BIFURCATION PHENOMENA. A SHORT INTRODUCTORY TUTORIAL WITH EXAMPLES

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1. INTRODUCTION

Many problems in the physical and the social sciences can be described (modelled) by equations or inequalities of one kind or another. E.g. simple polynomial equations such as

$x^3 - 2x^2 + 3x - 4 = 0$ or a difference equation $x(t+1) =$

$2y(t)+x(t)$, $y(t+1) = 2x(t)-y(t)$, or a differential equation

$\dot{x}(t) = - x^2(t) + \sin t$, or much more complicated equations

such as integro-differential equations, etcetera. In such a case a large part of solving the problem consists of solving the equation(s) and describing various properties of the nature of the solution (such as stability). Almost always such equations contain a number of parameters whose values are determined by the particular phenomenon being modelled. These are then usually not exactly known and may even change in time either in a natural way or because they are in the nature of control variables which can be adjusted to achieve certain goals.

Thus it becomes natural and important to study families of equations, e.g. $x(t+1) = f(x,\lambda)$ or $g(x,\lambda) = 0$ depending on a parameter λ and to study how the set of solutions of such an equation varies (in nature) as the parameters vary. This is the topic of bifurcation theory. Roughly speaking $\lambda_0 \in \Lambda$ (= parameter space) is a <u>bifurcation point</u> if the nature of the set of solutions of the family of equations changes at point λ_0.

This chapter tries mainly by means of a few examples from

M. Hazewinkel et al. (eds.), Bifurcation Analysis, 13–30.

physics and economics to give a first idea of what bifurcation theory is about.

2. EXAMPLE. BALL IN A HILLY LANDSCAPE

Consider a one dimensional landscape (depending on a parameter [1]) given by the (potential energy) expression

$$E(\lambda,x) = \tfrac{1}{4}x^4 - x^2 - (2\lambda - 4)\,x + 9 \tag{2.1}$$

E.g. if $\lambda = 2$, then $E(\lambda,x)$ looks as follows

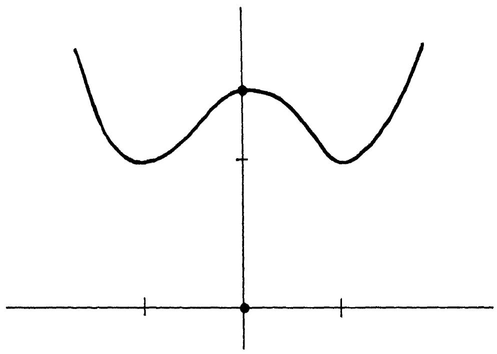

Figure 1.

We are interested in finding the equilibrium positions of a ball in this landscape. If $\lambda = 2$ these equilibrium positions are clearly $x = -\sqrt{2}$, $x = 0$ and $x = \sqrt{2}$.

In general these equilibrium positions are given by

$$E_x(\lambda,x) = x^3 - 2x - (2\lambda - 4) = 0 \tag{2.2}$$

Depending on λ this equation has one, two or three solutions. To be precise it has one solution for $\lambda < 2 - \frac{2}{9}\sqrt{6}$ and $\lambda > 2 + \frac{2}{9}\sqrt{6}$, it has two solutions for $\lambda = 2 \pm \frac{2}{9}\sqrt{6}$ and it has three solutions for $2 - \frac{2}{9}\sqrt{6} < \lambda < 2 + \frac{2}{9}\sqrt{6}$ A graph of the set of solutions as a function of λ looks roughly as in figure 2.

Bifurcation points are the points where the nature of the total set of solutions changes. Therefore there are two bifurcation points namely, the points $\lambda = 2 - \frac{2}{9}\sqrt{6}$ and $\lambda = 2 + \frac{2}{9}\sqrt{6}$

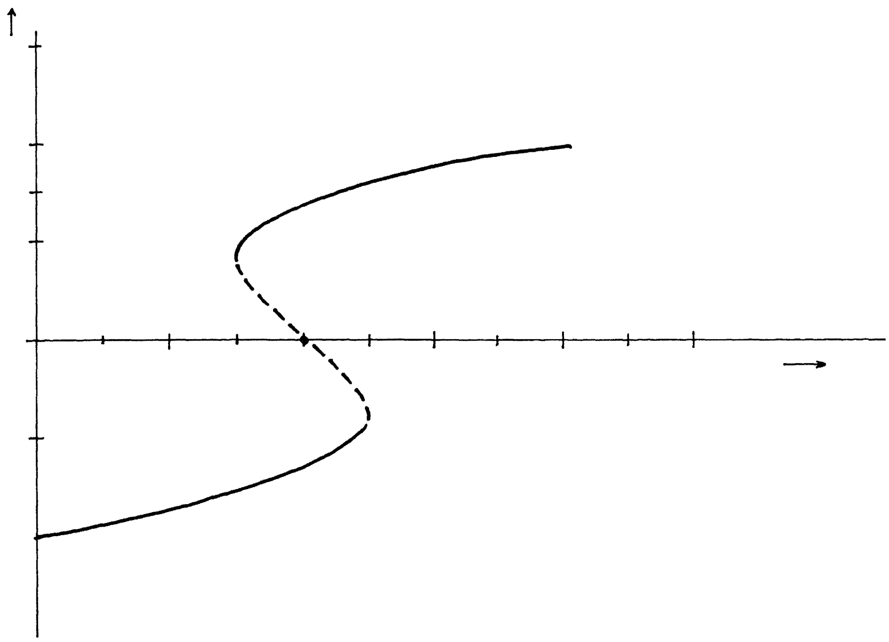

Figure 2.

Below is a picture (Figure 3) of $E(\lambda,x)$ as a function of both λ and x in which one can see how $E(\lambda,x)$ changes as λ varies.

Viewed from a different angle $E(\lambda,x)$ looks as below in Figure 4.
Here the dotted line indicates the equilibrium points: that is the bottoms of valleys and the tops of hills (see also figure 2!). As we walk through this two dimensional landscape in the direction of increasing λ we are first in a simple valley; then a new valley starts somewhere on the right hand slope so that shortly after we see on our right a moderate hill followed by the new valley followed by the original - so to speak - slope; going further still, the original valley peters out and only the new valley survives so that in the end we are again in a simple valley.

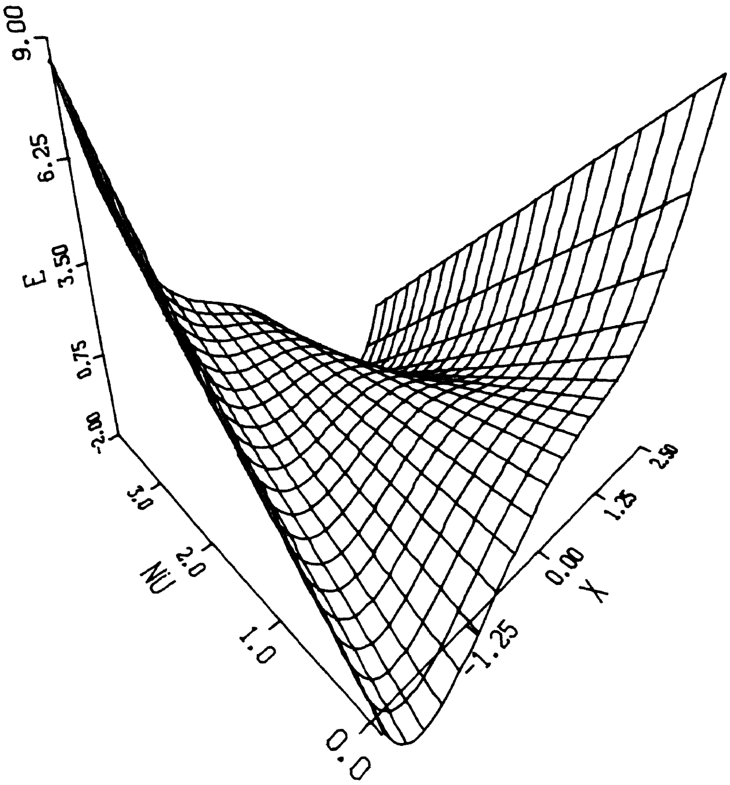

Figure 3.

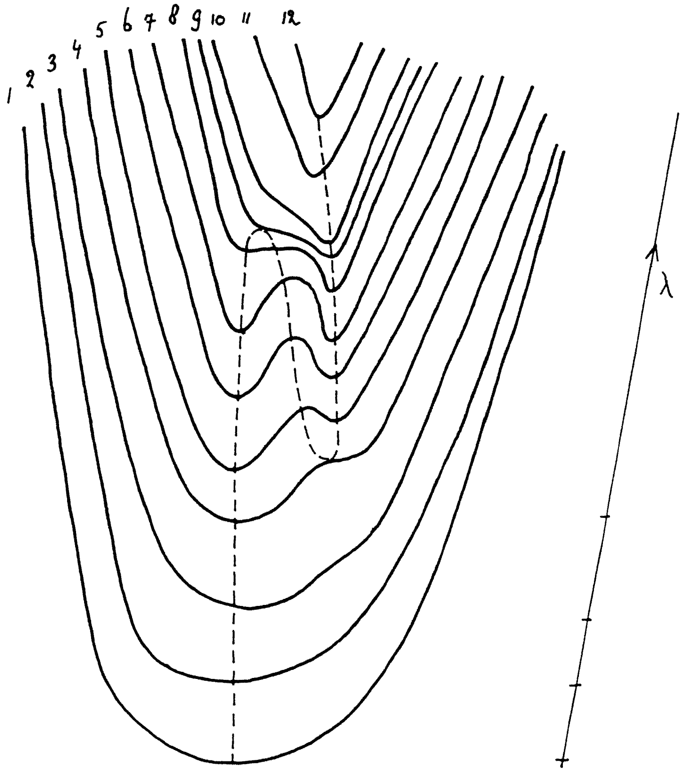

Figure 4.

3. STABILITY

Consider a ball in the landscape described by $E(\lambda,x)$. Its dynamics will be given by $\ddot{x} = -E_x(\lambda,x) = -(x^3 - 2x - (2\lambda-4))$. An equilibrium point x_0 will be called stable if for x' near enough x_0 the solution of this differential equation, starting in x' at time t=0 will remain near x_o for all $t > 0$.

Obviously in our example the equilibrium points given by the solid lines in figure 2 are stable (the bottoms of the valleys) and the ones given by the dotted part are unstable (tops of hills).

4. HYSTERESIS

It is easy to see what will happen if we kick a ball along as we are walking through the landscape of figure 4. The ball will roll along the bottom of the first valley until this one peters out (cross section number 9), there it will suddenly roll down the slope to come to rest at the bottom of the new valley and as λ increases further it will roll along the bottom of this second valley. If we then return, i.e. we let λ decrease, it will roll along the bottom of the second valley until this one peters out; i.e. its starting point is reached (cross section number 4) and there it will suddenly roll down the slope to come to rest at the bottom of the first valley.

Thus as λ moves back and forth the state of our system, i.e. the rest position of our ball, moves around a loop, Cf. figure 5. This phenomenon is known as hysteresis.

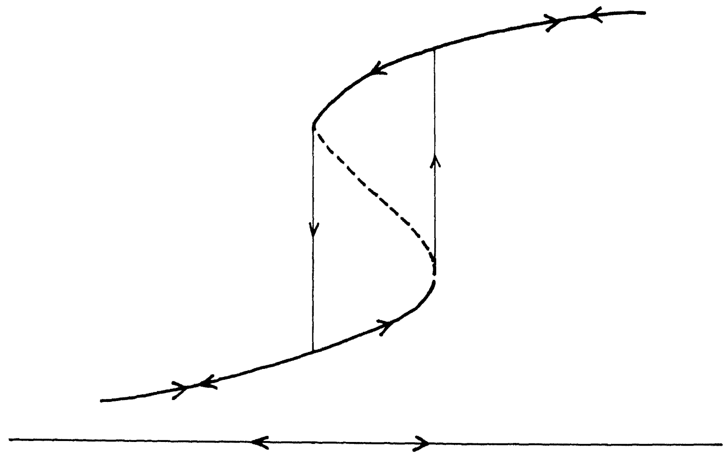

Figure 5.

5. EXAMPLE: A RUBBER BAR

Bifurcation and hysteresis phenomena like in the preceding three sections occur in many guises. As another example consider a stiffish rubber bar with both ends fixed in such a position that it bulges slightly upwards (or downwards). Now let a force F act on it in the transverse direction. The bar is supposed to be constrained to move in a plane. Positive F means it tries to push the bar downwards: negative F tries to push it upwards. As everybody knows (if not, it is a simple experiment to do) a small force F will push the bar very slightly downwards, as the force increases nothing much will

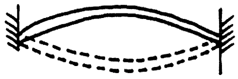

Figure 6.

happen until suddenly the bar snaps into the down position indicated by the dotted lines in Figure 6; diminishing the force after the snap will not do much, indeed an equal negative force is needed to make the bar snap back.

The potential energy of the bar at a given position x (of its central point) with force F acting on it is definitely not given by formula 2.1 with $F = \lambda - 2$. But as far as a qualitative description of its bifurcation, equilibrium points, stability and hysteresis properties go, formula 2.1 gives a correct description.

6. AN ECONOMIC EXAMPLE: EQUILIBRIUM PRICES

This example is a simple modification of one described by Wan in [7]. Consider an economy with two (groups of) traders 1 and 2 and two goods. The initial endowment of a trader of type 1 is $(t_1,0)$; the initial endowment of a trader of type 2 is $(0,t_2)$.

Let the utility function of a trader of type 1 be

$U_1(x_1,x_2) = \min(3x_1^3 - 6x_1^2 + 4x_1,\ x_2)$ and the utility function of a trader 2 be $U_2(x_1,\ x_2) = \min(3x_1,x_2)$.

There are as many traders of type 1 as of type 2 so that the whole situation can be conveniently depicted in a two person, two good box diagram. Cf. figure 7 below. The point P = (x,y) in the box represents the state of the economy where traders of type 1 have the bundle of goods (x,y) and the traders of type 2 $(t_1-x,\ t_2-y)$.

The Engel curve for traders of type 1 is the curved heavy line starting in the lefthand corner (the graph of $x_2 = 3x_1^3 - 6x_1^2 + 4x_1$ with a horizontal segment in the upper right hand corner added); the Engel curve for trades of type 2 is the slanted heavy dotted line starting in the upper right hand corner. Given prices (p_1,p_2) consider the line making an angle with the negative X-axis with $tg\gamma = p_1p_2^{-1}$. Then the bundles of goods which are within the budget of trader 1 are to the left of this line and the bundles within the budget of trader 2 to the right of this line. As both Engel curves have derivatives >0 (viewed from the respective origins) the optimal bundles of goods for traders 1 and 2 in the situation drawn in figure 7 are respectively B_1 and B_2. Assuming that excess demand for a good causes its price

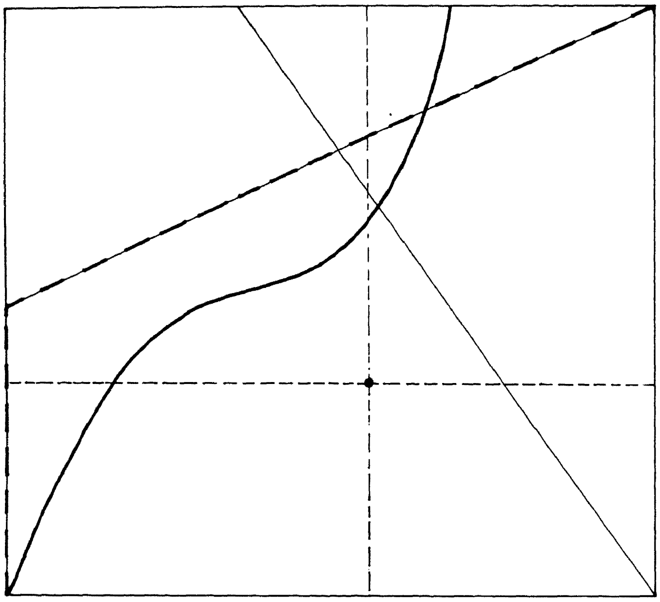

Figure 7.

to rise and vice versa, it is clear that the equilibrium states of the economy are the points where the Engel curves intersect (i.e. the point Q in the picture) and that the ratio of the equilibrium prices is given by the angle of the line from this point to R. Also such an equilibrium will be stable in the case of an intersection point as in Figure 8a and unstable in the case of Figure 8b.

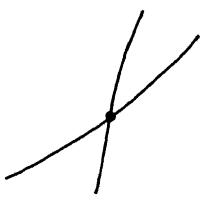

Figure 8a

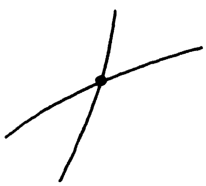

Figure 8b

Now consider how the equilibrium prices evolve as the economy grows and shrinks according to $t_1 = s$, $t_2 = \frac{2}{3}s$, $s \in [1,3]$. Cf. Figure 9. Then for small s and big s there is precisely one stable equilibrium price and in between them is a region where there are three equilibrium prices of which one is unstable and two are stable. The graph of the equilibrium price ratios as a function of s is sketched in figure 10. As the economy grows and shrinks again we see hysteresis phenomena occur.

REMARK. In this case equilibrium states where one of the goods is free (i.e. has price zero) do not occur. In case both Engel curves are convex there can be no more than two intersection points in the interior of the box and free goods (at equilibrium) do tend to occur, Cf. Wan loc. cit. In this case the equilibrium prices bifurcation diagram looks something like in Figure 11 below.

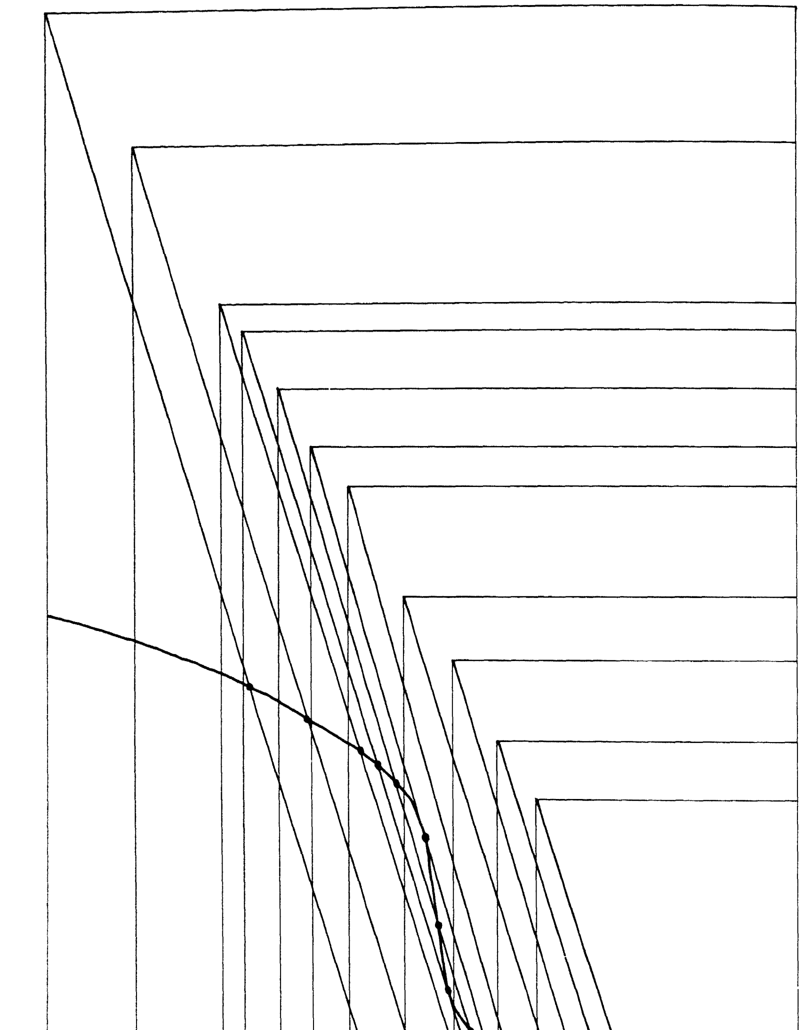

Figure 9.

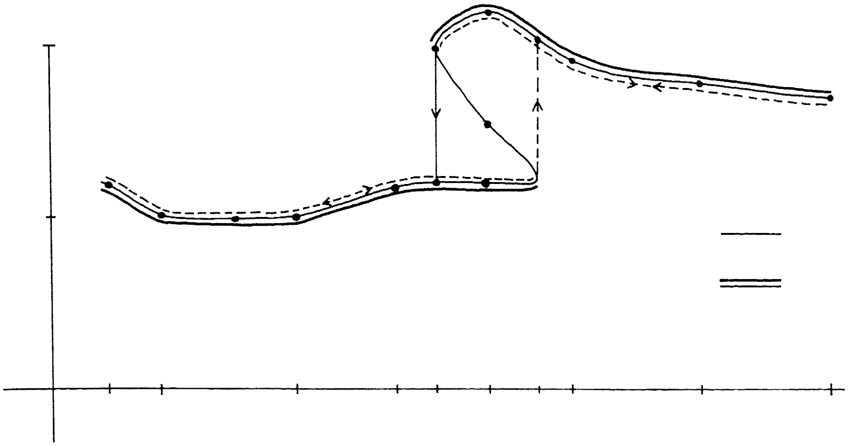

Figure 10.

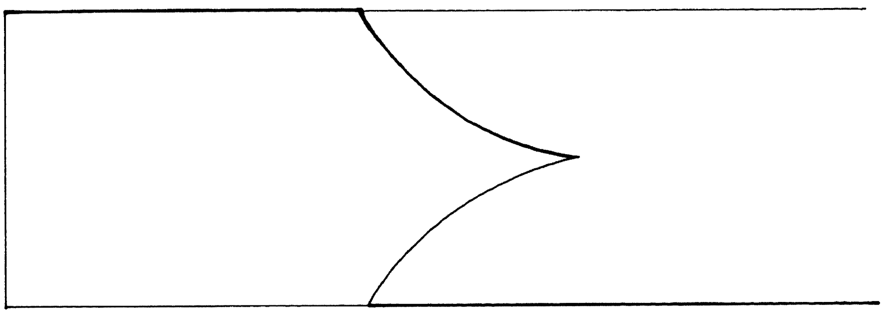

Figure 11.

7. THE PITCHFORK BIFURCATION

Now consider another family of potentials, namely

$$E(\lambda,x) = \tfrac{1}{4}x^4 - \tfrac{1}{2}\lambda x^2 \tag{7.1}$$

or slightly more generally

$$E(\lambda,x) = \tfrac{1}{4}x^4 - x^3 - \tfrac{1}{2}\lambda x^2 \tag{7.2}$$

The bifurcation diagrams for particles moving in these potentials are respectively sketched in the Figures 12a and b.

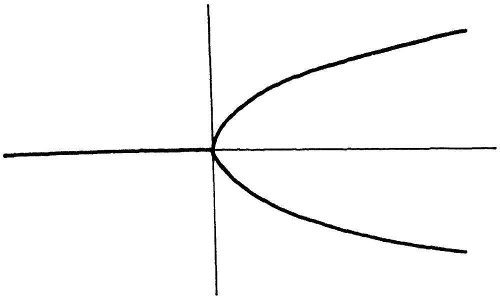

Figure 12a

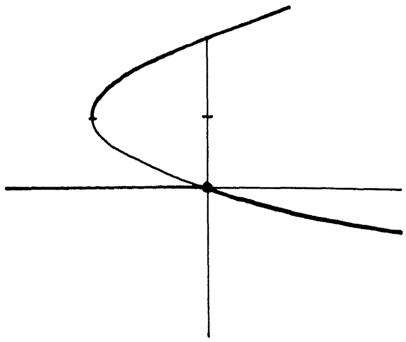

Figure 12b

Given these bifurcation diagrams the reader will have no difficulty in sketching the corresponding landscapes as in Figure 4 for the potential (2.1).

Note that here as λ grows a particle in the potential field (7.1) (or (7.2)) in stable equilibrium for $\lambda < 0$ has as λ crosses zero a "choice" of equilibria: it can either follow the left hand or the right hand valley. The smallest disturbance will be decisive here. Or, from a more positive point of view, if we are in a situation where we have extra control variables (besides possibly λ) the values of λ where there is a pitchfork bifurcation are particularly interesting. These are the hot spots or pressure points where very small controls (investments, ...) are influential out of all proportion.

The surfaces defined by (7.1) and (7.2) are equivalent in practically any sense of the word. Indeed (7.2) is obtained from (7.1) by the co-ordinate transformation $x \to x$, $E \to E$, $\lambda \to \lambda + 2x$. However from the bifurcation point of view the two potentials are quite different. Then we are interested in the solution set of $E_x(\lambda,x) = 0$ as a function of λ.

8. PHYSICAL EXAMPLE OF A PITCHFORK BIFURCATION

Consider again a stiffish rubber bar constrained to move in a plane. Now let the right-hand-end be fixed and the left-hand-end free to slide horizontally but otherwise fixed and let a force $F = \lambda$ act horizontally on the left end. Cf. Figure 13.

For negative F the bar will remain straight. The positive F it will either assume an upwards or downwards curved position. Although (7.1) does not give the potential energy of the bar in position x (the height of the mid-point of the bar above

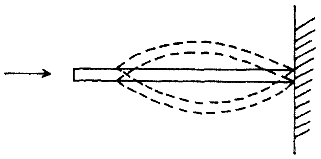

Figure 13.

the rest position) with force $\lambda = F$ acting on it, it does give (more or less) the right qualitative description of the phenomena.

Actually a small positive force F will simply shorten the bar a very slight bit and there will be a threshold which has to be passed before the bar suddenly snaps into either of the curved positions. So the true bifurcation diagram is more like the one in figure 14 below.

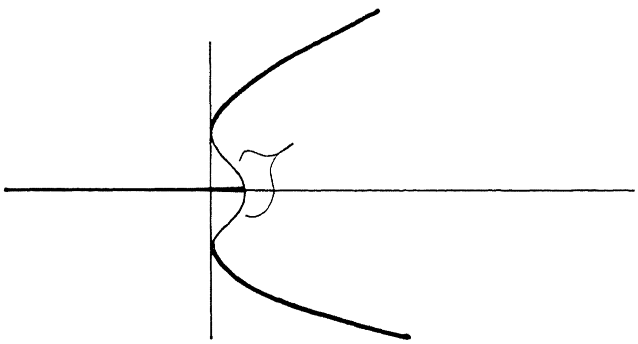

Figure 14.

9. GENERIC BIFURCATIONS

There are very few landscapes on earth which have the hill-valley configuration corresponding to the bifurcation diagrams of Figure 12. The reason is simple. These are not structurally stable patterns. This means that a small change in the potential $E(\lambda,x)$ will give a qualitatively different bifurcation diagram. E.g. if ε is small the perturbation

$$E(\lambda,x) = \tfrac{1}{4}x^4 - x^3 - \tfrac{1}{2}\lambda x^2 + \varepsilon x \qquad (9.1)$$

yields the bifurcation diagrams depicted in figure 15 below (the dotted lines give the diagram for $\varepsilon=0$; cf. Figure 12b).

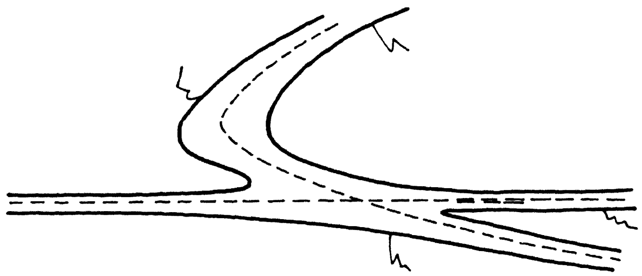

Figure 15.

This makes it important not only to study possible bifurcation but also deformations of bifurcations, that is families of bifurcating systems of equations. Some of the buzz words here are (universal) unfoldings of singularities, catastrophe theory (cf. [8]) and imperfect bifurcations.

10. DYNAMIC BIFURCATIONS. EXAMPLE: THE HOPF BIFURCATION

So far we have only considered static bifurcations so to speak. Now consider a family of ordinary differential equations (or difference equations)

$$\dot{x} = f(x,\lambda), \quad x \in R^n \tag{10.1}$$

Then it is entirely possible that the phase diagram of (10.1), i.e. the diagram of all solution curves of (10.1) changes in nature as λ varies. It is e.g. possible that for $\lambda \leq 0$ the phase diagram looks like Figure 16a (there is an equilibrium point which is a stable attractor) while for $\lambda > 0$ the equilibrium point becomes unstable and there is a stable limit cycle given by $x^2 + y^2 = \lambda$. Cf. Figure 16b. This is a so called Hopf-bifurcation where a periodic solution bifurcates from a stationary one. A set of equations exhibiting this Hopf bifurcation is

$$\begin{aligned} \dot{x} &= -y - x\,(x^2 + y^2 - \lambda) \\ \dot{y} &= x - y\,(x^2 + y^2 - \lambda) \end{aligned} \tag{10.2}$$

Hopf bifurcations occur very often (cf. e.g. [5]). A very simple example where it occurs is for an electrical RLC loop with a non-linear resistor, cf. [3, section 10.4].

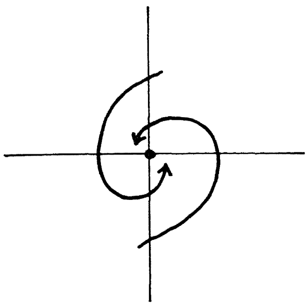

Figure 16a

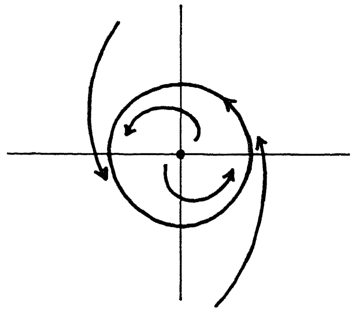

Figure 16b

11. RECOGNISING BIFURCATION POINT CANDIDATES

Consider a family of equations

$$F(x,\lambda) = 0 \tag{11.1}$$

where for simplicity $x \in R^n$, $\lambda \in R$. Here $F: R^{n+1} \to R^n$ is supposed to be differentiable. Assume that $(0,\lambda_0)$ is a solution. Consider $F_x(0,\lambda_0)$. If $F_x(0,\lambda_0) \neq 0$, then the implicit function theorem says that locally we can solve for x in terms of λ so that locally around $(0,\lambda_0)$ the solutions are a single valued function of λ. Thus for a point like P in Figure 12a or R or Q in Figure 12b to occur we need $F_x = 0$ (In the case of Figure 12a we have $F(\lambda,x) = x^3 - \lambda x$, $F_x(\lambda,x) = 3x^2 - \lambda$ so that indeed $F_x(0,0) = 0$; in case of Figure 12b $F(\lambda,x) = x^3 - 3x^2 - \lambda x$, $F_x(\lambda,x) = 3x^2 - 6x - \lambda$ so that $F_x(0,0) = 0$ and $F_x(2\frac{1}{4},1\frac{1}{2}) = 0$).

12. PITCHFORK BIFURCATION POINT THEOREM (VERY SIMPLE CASE). BIFURCATION FROM A SIMPLE EIGENVALUE

Consider again (11.1). Assume that $F(x,\lambda) = 0$, $F(0,\lambda) = 0$

for all λ. That is locally at least no serious restriction after a reparametrisation. Let $F_x(0,\lambda_0) : R^n \to R^n$ have a one dimensional kernel N spanned by $u \in R^n$ and let the range of $F_x(0,\lambda_0)$ be given by $\{v \in R^n : \langle v,w\rangle = 0\}$ for some $w \in R^n$, $w \neq 0$. Now assume that

$$\langle u, F_{x\lambda}(0,\lambda_0)\, w\rangle \neq 0 \qquad (12.1)$$

Then locally near $(0,\lambda_0)$ there is besides the branch $(0,\lambda_0 + t)$ of solutions a second branch through $(0,\lambda_0)$ of the form $(su + s^2\alpha_1(s),\ \lambda_0 + s\alpha_2(s))$, s small, for suitable continous functions α_1 and α_2 of s.

In case n = 1 condition (12.1) simply becomes $F_{x\lambda}(\lambda,x) \neq 0$. Thus e.g. in the case of Figure 12a we have $F_{x\lambda}(0,0) = -1$ so that P = (0,0) is indeed a pitchfork bifurcation point; and in case of Figure 12b $F_{x\lambda}(\lambda,x) = -1$ so that R is also a pitchfork bifurcation point. (Note that near Q the hypotheses of the theorem are not fulfilled, also after translating x so that Q becomes of the form $(0,\lambda_0)$).

13. HOPF BIFURCATION THEOREM

Let $F : R^n \times R \to R$ be analytic and let $F(0,\lambda) = 0$. Let there be an eigenvalue $\sigma(\lambda)$ of $F_x(0,\lambda) : R^n \to R^n$, depending smoothly on λ which crosses the imaginary axis at $\lambda = \lambda_0$ with non-vanishing velocity, i.e. such that $R\sigma'(\lambda) \neq 0$, $\sigma(\lambda) = i\omega_0$,

$\omega_0 \in R$, $\omega_0 > 0$. Assume that the eigenvalue $i\omega_0$ is simple and that there are no eigenvalues of $F_x(0,\lambda_0)$ of the form $ik\omega$, $k \in Z \setminus \{1,-1\}$. Then

$$\dot{x} = F(x,\lambda) \tag{13.1}$$

has a family of periodic solutions of the form $(u(\omega(\varepsilon)t, \varepsilon), \lambda(\varepsilon))$ with $u(s,\varepsilon)$ periodic of period 2π in s and $\lambda(0) = \lambda_0$, $\omega(0) = \omega_0$, $u(s,0) = 0$.

In the case of example (10.2)

$$F_x(0,\lambda) = \begin{pmatrix} \lambda & -1 \\ 1 & \lambda \end{pmatrix}$$

so that its eigenvalues are $\lambda \pm i$. And the Hopf theorem applies.

14. CONCLUDING REMARKS

The material described above only touches the outermost fringes of the subject. In particular there are far reaching generalisations of the two theorems in sections 12 and 13 above. Even in the simple cases described here there is more to be said. Also there exist non-local bifurcation theorems (the two given above are strictly local). A modern up-to-date high level textbook on bifurcation theory is [1]. For an introduction to the numerics of bifurcation the reader can e.g. consult [6]. A wealth of applications of bifurcation theory (and the related catastrophe theory cf. [8]) is contained in [2] and [4].

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