathbb{R}^m$, i.e. from states to inputs, which stabilizes the initial state. That is such that $\dot{x} = f(x, k(x))$ has x_0 as a stable and asymptotically stable equilibrium point.

2.2. Stabilization by output feedback. State feedback is a natural concept to use when one is involved with the design of control systems. In other settings it is much more natural to use output feedback. I.e. the question now is whether there exists a map $k : \mathbb{R}^p \to \mathbb{R}^m$ from output space to input space such that $\dot{x} = f(x, k(h(x)))$ has x_0 as a stable and asymptotically stable equilibrium point.

2.3. Reachability. Given any $x \in \mathbb{R}^n$, does there always exist a control $u(t)$ such that the solution of $\dot{x} = f(x, u(t))$, $x(0) = x_0$ passes through x.

2.4. Observability. Given $x_1, x_2 \in \mathbb{R}^n$ and a known input $u(t)$. Let $y_i(t)$ be the observed ouput of $\dot{x} = f(x, u(t))$, $x(0) = x_i$, $i = 1,2$. Is it true that $x_1 \neq x_2$ implies $y_1(t) \neq y_2(t)$ for some t? So that the fact that $x_1 \neq x_2$ can be observed.

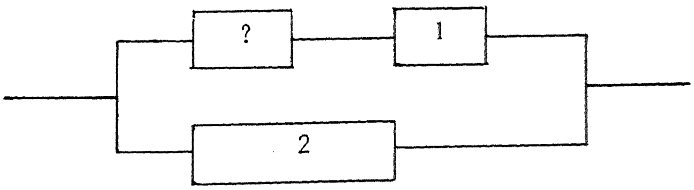
2.5. Realizability. The system (1.1) or (1.2) with initial state x_0 defines a (usually nonlinear) operator from a suitable space of input functions U (say compact support C^∞ functions $[0,\infty) \to \mathbb{R}^m$) to a suitable space of output functions Y (say C^∞ functions $[0,\infty) \to \mathbb{R}^p$). Indeed let $u(t) \in U$ be given. Let $x(t)$ be the solution of $\dot{x} = f(x, u(t)), x(0) = x_0$ (supposed to exist and to be unique). Let $y(t) = h(x(t))$. Then the associated operator V takes $u(t)$ to $y(t)$. Now suppose some operator $V : U \to Y$ given. When does there exist a system (of a certain specified type maybe) which has this operator as its associated input output operator[13].

2.6. Decoupling. Is it possible by, say, state-space feedback, to decompose the system into two subsystems such that the inputs of each subsystem only influence the outputs of the same subsystem.

2.7. Disturbance decoupling. Suppose now that in addition to the control's u there is a further set of input variables v to be regarded as disturbances so that now we are dealing with a system $\dot{x} = f(x,u,v)$, $y = h(x)$. Is it possible to find a $k(x)$ such that in $\dot{x} = f(x,k(x),v)$, $y = h(x)$ the influences of the v's do not anymore show up in the outputs y, or to find $k_\epsilon(x)$ for all ϵ such that the v's have only ϵ-influence on the outputs y.

2.8. Optimal control and synthesis. Suppose in addition to the control part of (1.1) we have a target set, say $x_1 \in \mathbf{R}^n$ and a cost functional $J\{x(\cdot),u(\cdot)\}$. Then the problem arises of finding the control $u(\cdot)$ which stears the system from x_0 to x_1 with minimal cost. For this type of problem the Pontryagin maximum principle is useful. This principle is not applicable if one asks for the least expensive feedback law $k(\ )$ which does the job. And the question arises, whether the optimal controls $u^{(x)}(\cdot)$ for each starting point x, which one finds for the usual optimal control problem, can be put together to yield an optimal feedback control law. The answer has much to do with whether the space of all $x \in \mathbf{R}^n$ which can be reached in time t from x_0 can be stratified (in the technical sense of differential topology) in a sufficiently nice manner [10], [11].

2.9. Model matching. Suppose given two systems 1 and 2, symbolically represented as in figure 1 below.



Can one find a system ? such that the serially connected pair: first ? then 1, reproduces exactly the input-output behaviour of system 2. This is of course but one example of a model matching problem.

All the questions so far, with the possible exception of 2.7 have dealt with deterministic questions, and systems. One also can and does consider systems (1.1) where now u is to be interpreted as a stochastic process, say white noise. This makes x and y stochastic processes and with one level of difficulty added many of the questions posed above can be repeated.

In addition there are a whole series of typically stochastic questions of which I mention only one.

2.10. Filtering. Consider the system $\dot{x} = f(x,u)$, where u is white noise, with observations corrupted by further noise $y = h(x) + v$. Given the observations y_s, $0 \leq s \leq t$, the problem is to calculate the conditional expectation of $x(t)$ given $y(s)$, $0 \leq s \leq t$.

3. Linear systems

A linear dynamical system (of finite dimension) is one of the form

$$\dot{x} = Ax + Bu, \ y = Cx, \ x_0 = 0. \tag{3.1}$$

These are so simple that an explicit formula can be written down for the associated input-output map. It is

$$y(t) = \int_0^t Ce^{(t-\tau)A} Bu(\tau)d\tau, \tag{3.2}$$

and it seems at first sight hard to believe that sophisticated mathematics will be needed to deal with such systems. By and large this is true in the sense that most of the questions listed in section 2 have satisfactory answers and that reasonable algorithms exist to compute these answers. A notable exception is the output feedback problem which is still mostly open.

Even so the answers we have are for single systems. Things change drastically as soon as families of linear systems (where now A, B, C depend on certain parameters) are considered. E.g. the question arises whether a stabilizing state-space feedback matrix K can be constructed depending continuously on the parameters such that $A + BK$ is stable for all parameters values[8]. Or whether it is always possible to find one fixed K which works for all parameters values in a certain compact range. The answer to that last question is: not always, and the analysis involves Stein spaces and complex interpolation theory [14].

Quite generally very little is known concerning which standard algorithms for solving various problems are continuous in the parameters and which problems admit continuous alternative algorithms.

This touches on both the local and global structure of the space of all linear systems. So let me take time out to say a few words about this space of all linear systems.

Let $L_{m,n,p}$ denote the space of all triples of matrices $\Sigma = (A, B, C)$ of sizes $n \times n$, $n \times m$, $p \times n$ respectively. Let $V(\Sigma)$ denote the associated input-output operator defined by (3.2). Then obviously if $S \in GL_n$,

$V(\Sigma^S)=V(\Sigma)$ if $\Sigma^S=(SAS^{-1},SB,CS^{-1})$. Thus Σ and Σ^S are indistinguishable from the input-output point of view and the space of all linear systems would be something like the orbit space $L_{m,n,p}\,/\,GL_n$, or even a quotient of this if the indeterminacy in the (A,B,C) description were even larger. This turns out to be mostly not the case in the following sense. For the dense open subspace $L^{co,cr}_{m,n,p}$ of $L_{m,n,p}$ of completely observable and completely reachable systems $V(\Sigma)=V(\Sigma^1)$ iff there is an $S\in GL_n$ such that $\Sigma^1=\Sigma^S$.

Thus the study arises of the orbit space $L^{cr,co}_{m,n,p}\,/\,GL_n$ which brings in invariant theory. It turns out that this space is far from trivial. E.g. $L^{cr,co}_{m,n,p}\longrightarrow L^{cr,co}_{m,n,p}\,/\,GL_n$ turns out to be a principal GL_n fibre bundle which is trivial iff $m=1$ or $p=1$. And even if $m=1=p$ the topology of this space of orbits still poses many questions [15]. And the matter is of some importance e.g. in connection with finding good identification procedures for finding the best linear system of type (3.1) for describing a given input-output map [16], [17][12)]. Here the Riemannian geometry of the orbit space is also important. Other approaches to identification involve Padé approximation and Nevalinna-Pick interpolation, cf. [19], and seem not to involve these topological and geometrical aspects. Which in itself is a bit surprising and calls for an explanation.

There are other groups acting on $L^{co,cr}_{m,n,p}$ whose invariants are obviously of interest in system theory. One is $GL_n\times GL_m\times GL_p$ where besides the action of GL_n already described (base change in state space), we have base change in input space $((A,B,C)\rightarrow(A,BT,C),T\in GL_m)$ and base change in output space $((A,B,C)\rightarrow(A,B,UC),U\in GL_p)$. Describing the orbit space $L^{co,cr}_{m,n,p}\,/\,GL_n\times GL_m\times GL_p$ is what is technically known as a wild problem (in the theory of representations of quivers and algebras) and practically nothing is known. The problem contains as a subproblem the one of classifying r-tuples of square matrices under similarity[11)].

Still another group is the feedback group acting on the space $L^{cr}_{m,n}$ of completely reachable pairs of matrices (A,B). This is the action of the Lie group generated by the transformations: base change in state space, base change in input space, and feedback: $(A,B)\rightarrow(A+BK,B)$. In this case it turns out that there are only discrete invariants and the quotient space is finite and identical with the space of all partitions into at most m parts of n. The topology of $L^{cr}_{m,n}$ induces a partial ordering on this set of partitions and this turns out to be a partial order which occurs all over mathematics [39], [40]. E.g. the same order occurs when studying orbits of nilpotent matrices, vectorbundles over the Riemann sphere and representations of the symmetric groups. It turns out that this is not an accident, [40].

There are still more reasons to study the orbit space $L_{m,n,p}^{cr,ro} / GL_n$. Realization theory with parameters is one and an other is the study of linear systems with delays, which, it turns out, can sometimes be profitably done by assigning a family of systems to them essentially by treating the delay operators as parameters. This trick, combined with an application of the Quillen-Suslin theorem on the triviality of algebraic vectorbundles over affine space, yields one of the better, known stabilization results for delay systems [24].

Many of the topics touched on in this section are discussed more fully in the survey papers [25,26]; and the lecture notes [27]; still more is contained in the volumes [28], [29].

To end this very sketchy, incomplete and global description of some of the interactions of linear systems and control theory with various parts of (pure) mathematics let me say a few words about Kalman filtering, cf. 2.10 above. The system under consideration is the one described by the stochastic differential equation

$$dx = Axdt + Bdw, \quad dy = Cxdt + dv \tag{3.3}$$

where w,v are independent Wiener noise processes. The problem is to calculate $\hat{x}(t) = E[x(t)|y(s), 0 \leqslant s \leqslant t]$, the conditional expectation of $x(t)$ given the observations $y(s)$, $0 \leqslant s \leqslant t$. This is solved by the Kalman-Bucy filter $d\hat{x} = A\hat{x}dt + PC^T(dy - C\hat{x}dt)$, $dP = (AP + PA^t + BB^T - PC^T CP)dt$.

The equation for P, which is an ordinary system of equations (no stochastic component, a remarkable fact), is the matrix Riccati equation, an equation which seems to have a habit of turning up everywhere in mathematics; e.g. in transport theory, in the theory of solutions, in factorization problems,...,so much so that a monograph has appeared on it [41] and another is in preparation [42]. I will come back both to filtering and the Riccati equation in section 4 below.

4. Nonlinear systems.

Given a quite successful linear theory and faced with having to deal with nonlinear objects there are two obvious things to try: (i) try to find good nonlinear analogues of the concepts which served well in the linear case and (ii) try to linearize in one way or another.

Both roads of investigation are being tried out with considerable energy in the systems and control world and I shall try to say a few things about both.

4.1. Nonlinear realization theory. The appropriate tool for linear realization theory were block Hankel matrices, or perhaps more precisely the power series development of the transfer function $T(s)=C(sI-A)^{-1}B$ which gives the relation between the Laplace transforms of the input and output functions. At least the criterium for realizability is formulated in terms of the block Hankel matrix associated to (the power series development in s^{-1}) of this proper rational matrix function. This part of realization theory seemes to have found a natural nonlinear extension in the framework of noncommutative power series and Volterra series expansions [31,32,33,34].

As usual when generalizing, more aspects emerge and for other parts of realization theory (existence and uniqueness results for realizations on smooth manifolds) a substantial amount of differential topology gets involved [35].

4.2. Controlability observability and feedback stabilizability. In linear system theory the socalled *A mod B* invariant subspaces and controllability subspaces play on important role, as well as certain generalizations [22] and the right dual notions [43]. Here a subspace $V \subset \mathbf{R}^n$ is *A mod B* invariant (for the control system $\dot{x}=Ax+Bu$) if there exists a feedback matrix K such that V is $A+BK$ invariant: in other words, if for any $x(0)\in V$, one can always find a $u(t)$ so that $x(t)$ stays in V. This notion is important in e.g. decoupling and disturbance decoupling, cf. 2.6 and 2.7 above.

One possible nonlinear substitute for linear vectorspace is vectorbundle with subspaces corresponding to subbundles. In this case it turned out that instead of looking at subspaces V of state space $\mathbf{R}^n$ one should consider distributions, i.e. subbundles of the tangent bundle TM to the state-space manifold, and of course also the corresponding foliations. And by now the outlines of a satisfactory nonlinear theory with respect to these problems is emerging, cf. e.g. [36], [46], [47], [48]. Assuming that everything is more or less the best in all possible worlds it is now perhaps reasonable to guess that the obstructions to stabilization lie in the cohomology of suitable foliation quotients. And it appears that indeed they do [56].

4.3. Nonlinear filtering. This is the nonlinear analogue of the problem briefly described in 2.10. Thus now we have a nonlinear system, given by stochastic differential equations

$$dx = f(x)dt + G(x)dw, \quad dy = h(x)dt + dv, \tag{Σ}$$

and the question is whether there exist finite dimensional recursive filters for calculating the conditional expectation $\hat{x}(t) = E[x(t)|y(s), 0 \leqslant s \leqslant t]$. Here by definition a finite dimensional recursive filter is itself a finite dimensional system that is driven by $y(t)$ and has output $\hat{x}(t)$:

$$dm = \alpha(m)dt + \beta(m)dy, \quad \hat{x} = \gamma(m).$$

Note that the Kalman-Bucy filter of 2.10 above is precisely such a machine. A beautiful idea due to Brockett, Clark and Mitter [60,61] states that the complexity of the filtering problem and the existence of such a filter are related to the structure of the Lie algebra of differential operators which occur in the stochastic evolution equation which governs the evolution of an unnormalized version of the probability density of $\hat{x}(t)$, the socalled Duncan-Mortensen-Zakai equation. The resulting Brockett-Clark homomorphism principle[14)] then rapidly leads to the study of things like the Lie algebra of all partial differential operators (any order) with polynomial coefficients and such questions as: which are its finite dimensional subagebras; what is the automorphism group; which are the maximal subalgebras; which of these can be embedded in a Lie algebra of vectorfields etc. [49,50,51,52,53].

Of great importance in this connection are "robustness" results, i.e. regularity and continuity results pertaining to solutions of such equations as the DMZ-equation. Here the Malliavin stochastic calculus plays an important role [58,57].

An idea of actual developments in filtering may be gleaned from the proceedings of the recent (Febr. 1983) Colloque $ENST-CNET$: "Developpments recents dans le filtrage et le contrôle des processus aléatoires" (H. Korezlioglu, G. Mazziotto, J. Szirglas (eds.)).

4.4. Linearization. One of the best known and most studied linearization problems, is the one tackled by Poincaré in his thesis. Consider a system of ordinary differential equations

$$\dot{x} = f(x), \ f(0) = 0, \ f(x) = Ax + \textit{higher order terms} \tag{*}$$

The question is a when there is nonlinear (local) diffeomorphism $\phi: \mathbb{R}^n \to \mathbb{R}^n, y = \phi(x)$ which makes $(*)$ equivalent to its linear part

$$\dot{y} = Ay$$

More generally one can ask this for an m-tuple of such differential equations

$$\dot{x}_i = f_i(x),\ f_i(0)=0,\ x_i \in \mathbb{R}^n,\ \dot{y}_i = A_i y_i,\ y_i \in \mathbb{R}^n.$$

Then a first obvious necessary condition is that the Lie algebra generated by the vectorfields $f_1,\ldots,f_n$ and the $A_1,\ldots,A_n$ are isomorphic (under $f_i \to A_i$). This problem is already quite close to the problem of when a nonlinear control system

$$\dot{x} = f(x,u),\ x \in \mathbb{R}^n,\ u \in \mathbb{R}^m,\ f(0,0) = 0,$$

can be linearized, especially in the case $f(x,u)$ is of the form $f(x)+\sum_{i=1}^{m} g_i(x)u_i$.

This question and various related questions such as when linearization is possible if one allows nonlinear feedback as well, or when a system is up to diffeomorphism or up to feedback embeddable in a linear one has had a great deal of attention recently [66-68], [71-72][15]. Moreover these linearization techniques are important in actual applications (to automatic flight control systems [69] and e.g. robotics[15].

The matrix Riccati equations, cf. 2.10 above, are linearizable in a covering sense as follows. Via $P \to$ space spanned by the rows of $[I\ P]$, where I is the $n \times n$ identity matrix, the space of all $n \times n$ matrices is densely embedded in the Grassmann manifold $G_n(\mathbb{R}^{2n})$ of all n-planes in $2n$-space. The Riccati flow extends to all of $G_n(\mathbb{R}^{2n})$ (and this enables one to get a handle on the finite escape time properties of the Riccati flow [77, 78]). Let V be the space of all rank n matrices of size $n \times 2n$. Then there exists a linear flow on V which descends to $G_n(\mathbb{R}^{2n})$. This is the same type of linearization which occurs in the context of completely integrable Hamiltonian systems (e.g. the Toda lattices) and it seems to be responsible for (unusually many) symmetries and conservation laws and superposition principles. Cf. [79, 80] for the matrix Riccati equation in this respect.

4.5. Other nice classes of systems. Linear systems are nice and we know a lot about them but they do not constitute a sufficiently rich class of models in many circumstances. Nonlinear systems in full generality are too general for the moment and often we do not even know what are the right questions to ask. Thus the search is on for a class of systems which is sufficiently regular to ressemble the linear class somewhat and also sufficiently different so that new phenomena may appear. Systems on Lie groups homogeneous w.r.t. the group translations in one way or another suggest themselves. As has been the case with several current topics in mathematical system theory the first important paper in this direction was written by Roger Brockett [59], cf. e.g. [74, 76] for more results on the topic. It turns out that the linear systems and the systems considered by Brockett are the extreme cases of a general class of systems. corresponding to respectively an abelian Lie group and a semisimple Lie group.

4.6. Concluding remarks. Practically all of the above has been written from the state-space point of view of mathematical system theory which owes a major debt to the ideas of R.E. Kalman and the enormous commercial and industrial sucess of the (state space based) Kalman-Bucy filter. Whether the minimal state-space point of view will still be the major paradigm when dealing (in the near future) with very large scale systems and very large number of (similar) component systems is debatable [65].

However it is clear that a large number of techniques, ideas and results from algebraic geometry, module and representation theory, Lie algebras and groups, operator theory, differential geometry and topology, algebraic topology, one and several complex variables, Hardy spaces, functional analysis,..., are finding interesting applications in system and control theory and that vice versa mathematical engineering is generating hard problems in these fields. This trend seems sure to persist.

And even though time, space, and knowledge ordained that I should neglect all the fascinating interrelations between mathematical system and control theory and e.g. functional analysis, H-spaces, and factorisation theory[16)] as well as many other topics[17)], I hope that the preceding pages may have given the reader some idea of why this might be the case.

Notes.

1) A quick glance of the table of contents of [1] gives already a fair indication to what extent this statement is quite simply true.
2) For some more remarks along these lines cf. e.g. [3] and [4].
3) Hamiltonian mechanical systems with inputs are the subject of thesis [2]
4) Think e.g. of a metallic rod, with temperature sensors attacked at a finite number of points and say a controllable heat source at one end.
5) All the same, one authoritative set of lectures [6] was colloquially announced as "redoing Whittaker and Watson with controls".
6) Any of these kinds of systems may be the most natural one in

modelling a given set of phenomena or processes. All are vigorously investigated as a glance at the contents of [5] and its predecessors will show.

7) This does not mean that e.g. when dealing with a system governed by a PDE such questions of conservation laws, shock waves, propagation of singularities etc. should not be important. They probably are. But, again, these matters have not yet received much attention in this context. Even such obviously important matters as symmetries on the one hand and the study of the natural quantum mechanical analogues of (1.1) have only recently begun to receive serious attention [7], [8], [9]. Before these matters came up there were other questions which were tackled first.

8) This touches on adaptive and self organizing control, a field which so far consists of an enormous collection of open problems, and a number of algorithms that work even if we don't know why precisely. For more cf. [12], [13].

9) Indeed there may very well exist an continuous algorithm for obtaining a certain matrix e.g. locally everywhere, without it being possible that global such algorithm exists. Think e.g. of sections of a fibre bun See also [18] and [16] for results on the topology of the orbit space.

11) For the classification of pairs of matrices under simultaneous similar also a wild problem, cf. [21].

12) There are also several reasons for studying various possible compactifications of the space of orbits. Identification is one (for obvious reasons). Others are high-gain feedback [22] and dynamic output feedback [23].

13) Incidentally, though finite dimensional linear realization theory is in a satisfactory state, there is still a lot to do (and going on) with respect to its stochastic cousin. Cf. e.g. [44], [45].

14) An approach to a general proof of the Brockett-Clark homomorphism principle is sketched in [54]. A verification of the principle for linear systems and the particular nonlinear filtering problem posed by the identification of linear systems can be found in [55].

15) A special session was devoted to this topic at the last CDC (San Antonio, Dec. 1983). A number of additional references can be found in [70].

16) Fortunately there is a nice survey paper [81] which deals with at least

some of these interrelations. And the related but rather differently oriented material of [82] is at least equally fascinating and promising.

17) The proceedings of the yearly IEEE CDC's (Conference on Decision and Control), published by the IEEE (Inst. of Electronic and Electrical Engineers) and the proceedings of the biannual MTNS conferences (Mathematical Theory of Networks and Systems) will give the interested mathematician a fair idea of what goes on in the mathematical (electrical) engineering world. The proceedings volumes [1,20,28,29,63,64] together will also give some idea.

References

[1] I.D. Landau (ed.), *Outils et modèles mathématiques pour l'analyse des système et le traitement du signal.* Trois volumes, CNRS 1981, 1982, 1984.

[2] A.J. van der Schaft, *System theoretic descriptions of physical systems.* Thesis, Groningen, 1983 (cf. also Math. Syst. Sept. Theory **15** (1982), 145-168; System and Control Letters **1** (1981), 108-115).

[3] M. Hazewinkel, *The art of applying mathematics,* Acta Appl. Math. **1** (1983), 1-3.

[4] M. Hazewinkel, *Experimental mathematics,* To appear in Computers & Math. and Appl. and in J.W. de Bakker, M. Hazewinkel, J.K. Lenstra (eds). Proceedings of the CWI Nov. 1983 Symposium on Mathematics and Computer Sci., CWI & North Holland Publ. G, 1984.

[5] Proceeding 22nd CDC, San Antonio 1983, IEEE, 1983.

[6] R.W. Brockett, *Control theory and analytical mechanics,* In [64], 1-48.

[7] M. Hazewinkel & C.F. Martin, *On decentralisation, symmetry and special structure in linear systems,* In [5].

[8] J. Grizzle & S.I. Marcus, *The structure of nonlinear control systems possessing symmetries.* In [5].

[9] G. Huang, T.J. Tarn & J.W. Clark, *On the controllability of quantum-mechanical systems,* J. Math. Physics **24** (1983), 2608-2618.

[10] P. Brunovsky, *On the structure of optimal feedback systems,* Proc. ICM Helsinski 1978, 841-846.

[11] H.J. Sussmann, *Analytic statifications and control theory,* Proc. ICM

Helsinki 1978, 865-872.

[12] I.D. Landau, *Deterministic and stochastic model reference adaptive control,* In [20], 387-420.

[13] G. Saridis, *Self-organizing control,* M. Dekker, 1981.

[14] A. Tannenbaum, *The blending problem and parameter uncertainty in control theory.* Preprint 1980. Cf. also Int. J. Control **32** (1980), 1-16.

[15] G. Segal, *The topology of spaces of rational functions,* Acta Math. **143** (1979), 39-72.

[16] D. Delchamps, Ph. D. Thesis, 1982, Harvard Univ.

[17] B. Hanzon, Thesis 1984, Erasmus Univ. Rotterdam.

[18] U. Helmke, *Geometry of the space of linear systems,* Proc. 21st CDC, Orlando, 1982, 948-949.

[19] P. de Wilde, J.T. Fokkema & I. Widya, *Inverse scattering and linear prediction,* In [20].

[20] M. Hazewinkel & J.C. Willems (eds), *Stochastic systems: th mathematics of filtering and identification and applications,* Reid Publ. Cy. 1981.

[21] S. Friedland, *Simultaneous similarity of matrices,* Adv. Math. **50** (1983), 189-265.

[22] J.C. Willems, *Almost invariant subspaces: an approach to high again feedback design I, II,* IEEE Trans. AC **26** (1981), 235-252, ibid. **27** (1982), 1071-1085.

[23] C. Byrnes, *Compactifications of spaces of systems and dynamic compensators,* Proc. 22nd CDC, San Antonio 1983.

[24] C. Byrnes, *On the control of certain deterministic infinite dimensional systems by algebro-geometric techniques,* Amer. J. Math. **100** (1979), 1333-1380.

[25] M. Hazewinkel, *(Fine) moduli (spaces) for linear systems; what are they and what are they good for,* [29], 125-193.

[26] M. Hazewinkel, *A partial survey of the uses of algebraic geometry in systems and control theory,* In: Symp. Math. INDAM **24**. Acad. Pr., 1981, 245-292.

[27] M. Hazewinkel, *Lectures on invariants, representations and Lie algebras in systems and control theory,* In: *Sém d'Algèbre* P. Dubreil,

M.-P. Malliavin 1981/1982. Lect. Notes Math., to appear.

[28] C.I. Byrnes, C.F. Martin (eds), *Linear systems theory,* AMS Lect. Notes Appl. Math. **18** , 1980

[29] C.I. Byrnes, C.F. Martin (eds), *Geometric methods for linear system theory,* Reidel Publ. Cy., 1980.

[30] R.E. Kalman, P.L. Falb & M.A. Arbib, *Topics in system theory, McGraw-Hill, 1966.*

[31] M. Fliess, *Nonlinear realization theory and abstract transitive Lie algebras,* Bull. Amer. Math. Soc. **2** (1980), 444-446.

[32] M. Fliess, *Fonctionelles causales nonlinéaires et indeterminés noncommutatives,* Bull. Soc. Math. de France **109** (1981).

[33] P.E. Crouch, *Polynomic systems theory; a review,* IEE Proc. **127** (1980), 220-228.

`4] M. Schetzen, *The Volterra and Wiener theories of nonlinear systems,* Wiley, 1980.

5] H.J. Sussmann, *Existence and uniqueness of minimal realizations of nonlinear systems,* Math. Sys. Theory **10** (1977), 263-284.

[36] R. Curtain & A.J. Pritchard, *Infinite dimensional linear systems theory,* Lect. N. Control and Inf. Sci. **8**, Springer, 1978.

[37] C. Byrnes & P.K. Stevens, *Pole placement by static and dynamic output feedback,* Proc. 21-st CDC, Orlando 1982, 130-137.

38] J. Baillieul & C.I. Byrnes, *Remarks on the number of solutions to the load flow equations for a power system with electrical losses,* Proc. 21-st CDC, Orlando 1982, 919-924.

39] A.W. Marshall & J. Olkin, *Inequalities: theory of majorization and its applications,* Acad. Pr., 1979.

)] M. Hazewinkel & C.F. Martin, *Representations of the symmetric groups, the specialization order, Schubert cells and systems,* Ens. Math. **29** (1983), 53-87.

] W.T. Reid, *Riccati differential equations,* Acad. Pr., 1972.

2] C.F. Martin, *The Riccati equation,* To appear Reidel Publ. Cy., 1985.

[43] J.M. Schumacher, *Dynamic feedback in finite and infinite dimensional linear systems,* Thesis, Amsterdam, 1981 cf. also IEEE Trans. AC **25** (1980), 1133-1138.

[44] A. Lindquist & G. Picci, *State space models for Gaussian stochastic processes,* In: [20], 169-204.

[45] G. Picci & J. van Schuppen, *On the weak finite stochastic realization problem,* preprint BW 184/83, CWI, Amsterdam.

[46] H. Nijmeijer, *Nonlinear multivariable control: a differential geometric approach,* Thesis, Groningen, 1983. Cf. also Syst. and Control Lett. **2** (1982), 122-129.

[47] A. Isidori, A.J. Krener, C. Cori-Giorgi & Z. Monaco, *Nonlinear decoupling via feedback: a differential approach,* IEEE Trans. AC **26**, 331.

[48] J.M. Schumacher, *Finite dimensional regulators for a class of infinite dimensional systems,* Systems & Control Lett. **3** (1983), 7-12.

[49] M. Hazewinkel & S.I. Marcus, *On Lie algebras and finite dimensional filtering,* Stocshastic **7** (1982), 29-62.

[50] R.W. Brockett, *Nonlinear systems and nonlinear estimation theory,* In [20], 441-478.

[51] S.K. Mitter, *Nonlinear filtering and stochastic mechanics,* In [20] 479-504.

[52] D. Ocone, *Topics in nonlinear filtering theory,* Thesis, M.I.I., 1980, Cf. also [20], 629-636.

[53] M. Hazewinkel, S.I. Marcus & H.J. Sussmann, *Nonexistence of finite dimensional filters for conditional statistics of the cubic sensor,* Systems and Control Letters **3** (1983), 331-340.

[54] O. Hijab, *Finite dimensional causal functionals of Brownian motion,* In: [63], 425-436.

[55] M. Hazewinkel, *The linear systems Lie algebra, the Segal-Weil representation and all Kalman-Bucy filters,* In: Proc. MTNS 83, Beer Sheva, Lecture Notes in Control and Information Sciece, Springer, to appear.

[56] C.I. Byrnes, In: Proc. MTNS 83, Lecture Notes in Control and Information Science, Springer, to appear.

[57] D. Michel, *Régularité des lois conditionelles en théorie du filtrage non-linéaire et calcul des variations stochastique,* J. of Funct. Anal. **41** (1981), 8-36.

[58] M. Chaleyat-Maurel & D. Michel, *Hypoellipticity theorems and*

conditional laws, preprint 1982.

[59] R.W. Brockett, *Systems theory on group manifolds and coset spaces,* SIAM J. Control **10** (1972), 265-284.

[60] R.W. Brockett & J.M.C. Clark, *The geometry of the conditional density equation,* In: O.L.R. Jacobs, a.o. (eds), Analysis and optimization of stochastic systems, Acad. Pr., 1980, 299-309.

[61] S.K. Mitter, *On the analogy between mathematical problems of nonlinear filtering and quantum physics,* Ricerche di Automatica **10** (1980), 163-216.

[62] G. Ruckebusch, *On the structure of minimal markovian representations,* In [63], 111-122.

[63] R.S. Bucy , J.M.F. Moura (eds), *Nonlinear stochastic problems,* Reidel Publ. Cy., 1983.

[64] C. Martin & R. Hermann (eds), *Geometric control theory,* Math. Sci. Press, 1977.

65] J. Zaborsky, *Development of systems science-past, present, and future,* Plenary address, IFAC, Aug. 1984, Budapest.

[66] B. Jakubczyk & W. Respondek, *On the linearization of control systems,* Bull. Acad. Pol. Sci. Ser. Sci. Math. Astron. Phys. **28** (1980), 517-522.

[67] R.W. Brockett, *Feedback invariants for nonlinear systems,* IFAC Congres, Helsinki 1978.

[68] L.R. Hunt & S. Su, *Linear equivalence of nonlinear time-varying systems,* Proc. MTNS 1981 (Santa Monica), 119-123.

[69] G. Meijer & L. Cicolani, *A formal structure for advanced automatic high control systems,* NASA TN D-7940 (1975).

[70] M. Hazewinkel, *Notes on (the philosophy of) linearization,* Preprint ZN, CWI Amsterdam, 1984.

[71] S. Monaco & D. Normand-Cyrot, *The immersion under feedback of a multidimensional discrete-time nonlinear system into a linear system,* Int. J. Control **38** (1983), 245-261.

[72] D. Claude, M. Fliess & A. Isidori, *Immersion, discrete et par bouclage, d'un systeme nonlinéaire dans un linéaire,* CR Acad. Sci. Paris **296** (1983), 237-240.

[73] M. Hazewinkel, *Control and filtering of a class of nonlinear but

homogenous systems, Lect. N. Control and Inf. Sci. **39** (1982), 123-146.

[74] V. Jurdjevic & I. Kupka, *Control systems on semi-simple Lie groups and their homogenous spaces*, Ann. Inst. Fourier **31** (1981), 151-179.

[75] B. Bonnard, *Controllabilité et observabilité d'une certain classe de systèmes nonlinéaires*, preprint.

[76] V. Jurdjevic & I. Kupka, *Polynomial control systems*, Preprint 1983.

[77] C. Martin, *Grassmann manifolds and global properties of the Riccati equation*, MTNS 1977 (Lubbock Texas), 82-85.

[78] R. Hermann & C. Martin, *Lie theoretic aspects of the Riccati equation*, CDC 1977 (New Orleans).

[79] M. Shayman, *A symmetry group for the matrix Riccati equation*, Systems and Control Lett. **2** (1982), 17-24.

[80] J. Harnad, P. Winternitz & R.L. Anderson, *Superposition for matrix Riccati equations*, J. Math. Phys. **24** (1983), 1062-1072.

[81] J.W. Helton, *Non-euclidean functional analysis and electronics*, Bull. Amer. Math. Soc. **7** (1982), 1-64.

[82] H. Bart, J.C. Gohberg & M. A. Kaashoek, *Minimal factorization of matrix and operator valued functions*, Birkhauser, 1979.