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SEMANTICS OF THE INFINITISTIC RULES OF PROOF

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§1. Introduction.[1] This paper is devoted to the study of the infinitistic rules of proof i.e. those which admit an infinite number of premises. The best known of these rules is the ω-rule. Some properties of the ω-rule and its connection with the ω-models on the basis of the ω-completeness theorem gave impulse to the development of the theory of models for admissible fragments of the language $L_{\omega_1\omega}$. On the other hand the study of representability in second order arithmetic with the ω-rule added revealed for the first time an analogy between the notions of recursivity and hyperarithmeticity which had an important influence on the further development of generalized recursion theory.

The consideration of the subject of infinitistic rules in complete generality seems to be reasonable for several reasons. It is not completely clear which properties of the ω-rule were essential for the development of the above-mentioned topics. It is also worthwhile to examine the proof power of infinitistic rules of proof and what distinguishes them from finitistic rules of proof.

What seemed to us the appropriate point of view on this problem was the examination of the connection between the semantics and the syntax of the first order language equipped with an additional rule of proof. In the case of the first order language the answer to this question is the Gödel completeness theorem and consistency theorem; in the case of the second order arithmetic with the ω-rule added it is the Henkin-Orey ω-completeness theorem.

In §2 we formulate the interesting notions in the general situation. The obvious Theorem 2 shows that for every class of structures one can construct a rule of proof for which this class of structures is a semantics. This theorem is false in the case of the finitistic rules of proof and it seems to indicate that in the completely general situation there exists too much freedom in the use of the notion of infinitistic rule of proof. What seems to be necessary is the restriction to rules of proof which are in a certain sense natural.

However, as long as the language is not sufficiently rich, it is hard to find in it natural infinitistic rules of the proof. The language to which we restrict our considerations in this case is the language of second order arithmetic. As it turns out, this language is sufficiently rich to be able to build in it various rules of proof in which the connections between the premises and conclusion are clear and simple. The required facts from the second order arithmetic are collected in §3.

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[1] This paper is based on our Ph.D. thesis which was submitted to the Institute of Mathematics of the Polish Academy of Science in December of 1973. The first approximation to this work was the paper Apt [1].

The basic problem from this subject for the language of second order arithmetic is the problem of the existence of a syntactical β-rule which was posed by Mostowski in his paper from 1959 in which the notion of a β-model was defined. §§4 and 5 are devoted to the study of this problem.

It is not completely clear when one should accept a rule of proof as a syntactical β-rule. In §4 we define three conditions which seem to be important here. The existence of a rule of proof which satisfies all the three conditions is a conclusion from the Theorem 2. Unfortunately this rule of proof is completely artificial and by no means can be considered as a satisfactory solution of this problem.

In order to find a simple rule of proof which satisfies the chosen three conditions, we examine the position of such rules of proof in the analytical hierarchy.

We show that no rule of proof with a Σ_2^1 graph can be a β-rule and we construct a rule of proof with a Π_2^1 graph which satisfies all three conditions from §4. This rule comes directly from the proof of Theorem 2.

What seemed to us a way out from this situation was the choosing of a class of rules which are, in an imprecise way, more natural. The proposed class of rules, which we call here regular ones, comes from the restriction of the class of rules given by Aczel [1] to the language of the second order arithmetic.

In §5 we prove a theorem which generalizes a particular unpublished result of Aczel—none of the regular rules of proof satisfies any of the three conditions from §4.

Thus syntactical β-rules exist but it is very doubtful whether one can find such a rule of proof among natural ones.

§6 is devoted to the study of the semantical properties of the regular rules of proof. One of the known regular rules besides the ω-rule is Enderton's $\mathscr{A}$-rule. This rule plays an important role among regular rules—under the assumption $P(\omega) \subseteq L$ the regular rules stronger than the $\mathscr{A}$-rule do not satisfy the completeness theorem. For some rules (e.g. for the $\mathscr{A}$-rule) we are able to omit the assumption $P(\omega) \subseteq L$.

In §7 we prove a theorem which puts essential restrictions on the semantics of the regular rules which consist of ω-models. Namely it turns out (under the assumption $P(\omega) \subseteq L$) that the ω-semantics of the regular rules are not well-founded in the sense of inclusion. In order to prove this theorem we carry out the key part of the proof of Friedman's theorem about nonexistence of minimal ω-models of A_2.

It is not clear to us whether our restriction in this study to the language of second order arithmetic and then to regular rules of proof is not an arbitrary decision. It seems to us that in order to find an appropriate point of view on this problem one should first define some criteria for naturalness of a rule of proof. Unfortunately we were unable to do this.

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§2. The basic notions and definitions. Let L be an arbitrary, fixed language of first order. By Sent we denote the set of all sentences of L. If $T \subset$ Sent, then

Cn(T) denotes the set of logical consequences of T. $P(X)$ denotes the power set of the set X.

DEFINITION 1. f is a rule of proof in L if f is a subset of $P(\text{Sent}) \times \text{Sent}$.

The above definition seems to contain in the whole generality that which is intuitively understood under the term of a rule of proof.

DEFINITION 2. Let $\mathfrak{A}$ be a structure for L and f a rule of proof in L. The rule f is sound in $\mathfrak{A}$ if, for every $T \subset \text{Sent}$,

$$\mathfrak{A} \models T \rightarrow \forall \phi(\langle T, \phi \rangle \in f \rightarrow \mathfrak{A} \models \phi).$$

By $K(f)$ we denote the class of all structures in which f is sound.

DEFINITION 3. (i) A set of sentences T is f-complete if $T = \text{Cn}(T)$ and, for every $S \subset T$, $\forall \phi(\langle S, \phi \rangle \in f \rightarrow \phi \in T)$.

(ii) A set of sentences T is f-consistent if $T = \text{Cn}(T)$ and, for every $S \subset T$, $\forall \phi(\langle S, \phi \rangle \in f \rightarrow \neg \phi \notin T)$.

By $(T)_f$ we denote the least f-complete set of sentences which contains T. By $T \vdash_f \phi$ we mean that $\phi \in (T)_f$ and we say then that ϕ is f-provable from T.

DEFINITION 4. Let K be a class of structures for L. If $T \subset \text{Sent}$ then

$$(T)_K = \{\phi : \phi \in \text{Sent} \wedge \forall \mathfrak{A}[\mathfrak{A} \models T \wedge \mathfrak{A} \in K \rightarrow \mathfrak{A} \models \phi]\}.$$

K is a semantics for a rule f if, for every $T \subset \text{Sent}$, $(T)_f = (T)_{K(f)}$.

DEFINITION 5. A rule f satisfies the deduction theorem if, for every $T \cup \{\phi, \psi\} \subset \text{Sent}$,

$$T \vdash_f \phi \rightarrow \psi \quad \text{iff} \quad T \cup \{\phi\} \vdash_f \psi.$$

The term "model" is analogous here to the term "model in which f is sound". Gödel's completeness theorem says that a sentence is provable iff it is true. The corresponding statement in our case presents the following definition.

DEFINITION 6. A rule f satisfies the completeness theorem if, for every $T \subset \text{Sent}$, $(T)_f = (T)_{K(f)}$.

In the above terminology the ω-rule satisfies the completeness theorem and the deduction theorem and ω-models form a semantics for the ω-rule.

The easy connections between the notions introduced here shows

THEOREM 1. (i) *Suppose that K is a semantics for a rule f. If $\mathfrak{A} \in K$ then f is sound in $\mathfrak{A}$.*

(ii) *If a rule f has a semantics then it satisfies the completeness theorem.*

(iii) *If a rule f satisfies the completeness theorem then every consistent, f-complete set of sentences has a model in which f is sound.*

PROOF. (i) Let $\mathfrak{A} \in K$. Then $(\text{Th}(\mathfrak{A}))_f = (\text{Th}(\mathfrak{A}))_K = \text{Th}(\mathfrak{A})$ which concludes the proof.

(ii) Let $T \subset \text{Sent}$. By induction with respect to the structure of the set $(T)_f$ we prove that $(T)_f \subset (T)_{K(f)}$. On the other hand it follows from (i) that $(T)_{K(f)} \subset (T)_K = (T)_f$.

(iii) Let T be a consistent, f-complete set of sentences. Then $T = (T)_f = (T)_{K(f)}$, i.e. there exists a structure $\mathfrak{A}$ which is a model of T in which f is sound.

Sometimes instead of the questions of whether a rule of proof has a semantics, etc., one poses the converse question: whether to a given class of structures can one find a rule of proof for which this class of structures is a semantics. A positive answer is given by the following.

THEOREM 2. *Let K be a class of structures for L. Then there exists a rule f such that*

(i) *K is a semantics for f,*

(ii) *every f-consistent set of sentences has a model belonging to K.*

PROOF. Let $f = \{\langle T, \phi\rangle : \phi \in (T)_K\}$. Then, for every $T \subset \text{Sent}$, $(T)_f = (T)_K$. Suppose now that T is an f-consistent set of sentences. Then by definition $\forall\phi(\langle T, \phi\rangle \in f \to \neg\phi \notin T)$, i.e. $\forall\phi(\phi \in (T)_K \to \neg\phi \notin T)$. Thus $(T)_K$ is a consistent set of sentences, so there exists $\mathfrak{A}$ such that $\mathfrak{A} \models T$ and $\mathfrak{A} \in K$.

In the above theorem one cannot demand that f be a finitistic rule of proof, i.e. one such that the sets of its premises are always finite. The following example gives evidence for this.

EXAMPLE. Let L be the language of Peano arithmetic. For the purposes of the above theorem one may identify every class of structures with the set of the theories of the structures belonging to this class. Thus each class of structures K we identify with a set $F(K) = \{T: T = \text{Th}(\mathfrak{A}) \text{ for some } \mathfrak{A} \in K\}$.

It is well known that there exist continuum extensions of the Peano arithmetic to a complete theory, i.e. L has exactly continuum complete theories. Thus the number of the sets $F(K)$ is equal to $2^{\aleph_0}$. If $F(K_1) \neq F(K_2)$ then there exists a complete theory T such that e.g. $T \in F(K_1) - F(K_2)$. Then $(T)_{K_1} = T$ but $(T)_{K_2}$ is the set of all sentences. Thus in order that the Theorem 2 might be true for the finitistic rules of proof there must be at least $2^{\aleph_0}$ of them, because every set $F(K)$ requires another rule of proof. But there is only a continuum of the finitistic rules of proof, thus too few.

Theorem 2 thus indicates an essential difference between the finitistic and infinitistic rules of proof.

It is hard to accept the rules of proof constructed in the Theorem 2. Although they satisfy our definition of a rule of proof they are completely artificial and it is hard to use them in concrete cases. What seems to be worthwhile is the consideration of those rules of proof which are in a certain sense natural, in which there is clear and simple connection between the set of premises and consequences.

If in the language there exists an infinite number of constants then we can formulate the following rule of proof:

> If for every constant term t we have $\vdash\phi(t)$, then $\vdash\forall x\phi(x)$, where ϕ is a formula with one free variable.

If the language is countable then this rule of proof satisfies the completeness theorem and the deduction theorem. The proof is exactly the same as that of the Henkin-Orey ω-completeness theorem (see Orey [1]). Instead of the type $\{x \neq \boldsymbol{n} : n < \omega\}$ we omit here the type $\Gamma = \{x \neq t : t \text{ is a constant term}\}$. The models which omit the type Γ correspond to the ω-models.

However as long as the language is not rich enough it is hard to find in it other interesting infinitistic rules of proof.

The remaining part of the paper is devoted to considerations of infinitistic rules of proof formulated in the language of second order arithmetic. As we shall see this language is rich enough so the problem does not reduce here to considerations concerning the ω-rule.

§3. Terminology and notations from second order arithmetic and recursion theory. By second order arithmetic A_2 we mean the theory described in the paper Apt-Marek [1]. A_2^- denotes A_2 without choice scheme. Recall that set variables are denoted by letters X, Y, Z and number variables are denoted by letters x, y, z, a. A relation $P(X, a)$ is a Σ_n^1 relation if, for some recursive relation R,

$$P(X, a) \leftrightarrow \exists X_1 \forall X_2 \cdots QX_n \tilde{Q}yR(\bar{X}(y), \bar{X}_1(y), \cdots, \bar{X}_n(y)),$$

where Q is an appropriate quantifier and $\tilde{Q}$ is the quantifier dual to Q. Here $\bar{X}(y) = \bar{\chi}_X(y)$, where χ_X is the characteristic function of the set X. All the other notations from recursion theory are taken from Shoenfield's book [1].

ω denotes the set of natural numbers and $P(\omega)$ its power set. Each ω-model we identify with its family of sets, i.e. with a subset of $P(\omega)$.

A β-model (see Mostowski [1]) is a structure for the language of second order arithmetic $L(A_2)$ for which the notion of well-ordering is absolute.

Being a Σ_n^1 or Π_n^1 formula is defined in a natural way. An ω-model is a β_n-model (see Enderton-Friedman [1]) if, for every Σ_n^1 formula ϕ with parameters from $\mathfrak{A}$, $P(\omega) \models \phi \rightarrow \mathfrak{A} \models \phi$. The notion of β_1-model of A_2^- is identical with the notion of β-model of A_2^- (see Mostowski [1]).

Now Sent denotes the set of sentences of $L(A_2)$. If $T \subset$ Sent then T_β (or T_{β_n}) denotes the set of all sentences true in all β-models (or β_n-models) of T. For simplicity let $A_\beta = (A_2^-)_\beta$ and $A_{\beta_n} = (A_2^-)_{\beta_n}$.

If $X \subset \omega$ then $\Pi_X = \{(X)_n : n < \omega\}$, where $(X)_n = \{x : \langle n, x\rangle \in X\}$. By Constr($X$) we mean the Σ_2^1 formula of Addison [1] which is satisfied in $P(\omega)$ exactly by the constructible subsets of ω.

Sometimes we do not distinguish analytical relations from the formula of $L(A_2)$ which define them.

We shall use some facts which concern inductive definitions.

Let Γ be a monotone operator on $P(\omega)$, i.e. $\Gamma: P(\omega) \rightarrow P(\omega)$ and $X \subset Y \rightarrow \Gamma(X) \subset \Gamma(Y)$. A set Z is inductively defined by Γ if $S_\alpha = \bigcup_{\beta<\alpha}\Gamma(S_\beta)$ and $Z = \bigcup_\alpha S_\alpha$. Then $Z = S_{|\Gamma|}$ where $|\Gamma|$ is the least α such that $S_\alpha = S_{\alpha+1}$.

Γ is said to be Σ_n^1 if the relation $n \in \Gamma(Y)$ is Σ_n^1. We shall need the well-known fact that if Γ is Σ_n^1 $(n > 1)$ then $S_{|\Gamma|}$ is Σ_n^1, too.

§4. The problem of the existence of a β-rule. The question of whether there exists a syntactical β-rule was raised by Mostowski in the last two sentences of the paper Mostowski [1]: "The rule β treated in this paper has been defined by means of semantical notions. It would be interesting to find an equivalent definition formulated in a syntactical (although infinitistic) manner."

The equivalence of the semantical and syntactical definitions can be understood here in several ways.

In the most moderate way one should consider f as a syntactical β-rule if

$$(*) \qquad (A_2^-)_f = A_\beta.$$

If one wants to connect the notion of a β-rule not with the second order arithmetic but with its language then instead of condition (*) one should take the condition:

(**) For every $T \subset \text{Sent}$, $(T)_f = T_\beta$.

In order to understand better which properties should be satisfied by the β-rule and how they should be connected with β-models, it will be best to inspect the connections between the ω-rule and ω-models. The appropriately reformulated condition (**) is simply the Henkin-Orey ω-completeness theorem. This theorem is a crucial property of the ω-rule. It enables in many cases the building of ω-models and in fact it is used in almost all theorems about ω-models. If one wants to have a syntactical β-rule in order to be able to build β-models then one should demand that:

(***) every consistent, β-complete set of sentences has a β-model.

Of course this condition is trivially satisfied by every rule of proof leading to the contradiction and therefore we shall consider this condition in a connection with one of the conditions (*) and (**).

Let us observe that Theorem 2 applied to the language of second order arithmetic and the class of all β-models gives a rule which satisfies the conditions (*)–(***). It turns out to be a very unsatisfactory solution if one notices how artificially this rule is built. The satisfactory solution would be finding a rule which satisfies at least one of the conditions (*)–(***) but which is in a certain sense natural.

The natural criterion of the complexity of the structure of a set is its position in the arithmetical or analytical hierarchy. We can apply this criterion to the rules of proof formulated in $L(A_2)$ after Gödelization of the language. Using this criterion we shall find a syntactical β-rule least complicated in this sense.

The theorem below fixes the level of the analytical hierarchy on which there appear rules which satisfy the conditions (*)–(***).

A rule of proof f is Σ^1_n if after Gödelization of the language it is a Σ^1_n relation. We make the Gödelization in such a way that every natural number is the Gödel number of a formula.

$\ulcorner\phi\urcorner$ denotes the Gödel number of the formula ϕ. If $T \subset \text{Sent}$ then $\ulcorner T\urcorner$ denotes the set of Gödel numbers of the sentences from T.

THEOREM 3. (i) *If f is a Σ^1_2 rule then f does not satisfy the condition* (*).

(ii) *There exists a Π^1_2 rule which satisfies the conditions* (*)–(***).

PROOF. (i) Suppose that f is Σ^1_2. Let

$$a \in \Gamma_f(X) \leftrightarrow a \in \ulcorner\text{Cn}\urcorner(X) \vee a \in \ulcorner A_2^-\urcorner \vee \exists Y(Y \subset X \wedge \langle Y, a\rangle \in f)$$

where $\ulcorner\text{Cn}\urcorner(X) = \{\ulcorner\phi\urcorner : \{\psi : \ulcorner\psi\urcorner \in X\} \vdash \phi\}$. Then Γ_f is a monotone operator such that $S_{|\Gamma_f|} = \ulcorner(A_2^-)_f\urcorner$ and $\Gamma_f \in \Sigma^1_2$. By virtue of the theorem mentioned in the end of §3, $\ulcorner(A_2^-)_f\urcorner \in \Sigma^1_2$. But $\ulcorner A_\beta\urcorner \in \Pi^1_2 - \Sigma^1_2$ (see Mostowski [1]), so $(A_2^-)_f \neq A_\beta$.

(ii) Let $f = \{\langle T, \phi\rangle : \phi \in (T)_\beta\}$. Then f satisfies the conditions (*)–(***) by Theorem 2 applied to $L(A_2)$ and the class of all β-models. Also f is Π^1_2.

Thus on the basis of the criterion we chose here, namely the position in the analytical hierarchy, we are unable to decide whether there exists a syntactical

β-rule that is in a certain sense natural. This criterion turns out to be inappropriate and the only thing which is left in this situation is to attempt to make precise of the notion of naturalness.

§5. There is no syntactical β-rule among regular rules.

Instead of looking for some doubtful criteria for naturalness of a rule of proof we introduce now a class of rules of proof which characterize themselves by a concrete, natural connection of the set of premises with the conclusion. This class is the restriction to the language of the second order arithmetic of the class of rules of proof given by Aczel (see Aczel [1]). These rules come from the notion of a generalized quantifier.

We shall check in this section whether there exist in this class any rules of proof which satisfy the conditions (*)–(***).

DEFINITION 7. Let R be an analytical subset of $P(\omega)$ which is monotone, i.e. $X \subset Y \wedge R(X) \rightarrow R(Y)$.

R as a formula of $L(A_2)$ determines the following rule of proof f_R:

$$\operatorname{dom} f_R = \{\{\phi(\boldsymbol{n}) : n \in X\} : X \in R \text{ and } \phi \text{ is a formula of } L(A_2) \text{ with exactly one free variable}\} \text{ and}$$

$$f_R(\{\phi(\boldsymbol{n}) : n \in X\}) = (Rx)\phi(x),$$

where $(Rx)\phi(x) = \exists Y(R(Y) \wedge \forall x(x \in Y \leftrightarrow \phi(x)))$. Thus if $\{n : \vdash_{f_R} \phi(\boldsymbol{n})\} \in R$ then $\vdash_{f_R}(Rx)\phi(x)$. Every rule of proof of the above kind we call a regular rule.

Observe that for different definitions R_1, R_2 of the relation R we obtain different rules of proof f_{R_1} and f_{R_2}. If we take for R the relation $R(X) \leftrightarrow \forall x(x \in X)$ we simply obtain the ω-rule.

The following theorem shows that there is no syntactical β-rule among the regular rules.

THEOREM 4. *If f_R is a regular rule of proof then*

(i) $(A_\beta \not\subset (A_2^-)_{f_R})$,

(ii) *there exists a consistent, f_R-complete set of sentences which has no β-model.*

We precede the proof of this theorem by the proof of some lemmas.

The following general criterion found by Aczel turns out to be very useful here.

LEMMA 1 (ACZEL). *Let f be a rule of proof with analytical graph. If*

$$(\Delta) \qquad A_\beta \vdash \ulcorner\phi\urcorner \in \ulcorner(A_2^-)_f\urcorner \rightarrow \phi \quad \textit{for each sentence } \phi$$

then $(A_\beta \not\subset (A_2^-)_f)$.

PROOF. Let ψ be a sentence such that

$$A_2^- \vdash \psi \leftrightarrow \ulcorner\psi\urcorner \notin \ulcorner(A_2^-)_f\urcorner$$

(for the existence of such sentences see Feferman [1]). From the assumption,

$$A_\beta \vdash \ulcorner\psi\urcorner \in \ulcorner(A_2^-)_f\urcorner \rightarrow \psi.$$

From the above two facts $\psi \in A_\beta$.

In particular $P(\omega) \models \psi$, i.e. $P(\omega) \models \ulcorner\psi\urcorner \notin \ulcorner(A_2^-)_f\urcorner$ so $\psi \notin (A_2^-)_f$, which concludes the proof.

We cannot use directly this criterion to our situation because there exist regular rules of proof which do not satisfy the condition (Δ). The following example gives evidence for this.

EXAMPLE. Let $R(X) \leftrightarrow (\phi \wedge X = X) \vee X = \{0\}$, where ϕ is a sentence such that $P(\omega) \models \phi$ and $\phi \notin A_\beta$.

Of course $P(\omega) \models \forall X R(X)$ so R is monotone. Let $\mathfrak{A}$ be a β-model of A_2^- such that $\mathfrak{A} \models \neg\phi$. Observe that R is satisfied in $\mathfrak{A}$ only by the set $\{0\}$.

Observe that $\mathfrak{A} \models \ulcorner 0 = 0 \urcorner \in \ulcorner (A_2^-)_{f_R} \urcorner$ and $\mathfrak{A} \models R(\{0\})$ so $\mathfrak{A} \models \ulcorner (Rx)(x = x) \urcorner \in \ulcorner (A_2^-)_{f_R} \urcorner$. Suppose now that f satisfies the condition (Δ). Then $\mathfrak{A} \models (Rx)(x = x)$, i.e. $\mathfrak{A} \models \exists X(R(X) \wedge \forall x(x \in X \leftrightarrow x = x))$, i.e. $\mathfrak{A} \models R(\omega)$ which is the contradiction.

In order to overcome this difficulty we prove that every regular rule of proof can be "majorized" by a regular rule which satisfies the condition (Δ). This will already suffice to prove part (i) of Theorem 4.

DEFINITION 8. Let

$$\Sigma(n)(X) \leftrightarrow \exists X_1 \forall X_2 \cdots Q X_n \tilde{Q} y(\langle \bar{X}_1(y), \cdots, \bar{X}_n(y)\rangle \in X),$$
$$\Pi(n)(X) \leftrightarrow \forall X_1 \exists X_2 \cdots \tilde{Q} X_n Q y(\langle \bar{X}_1(y), \cdots, \bar{X}_n(y)\rangle \in X),$$

where Q and $\tilde{Q}$ are the appropriate quantifiers. Then $\Sigma(n)$ and $\Pi(n)$ are the monotone subsets of $P(\omega)$. Observe that for example if $\forall X_1 \exists X_2 \forall y T \vdash_{f_{\Pi(2)}} \phi(\bar{X}_1(y), \bar{X}_2(y))$ then

$$T \vdash_{f_{\Pi(2)}} \forall X_1 \exists X_2 \forall y \phi(\bar{X}_1(y), \bar{X}_2(y)).$$

LEMMA 2. *Let R be a Σ_m^1 (Π_m^1) monotone subset of $P(\omega)$. Then, for every T such that $A_2^- \subset T \subset$ Sent,*

$$(T)_{f_R} \subset (T)_{f_{\Sigma(m+2)}} \qquad ((T)_{f_R} \subset (T)_{f_{\Pi(m+1)}}).$$

PROOF. We give the proof only in the case $R \in \Sigma_m^1$. The case $R \in \Pi_m^1$ one proves analogously.

Assume for simplicity that m is odd; $m = 2k + 1$. Let ϕ be a Σ_{m+2}^1 sentence. Then, for a recursive relation S,

$$A_2^- \vdash \phi \leftrightarrow \exists X_1 \forall X_2 \cdots \exists X_{m+2} \forall y S(\bar{X}_1(y), \cdots, \bar{X}_{m+2}(y)).$$

Suppose that $P(\omega) \models \phi$. Then

$$\exists X_1 \forall X_2 \cdots \exists X_{m+2} \forall y A_2^- \vdash S(\bar{X}_1(y), \cdots, \bar{X}_{m+2}(y)),$$

because true open sentences are provable in A_2^-.

Using the $f_{\Sigma(m+2)}$-rule we obtain

$$A_2^- \vdash_{f_{\Sigma(m+2)}} \exists X_1 \forall X_2 \cdots \exists X_{m+2} \forall y S(\bar{X}_1(y), \cdots, \bar{X}_{m+2}(y)),$$

i.e. $\phi \in (A_2^-)_{f_{\Sigma(m+2)}}$. This shows that true Σ_{m+2}^1 sentences are $f_{\Sigma_{m+2}}$-provable from A_2^-. Observe also that for every $T \subset$ Sent, $(T)_{f_{\Sigma(m)}} \subset (T)_{f_{\Sigma(m+1)}}$ because every $f_{\Sigma(m)}$-proof can be changed into a $f_{\Sigma(m+1)}$-proof by adding in the appropriate places additional quantifiers.

For a recursive relation W,

(1) $$R(X) \leftrightarrow \exists X_1 \forall X_2 \cdots \exists X_{2k+1} \forall y W(\bar{X}(y), \bar{X}_1(y), \cdots, \bar{X}_{2k+1}(y)).$$

This equivalence is provable in A_2^- (see Lemma 2.16 in Mostowski [1]).

Assume that $A_2^- \subset T \subset \text{Sent}$. By induction on proofs we prove that $(T)_{f_R} \subset (T)_{f_{\Sigma(m+2)}}$.

The only nontrivial case occurs when $\sigma = (Rx)\phi(x) \in (T)_{f_R}$. From the monotonicity of R and the inductive assumption

$$A = \{n : \phi(\mathbf{n}) \in (T)_{f_{\Sigma(2k+3)}}\} \in R.$$

Thus

$$P(\omega) \models \exists X_1 \forall X_2 \cdots \exists X_{2k+1} \forall y W(\bar{A}(y), \bar{X}_1(y), \cdots, \bar{X}_{2k+1}(y)). \tag{2}$$

We prove now that

$$\begin{aligned} &\exists Z \exists X_1 \forall X_2 \cdots \exists X_{2k+1} \forall y \\ &\quad T \vdash_{f_{\Sigma(2k+3)}} W(\bar{Z}(y), \bar{X}_1(y), \cdots, \bar{X}_{2k+1}(y)) \wedge (((\bar{Z}(y+1))_y = 1 \to \phi(y)). \end{aligned} \tag{3}$$

It follows from (2) that

$$\exists X_1 \forall X_2 \cdots \exists X_{2k+1} \forall y T \vdash_{f_{\Sigma(2k+3)}} W(\bar{A}(y), \bar{X}_1(y), \cdots, \bar{X}_{2k+1}(y)), \tag{4}$$

because true open sentences are provable in A_2^-.

In order to prove (3) it is left to prove

$$\forall y T \vdash_{f_{\Sigma(2k+3)}} (\bar{A}(y+1))_y = 1 \to \phi(y). \tag{5}$$

But from the definition of the set A the following equivalence holds for any y:

$$(\bar{A}(y+1))_y = 1 \leftrightarrow y \in A \leftrightarrow T \vdash_{f_{\Sigma(2k+3)}} \phi(y)$$

which proves (5).

Having already proved (3) we use the $f_{\Sigma(2k+1)}$-rule (which is, as we observed, derivable from the $f_{\Sigma(2k+3)}$-rule) and obtain

$$\begin{aligned} &T \vdash_{f_{\Sigma(2k+3)}} \exists Z \exists X_1 \forall X_2 \cdots \exists X_{2k+1} \forall y \\ &\quad W(\bar{Z}(y), \bar{X}_1(y), \cdots, \bar{X}_{2k+1}(y)) \wedge ((\bar{Z}(y+1))_y = 1 \to \phi(y)). \end{aligned} \tag{6}$$

On the basis of (1) the above sentence is equivalent in A_2^- to the sentence

$$\exists Z(R(Z) \wedge \forall y(y \in Z \to \phi(y)).$$

The sentence $\forall X \forall Y(R(X) \wedge X \subset Y \to R(Y))$ is a true Π^1_{2k+2} sentence and so, as we observed, it belongs to $(A_2^-)_{f_{\Sigma(2k+3)}}$.

From the last two facts using the comprehension schema we obtain that

$$T \vdash_{f_{\Sigma(2k+3)}} \exists X(R(X) \wedge \forall y(y \in X \leftrightarrow \phi(y)))$$

which concludes the proof.

LEMMA 3. *The rules $f_{\Sigma(m)}$ and $f_{\Pi(m)}$ satisfy the condition* (Δ) *from Lemma* 1.

PROOF. We prove that if R is an analytical, monotone subset of $P(\omega)$ such that

$$A_\beta \vdash \forall X \forall Y(R(X) \wedge X \subset Y \to R(Y)) \tag{∇}$$

then f_R satisfies the condition (Δ) from Lemma 1. The idea of this proof comes from Aczel's proof of the fact that the $f_{\Pi(2)}$-rule satisfies the condition (Δ) from Lemma 1. The rules $f_{\Sigma(n)}$ and $f_{\Pi(n)}$ of course satisfy the condition (∇).

Let $\mathfrak{A}$ be a β-model of A_2^-. Suppose that $\mathfrak{A} \models \ulcorner\phi_1\urcorner \in \ulcorner(A_2^-)_{f_R}\urcorner$. Then in $\mathfrak{A}$ the following sentence holds:

$\exists X\{(X$ is a tree) $\wedge$ (every minimal vertex of X is a Gödel number of an axiom of A_2^-) $\wedge$ [(every vertex $\ulcorner\phi\urcorner$ in X (as a formula) follows from the one directly below by a logical rule of proof) $\vee$ $(\exists Y \exists \ulcorner\psi\urcorner(\ulcorner\phi\urcorner = \ulcorner(Rx)\psi(x)\urcorner \wedge \forall n(n \in Y \leftrightarrow \ulcorner\psi(\boldsymbol{n})\urcorner$ lies in X directly below $\ulcorner(Rx)\psi(x)\urcorner) \wedge R(Y)))] \wedge (\ulcorner\phi_1\urcorner$ is "on the top" of the tree)$\}$.

"To be a tree" is a Π_1^1 formula and so absolute with respect to the β-models of A_2^-.

Thus the above proof is a tree in $P(\omega)$, too. We prove now by induction with respect to the range of the elements of this tree X that

$$\ulcorner\psi\urcorner \text{ is a vertex in } X \rightarrow \mathfrak{A} \models \psi.$$

The only nontrivial case occurs when $\psi = (Rx)\sigma(x)$. Then

$$\mathfrak{A} \models \exists Y[\forall n(n \in Y \leftrightarrow \ulcorner\sigma(\boldsymbol{n})\urcorner \text{ lies directly below } \ulcorner(Rx)\sigma(x)\urcorner) \wedge R(Y)].$$

From the inductive assumption $n \in Y \rightarrow \mathfrak{A} \models \sigma(\boldsymbol{n})$. Thus $A = \{n: \mathfrak{A} \models \sigma(\boldsymbol{n})\} \supset Y \in \mathfrak{A}$. From the condition ($\nabla$) we obtain that $\mathfrak{A} \models R[A]$, i.e. $\mathfrak{A} \models (Rx)\sigma(x)$. In particular $\mathfrak{A} \models \phi_1$.

ϕ_1 was arbitrary so the lemma is proved.

PROOF OF THEOREM 4(i). Let f_R be a regular rule of proof. For some m, $R \in \Sigma_m^1$. By virtue of Lemma 2, $(A_2^-)_{f_R} \subset (A_2^-)_{f_{\Sigma(m+2)}}$. From Lemma 1 and Lemma 3, $(A_\beta \not\subset (A_2^-)_{f_{\Sigma(m+2)}})$ so $(A_\beta \not\subset (A_2^-)_{f_R})$.

In order to prove part (ii) of the Theorem 4 we prove first the following:

LEMMA 4. *Every regular rule of proof satisfies the deduction theorem.*

PROOF. Let f_R be a regular rule of proof and let $T \cup \{\phi, \psi\} \subset$ Sent. Always $T \vdash_{f_R} \phi \rightarrow \psi$ implies $T \cup \{\phi\} \vdash_{f_R} \psi$. So suppose that $T \cup \{\phi\} \vdash_{f_R} \psi$. By induction with respect to the range of the elements of the f-proof tree X of ϕ we show that $\chi \in X$ implies $T \vdash_{f_R} \phi \rightarrow \chi$. The only interesting case occurs when $\chi = (Rx)\sigma(x) \in X$. We have $\{n: T \cup \{\phi\} \vdash_{f_R} \sigma(\boldsymbol{n})\} \in R$, so, by the induction hypothesis and monotonicity of R, $\{n: T \vdash_{f_R} \phi \rightarrow \sigma(\boldsymbol{n})\} \in R$. Hence by the f_R-rule $T \vdash_{f_R} (Rx)(\phi \rightarrow \psi(x))$. But by virtue of the deduction theorem the sentence $Rx(\phi \rightarrow \psi(x)) \rightarrow (\phi \rightarrow (Rx)\psi(x))$ is logically provable. Thus $T \vdash_{f_R} \phi \rightarrow (Rx)\psi(x)$, which concludes the proof.

PROOF OF THEOREM 4(ii). From part (i) of Theorem 4 there exists a sentence ϕ such that $\phi \in A_\beta - (A_2^-)_{f_R}$. From Lemma 4, $\mathrm{Cn}((A_2^-)_{f_R} \cup \{\phi\})$ is a consistent, f_R-complete set of sentences which by the choice of ϕ has no β-model.

COROLLARY. *If F is a class of regular rules such that for some n whenever $f_R \in F$ then $R \in \Sigma_n^1$, then*

(i) $(A_\beta \not\subset (A_2^-)_F)$, *where $(A_2^-)_F$ is the closure of A_2^- under all the rules which belong to F,*

(ii) *there exists a consistent set of sentences T which is f-complete for all $f \in F$ but which has no β-model.*

PROOF. From Lemma 2 it is easy to prove by induction on proofs that $(A_2^-)_F \subset (A_2^-)_{f_{\Sigma(n+2)}}$ which by virtue of Theorem 4(i) proves (i).

Let $\phi \in A_\beta - (A_2^-)_{f_{\Sigma(n+2)}}$. For every $f \in F$ the set $\mathrm{Cn}((A_2^-)_{f_{\Sigma(n+2)}} \cup \{\neg\phi\})$ is f-complete. Indeed, if $f \in F$ then, from Lemmas 2 and 4,

$$\mathrm{Cn}((A_2^-)_{f_{\Sigma(n+2)}} \cup \{\neg\phi\})_f \subset \mathrm{Cn}((A_2^-)_{f_{\Sigma(n+2)}} \cup \{\neg\phi\})_{f_{\Sigma(n+2)}} = \mathrm{Cn}((A_2^-)_{f_{\Sigma(n+2)}} \cup \{\neg\phi\}).$$

From the choice of ϕ the set $\mathrm{Cn}((A_2^-)_{f_{\Sigma(n+2)}} \cup \{\neg\phi\})$ is consistent and has no β-model.

As we shall see in §6 the above corollary is not true in the case of arbitrary classes of the regular rules. On the other hand the class of regular rules F found in §6 such that $(A_2^-)_F = A_\beta$ is of no importance for the problem of finding a natural syntactical β-rule.

The results of this section suggest that the existence of a syntactical β-rule with some natural properties is very doubtful. It is difficult to find any interesting rules of proof different from the regular ones. The only rule worthy of mention and different from the regular rules seems to be the Def-rule defined in the end of this paper. It does not satisfy any of the conditions (*)–(***) because its graph is arithmetical.

The above considerations indicate the essential difference between the notion of the ω-model and the β-model. It is easy to build ω-models by using the ω-completeness theorem. The lack of a natural syntactical β-rule reveals, at least partially, why it is so difficult to build β-models.

It is easy to carry over the results of this section to the analogous ones which concern the syntactical β_n-rule ($n > 1$). The interesting conditions will be the conditions (*)–(***) in which instead of the notion of a β-model the notion of a β_n-model occurs.

The only additionally needed fact here is the following one: $A_{\beta_n} \in \Pi^1_{n+1} - \Sigma^1_{n+1}$.

§6. **The regular rules in relation to the completeness theorem.** The regular rules f_R such that $R \in \Pi^1_1$ fairly often satisfy the completeness theorem as the following shows:

THEOREM 5. (i) *If $R \in \Sigma^0_1$ then, for every T such that $A_2^- \subset T \subset$ Sent, $\mathrm{Cn}(T) = (T)_{f_R}$.*

(ii) *If $R \in \Pi^1_1$ then, for every T such that $A_2^- \subset T \subset$ Sent, $(T)_{f_R} \subset (T)_\omega$.*

PROOF. (i) For a recursive relation P, $R(X) \leftrightarrow \exists y P(\overline{X}(y))$. Suppose that $A_2^- \subset T \subset$ Sent. Let ϕ be a formula with one free variable such that $X_0 = \{n: T \vdash \phi(\mathbf{n})\} \in R$. Let $\mathfrak{A}$ be a model of T. We may assume that the interpretation of $\in$ in $\mathfrak{A}$ is the natural one, and that the standard model of Peano arithmetic is a substructure of the first order part of $\mathfrak{A}$ (see e.g. Shoenfield [1, p. 231]). Then ω is an initial segment of the numerical universe of $\mathfrak{A}$.

There is a unique set X_1 in $\mathfrak{A}$ such that $\mathfrak{A} \models \forall x(x \in X_1 \leftrightarrow \phi(x))$.

Let $A = \{n: \mathfrak{A} \models n \in X_1\}$. Then $X_0 \subset A$, so by the monotonicity of R we get $R(A)$, i.e., for some n_0, $P(\overline{A}(n_0))$.

$P(\overline{A}(n_0))$ is a true first order open sentence, so it is provable in Peano arithmetic. Hence $\mathfrak{A} \models P(\overline{A}(n_0))$. But, by the definition of A, $\mathfrak{A} \models \overline{X}_1(n_0) = \overline{A}(n_0)$, so $\mathfrak{A} \models P(\overline{X}_1(n_0))$, i.e. $\mathfrak{A} \models R(X_1)$.

This means that $\mathfrak{A} \models (Rx)\phi(x)$. $\mathfrak{A}$ was arbitrary, so $T \vdash (Rx)\phi(x)$, which concludes the proof.

(ii) Let $A_2^- \subset T \subset$ Sent. We prove by induction on the proof that $(T)_{f_R} \subset (T)_\omega$. Suppose that $(Rx)\phi(x) \in (T)_{f_R}$. From the monotonicity of R and the inductive assumption,

$$A = \{n: T \vdash_\omega \phi(\mathbf{n})\} \in R.$$

Let $\mathfrak{A}$ be an ω-model of T. Let $B = \{n: \mathfrak{A} \models \phi(\boldsymbol{n})\}$. Then $A \subset B$, hence by the monotonicity of R, $P(\omega) \models R[B]$.

From comprehension $B \in \mathfrak{A}$, and so $\mathfrak{A} \models R[B]$ because R is a Π^1_1 relation. But this means that $\mathfrak{A} \models (Rx)\phi(x)$.

$\mathfrak{A}$ was arbitrary, so by the ω-completeness theorem $(Rx)\phi(x) \in (T)_\omega$, which concludes the proof.

We shall now describe an easy method for obtaining regular rules which are different from the ω-rule and which satisfy the completeness theorem.

Let ψ be a true sentence of $L(A_2)$ and R an analytical monotone subset of $P(\omega)$. Then $R \wedge \psi$ as a formula determines the rule of proof $f_{R \wedge \psi}$.

THEOREM 6. *Let ψ be a true sentence of $L(A_2)$. If a regular f_R satisfies the completeness theorem then $f_{R \wedge \psi}$ does also.*

PROOF. Let $\mathfrak{A}$ be a structure for $L(A_2)$. It is easy to see that $f_{R \wedge \psi}$ is sound in $\mathfrak{A} \leftrightarrow f_R$ is sound in $\mathfrak{A}$ and $\mathfrak{A} \models \psi$. Thus, for every $T \subset$ Sent,

$$(T)_{K(f_{R \wedge \psi})} = (T \cup \{\psi\})_{K(f_R)} = (T \cup \{\psi\})_{f_R}.$$

On the other hand it is easy to prove by induction on the proofs that $(T)_{f_{R \wedge \psi}} = (T \cup \{\psi\})_{f_R}$, which concludes the proof.

And so if we take for ψ a true sentence of $L(A_2)$ such that $\psi \notin A_\omega$, then the $\omega \wedge \psi$-rule is a regular rule not equivalent to the ω-rule and which satisfies the completeness theorem.

We are now able to construct a class F of regular rules such that $(A_{\bar{2}})_F = A_\beta$. Namely $F = \{\omega \wedge \psi : \psi \in A_\beta\}$. Then $A_\beta \subset (A_{\bar{2}})_F \subset (A_\beta)_F = A_\beta$, so $(A_{\bar{2}})_F = A_\beta$. However this fact is of no importance to us, because the construction of this class directly depends on the notion of a β-model.

The results we have proved up to now seem to suggest that regular rules have properties similar to the ω-rule.

This however turns out to be not true.

Enderton introduced in Enderton [1] the $\mathscr{A}$-rule of proof. In our terminology it is the f_{Σ_1}-rule:

$$\text{if } \exists X \forall n \vdash_{\mathscr{A}} \phi(\bar{X}(\boldsymbol{n})) \text{ then } \vdash_{\mathscr{A}} \exists X \forall x \phi(\bar{X}(x)).$$

DEFINITION 9 (ENDERTON [1]). Let $\mathfrak{A}$ be a structure for $L(A_2)$. $\mathfrak{A}$ is a $d\beta$-model if

$$\mathfrak{A} \models \mathrm{Bord}[X] \rightarrow (X \text{ codes a well-ordering})$$

for every X which is definable in $\mathfrak{A}(X \in \mathrm{Def}(\mathfrak{A}))$.

The following properties of the $d\beta$-models connect them with the $\mathscr{A}$-rule (see Enderton [1]):

1°. Every $d\beta$-model of $A_{\bar{2}}$ is an ω-model.

2°. An ω-model of $A_{\bar{2}}$ is a $d\beta$-model iff the $\mathscr{A}$-rule is sound in it.

3°. The ω-rule is derivable in $A_{\bar{2}}$ from the $\mathscr{A}$-rule.

Using this we proved in Apt [1] that the $\mathscr{A}$-rule does not satisfy the completeness theorem. Namely we proved that

$$\ulcorner (A_2)_{K(\mathscr{A})} \urcorner = \ulcorner \{\phi : \mathfrak{A} \models \phi \text{ for every } d\beta\text{-model } \mathfrak{A} \text{ of } A_2\} \urcorner \notin \Sigma^1_2$$

whereas (see Enderton [1]) $\ulcorner (A_2)_{\mathscr{A}} \urcorner \in \Sigma^1_2$.

We prove now a stronger theorem.

THEOREM 7. *Assume that* $P(\omega) \subseteq L$. *If, for every* T *such that* $A_2^- \subset T \subset$ Sent, $(T)_{\mathscr{A}} \subset (T)_{f_R}$, *then the* f_R*-rule does not satisfy the completeness theorem.*

PROOF. It is easy to see that in Lemma 1 we can replace A_2^- by an arbitrary T such that $A_2^- \subset T \subset$ Sent, T is recursive and $P(\omega) \models T$. Then Theorem 4(i) is true after replacing A_2^- by T.

By assumption $P(\omega) \models A_2^- \cup \{\forall X\, \mathrm{Constr}(X)\}$, so by the above

$$((A_2^- \cup \{\forall X\, \mathrm{Constr}(X)\})_\beta \not\subset (A_2^- \cup \{\forall X\, \mathrm{Constr}(X)\})_{f_R}).$$

Let $\mathfrak{A} \in K(f_R)$ and $\mathfrak{A} \models A_2^- \cup \{\forall X\, \mathrm{Constr}(X)\}$. Then by assumption the $\mathscr{A}$-rule is sound in $\mathfrak{A}$. From facts 1°–3° it is easy to see (cf. e.g. Apt [1]) that there exists a $d\beta$-model $\mathfrak{B}$ which is elementarily equivalent to $\mathfrak{A}$. But $\mathfrak{B} \models A_2^- \cup \{\forall X\, \mathrm{Constr}(X)\}$, so by the definability of the Skolem functions in $\mathfrak{B}$, $\mathrm{Def}(\mathfrak{B}) \prec \mathfrak{B}$. Thus $\mathrm{Def}(\mathfrak{B})$ is a β-model which is elementarily equivalent to $\mathfrak{A}$. This shows that

$$(A_2^- \cup \{\forall X\, \mathrm{Constr}(X)\})_\beta \subset (A_2^- \cup \{\forall X\, \mathrm{Constr}(X)\})_{K(f_R)}.$$

This fact together with the above one gives that

$$(A_2^- \cup \{\forall X\, \mathrm{Constr}(X)\})_{K(f_R)} \neq (A_2^- \cup \{\forall X\, \mathrm{Constr}(X)\})_{f_R}$$

which concludes the proof.

In the case of the $f_{\Sigma(n)}$-rules $(n \geq 1)$ we can omit the assumption $P(\omega) \subseteq L$. Namely the following holds:

THEOREM 8. *None of the* $f_{\Sigma(n)}$*-rules* $(n \geq 1)$ *satisfies the completeness theorem.*

We shall need in the proof the following obvious lemma.

LEMMA 5. *Let* f_R *be a regular rule. If* $\mathfrak{A}$ *is an* ω*-model of* A_2^-, *then* f_R *is sound in* $\mathfrak{A} \leftrightarrow$ *(for every* $A \in \mathrm{Def}(\mathfrak{A})(P(\omega) \models R[A] \rightarrow \mathfrak{A} \models R[A])$*).*

PROOF. f_R is sound in $\mathfrak{A}$ iff, for every formula ϕ with one free variable, $\{n : \mathfrak{A} \models \phi(\boldsymbol{n})\} \in R \rightarrow \mathfrak{A} \models (Rx)\phi(x)$, which concludes the proof.

PROOF OF THE THEOREM 8. Since we already know that the $\mathscr{A}$-rule does not satisfy the completeness theorem, we may assume that $n > 1$.

Define the monotone operator Γ_n as follows:

$$x \in \Gamma_n(X) \leftrightarrow x \in \ulcorner \mathrm{Cn} \urcorner(X) \vee x \in \ulcorner A_2^- \cup \{\forall X\, \mathrm{Constr}(X)\}\urcorner \\ \vee \exists \psi \exists Y(\Sigma(n)(Y) \wedge \forall y(y \in Y \leftrightarrow \ulcorner \psi(y) \urcorner \in X) \wedge x = \ulcorner(\Sigma(n)z)\psi(z)\urcorner).$$

Then $(A_2^- \cup \{\forall X\, \mathrm{Constr}(X)\})_{f_{\Sigma(n)}}$ is the set which is inductively defined with respect to Γ_n. Since $\Gamma_n \in \Sigma_n^1$, $\ulcorner(A_2^- \cup \{\forall X\, \mathrm{Constr}(X)\})_{f_{\Sigma(n)}}\urcorner \in \Sigma_n^1$.

We now prove that $(A_2^- \cup \{\forall X\, \mathrm{Constr}(X)\})_{K(f_{\Sigma(n)})} \notin \Sigma_{n+1}^1$ which will conclude the proof.

Suppose that $\mathfrak{A} \in K(f_{\Sigma(n)})$ and $\mathfrak{A} \models A_2^- \cup \{\forall X\, \mathrm{Constr}(X)\}$. Since the ω-rule is derivable in A_2^- from the $f_{\Sigma(n)}$-rule, it is sound in $\mathfrak{A}$. By the ω-completeness theorem there exists an ω-model $\mathfrak{B}$ which is elementarily equivalent to $\mathfrak{A}$. But $\mathfrak{B} \models A_2^- \cup \{\forall X\, \mathrm{Constr}(X)\}$, thus $\mathrm{Def}(\mathfrak{B}) \prec \mathfrak{B}$.

Let now R be a Σ_n^1 monotone subset of $P(\omega)$. We prove that f_R is sound in $\mathrm{Def}(\mathfrak{B})$. In order to prove this it is sufficient to prove that if S is a complete set of sentences of $L(A_2)$ which contains A_2^- then $(S)_{f_R} \subset (S)_{f_{\Sigma(n)}}$.

We prove it by induction on the proofs. For simplicity of notation assume that n is odd; $n = 2k + 1$.

For a recursive relation P we have

$$A_2^- \vdash R(X) \leftrightarrow \exists X_1 \forall X_2 \cdots \exists X_{2k+1} \forall y P(\bar{X}(y), \bar{X}_1(y), \cdots, \bar{X}_{2k+1}(y)).$$

Assume that $(Rx)\phi(x) \in (S)_{f_R}$. Using the inductive assumption and repeating the reasoning from Lemma 2 we obtain

$$\exists Z \exists X_1 \forall X_2 \cdots \exists X_{2k+1} \forall y \quad S \vdash_{f_{\Sigma(2k+1)}} P(\bar{Z}(y), \bar{X}_1(y), \cdots, \bar{X}_{2k+1}(y)) \wedge ((\bar{Z}(y+1))_y = 1 \leftrightarrow \phi(y))$$

(equivalence in the second part of the conjunction comes from the fact that S is complete).

Hence by the $f_{\Sigma(2k+1)}$-rule

$$S \vdash_{f_{\Sigma(2k+1)}} \exists Z \exists X_1 \forall X_2 \cdots \exists X_{2k+1} \forall y \; P(\bar{Z}(y), \bar{X}_1(y), \cdots, \bar{X}_{2k+1}(y)) \wedge ((\bar{Z}(y+1))_y = 1 \leftrightarrow \phi(y)).$$

But $A_2^- \subset S$, thus from the form of R we obtain

$$S \vdash_{f_{\Sigma(2k+1)}} \exists Z(R(Z) \wedge \forall y(y \in Z \leftrightarrow \phi(y)), \quad \text{i.e. } S \vdash_{f_{\Sigma(2k+1)}} (Rx)\phi(x).$$

All the other cases are trivial, so indeed $(S)_{f_R} \subset (S)_{f_{\Sigma(n)}}$.

Let now ψ be a Π^1_{n+1} true sentence. For a Σ^1_n formula

$$A_2^- \vdash \psi \leftrightarrow \forall X \phi(X),$$

ϕ as a relation is a Σ^1_n monotone subset of $P(\omega)$, so by the above the f_ϕ-rule is sound in Def($\mathfrak{B}$). By Lemma 5 and the form of ψ we obtain Def($\mathfrak{B}$) $\models \psi$.

But Def($\mathfrak{B}$) is elementarily equivalent to $\mathfrak{A}$, so $\psi \in (A_2^- \cup \{\forall X \operatorname{Constr}(X)\})_{K(f_{\Sigma(n)})}$. Thus (see Rogers [1, p. 390]) $(A_2^- \cup \{\forall X \operatorname{Constr}(X)\})_{K(f_{\Sigma(n)})} \notin \Sigma^1_{n+1}$.

One can also omit in Theorem 7 the assumption $P(\omega) \subseteq L$ when $R \in \Pi^1_3$. Namely it holds.

THEOREM 9. *Suppose that, for every T such that $A_2^- \subset T \subset$ Sent, $(T)_{\mathscr{A}} \subset (T)_{f_R}$ and that $R \in \Pi^1_3$. Then the f_R-rule does not satisfy the completeness theorem.*

PROOF. The set $\ulcorner(A_2^- \cup \{\forall X \operatorname{Constr}(X)\})_{f_R}\urcorner$ is a Π^1_3 one, because

$$\phi \in (A_2^- \cup \{\forall X \operatorname{Constr}(X)\})_{f_R}$$
$$\leftrightarrow \forall T[(A_2^- \cup \{\forall X \operatorname{Constr}(X)\} \subset T \wedge \operatorname{Cn}(T) = T \wedge T \text{ is } f_R\text{-complete}) \rightarrow \phi \in T]$$

and a set $\{\phi \colon \ulcorner\phi\urcorner \in T\}$ is f_R-complete if

$$\forall \psi[\forall Y(\forall n(n \in Y \leftrightarrow \psi(\mathbf{n}) \in T) \rightarrow R(Y)) \rightarrow (Rx)\phi(x) \in T].$$

Assume now the axiom of constructibility. From the proof of Theorem 7 it follows that there exists a natural number n such that

$$n \in \ulcorner(A_2^- \cup \{\forall X \operatorname{Constr}(X)\})_\beta\urcorner \wedge n \notin \ulcorner(A_2^- \cup \{\forall X \operatorname{Constr}(X)\})_{f_R}\urcorner.$$

Since the sentence $n \in \ulcorner(A_2^- \cup \{\forall X \operatorname{Constr}(X)\})_\beta\urcorner$ is a Π^1_2 one (see Mostowski [1]), the above conjunction is a Σ^1_3 sentence. From Shoenfield's lemma (see Shoenfield [1]) it follows that the fact that $ZFC + V = L \vdash \phi$ where ϕ is a Σ^1_3 sentence implies that $ZFC \vdash \phi$. Thus

$$((A_2^- \cup \{\forall X \operatorname{Constr}(X)\})_\beta \not\subset (A_2^- \cup \{\forall X \operatorname{Constr}(X)\})_{f_R}).$$

On the other hand from the proof of the Theorem 7 it also follows that

$$(A_2^- \cup \{\forall X\, \mathrm{Constr}(X)\})_\beta \subset (A_2^- \cup \{\forall X\, \mathrm{Constr}(X)\})_{K(f_R)}.$$

The above two facts indicate that the f_R-rule does not satisfy the completeness theorem.

Thus in particular the rules $f_{\pi(2)}$ and $f_{\pi(3)}$ do not satisfy the completeness theorem. In the next section we come to the problems concerning the completeness theorem for regular rules from another point of view—we examine which properties must be satisfied by the semantics of a regular rule which consist of ω-models.

§7. Restrictions on the ω-semantics of the regular rules.

DEFINITION 10. Let f be a rule of proof formulated in the language $L(A_2)$. Suppose that K is a semantics for f. K is an ω-semantics for f if every structure belonging to K is an ω-model and if K is closed under elementary substructures.

THEOREM 10. *Assume that $P(\omega) \subseteq L$. If K is an ω-semantics for a regular rule f_R then there exists in K an $\subseteq$-descending ω-sequence of models of $A_2^- \cup \{\forall X\, \mathrm{Constr}(X)\}$.*

Observe that a regular rule f_R has an ω-semantics iff it satisfies the completeness theorem and, for every set of sentences T, $(T)_\omega \subset (T)_{f_R}$.

For the proof of the theorem we need the following lemma:

LEMMA 6. *Suppose that K is an ω-semantics for a f_R-rule. Then*

$$[\mathfrak{A} \in K \wedge \mathfrak{A} = \mathrm{Def}(\mathfrak{A}) \wedge \mathfrak{A} \models A_2^- \cup \{\forall X\, \mathrm{Constr}(X)\} \wedge \mathfrak{A} \textit{ is not a } \beta\textit{-model}]$$
$$\rightarrow \exists \mathfrak{B}[\mathfrak{B} \in K \wedge \mathfrak{B} \subsetneq \mathfrak{A} \wedge \mathfrak{B} = \mathrm{Def}(\mathfrak{B})$$
$$\wedge\, \mathfrak{B} \models A_2^- \cup \{\forall X\, \mathrm{Constr}(X)\} \wedge \mathfrak{B} \textit{ is not a } \beta\textit{-model}].$$

PROOF. We use here the idea of the proof of Friedman's theorem about the nonexistence of a minimal ω-model of A_2 (Friedman [1]; an idea of this proof is presented in Apt-Marek [1]).

Since $\mathfrak{A}$ is not a β-model, there exists $X_0 \in \mathfrak{A}$ such that $\mathfrak{A} \models \mathrm{Bord}[X_0]$ but $P(\omega) \models \neg\mathrm{Bord}[X_0]$. For some n_0 there exists a $\Sigma^1_{n_0}$ formula ϕ_0 such that $\mathfrak{A} \models \forall x(x \in X_0 \leftrightarrow \phi_0(x))$ (because $\mathfrak{A} = \mathrm{Def}(\mathfrak{A})$).

Let Tr_{n_0} be a definition of truth for $\Sigma^1_{n_0} \cup \Pi^1_{n_0}$ sentences, i.e. a formula with one free variable such that, for every $\Sigma^1_{n_0} \cup \Pi^1_{n_0}$ sentence ϕ, $A_2^- \vdash \phi \leftrightarrow \mathrm{Tr}_{n_0}(\ulcorner\phi\urcorner)$.

DEFINITION. A theory T is f_R-n-complete if:

(a) $T = (\{\phi : \phi \text{ is a } \Sigma^1_n \cup \Pi^1_n \text{ sentence } \wedge \phi \in T\} \cup A_2^- \cup \{\forall X\, \mathrm{Constr}(X)\})_{f_R}$.

(b) T decides all $\Sigma^1_n \cup \Pi^1_n$ sentences.

(c) T contains all true $\Sigma^1_{n_0} \cup \Pi^1_{n_0}$ sentences.

(d) T is consistent.

Let $V = \{\langle T_0, T_1, \cdots, T_n\rangle : T_i \text{ is } f_R\text{-}i\text{-complete and } i < j \rightarrow T_i \subseteq T_j\}$. If $X \in V$ and $Y \in V$ then $X \prec Y$ denotes that Y, as a sequence, extends X. It is easy to see by using the formula $\mathrm{Tr}_{n_0}(x)$ that V is an analytical subset of $P(\omega)$. Let $\tilde{V} = \{X : \mathfrak{A} \models V[X]\}$. $\tilde{V}$ is a tree with the relation $\prec$ restricted to the elements of $\tilde{V}$.

We prove now the following three facts which suffice for the proof of the lemma.

Fact 1. Every infinite branch in $\tilde{V}$ determines an ω-model $\mathfrak{B}$ such that $\mathrm{Def}(\mathfrak{B}) = \mathfrak{B}$, $\mathfrak{B} \subseteq \mathfrak{A}$, $\mathfrak{B} \in K$, $\mathfrak{B} \models A_2^- \cup \{\forall X\, \mathrm{Constr}(X)\}$, and $\mathfrak{B}$ is not a β-model.

Fact 2. The different infinite branches in $\tilde{V}$ determine different submodels of $\mathfrak{A}$.

Fact 3. There exist two different infinite branches in $\tilde{V}$.

PROOF OF THE FACT 1. Let $\{\langle Y_0, Y_1, \cdots, Y_n\rangle\}_{n<\omega}$ be an infinite branch in $\tilde{V}$. Let $T = \bigcup_{n<\omega} Y_n$. Then since the conditions (b) and (d) are arithmetical and thus absolute with respect to $\mathfrak{A}$, T is a complete, consistent set of sentences. We prove now that T is f_R-complete.

Suppose that $A = \{n : \phi(\boldsymbol{n}) \in T\} \in R$, where ϕ is a formula with one free variable. For some k, ϕ is a Σ_k^1 formula, thus $A = \{n : \phi(\boldsymbol{n}) \in Y_k\}$, since Y_k decides all $\Sigma_k^1 \cup \Pi_k^1$ sentences. Thus $n \in A \leftrightarrow \mathfrak{A} \models \ulcorner\phi(\boldsymbol{n})\urcorner \in Y_k$. From comprehension, $A \in \mathfrak{A}$.

Since $\mathfrak{A} \in K$, f_R is sound in $\mathfrak{A}$ (Theorem 1(i)). We have $P(\omega) \models R[A]$ and $A \in \mathrm{Def}(\mathfrak{A}) = \mathfrak{A}$. From this and from Lemma 5, $\mathfrak{A} \models R[A]$, i.e.

$$\mathfrak{A} \models \exists X(R(X) \wedge \forall x(x \in X \leftrightarrow \ulcorner\phi(x)\urcorner \in \ulcorner Y_k\urcorner)).$$

But the sentence "Y_k is f_R-complete" is true in $\mathfrak{A}$, thus from the above $\mathfrak{A} \models \ulcorner(Rx)\phi(x)\urcorner \in \ulcorner Y_k\urcorner$. From the definition of T, $(Rx)\phi(x) \in T$, i.e. T is indeed f_R-complete.

By the assumption about K, $(T)_K = (T)_{f_R} = T$, so T has an ω-model $\mathfrak{C} \in K$. Since $A_2^- \cup \{\forall X\,\mathrm{Constr}(X)\} \subseteq T$, $\mathfrak{B} = \mathrm{Def}(\mathfrak{B}) \prec \mathfrak{C}$. Thus $\mathfrak{B} \in K$.

Let $X \in \mathfrak{B}$. For some Σ_m^1 formula ϕ with one free variable $X = \{n : \mathfrak{B} \models \phi(\boldsymbol{n})\}$. $T = \mathrm{Th}(\mathfrak{B})$, so $X = \{n : \phi(\boldsymbol{n}) \in T\}$. But $T = \bigcup_{k<\omega} Y_k$ and Y_m decides all $\Sigma_m^1 \cup \Pi_m^1$ sentences, so $\forall n(n \in X \leftrightarrow \phi(\boldsymbol{n}) \in Y_m)$. Since $\ulcorner Y_m\urcorner \in \mathfrak{A}$, by comprehension, $X \in \mathfrak{A}$. Thus $\mathfrak{B} \subseteq \mathfrak{A}$.

It is left to check that $\mathfrak{B}$ is not a β-model. Let ϕ be a $\Sigma_{n_0}^1 \cup \Pi_{n_0}^1$ sentence which is true in $\mathfrak{A}$. Then $\mathfrak{A} \models \mathrm{Tr}_{n_0}(\ulcorner\phi\urcorner)$, because $A_2^- \vdash \phi \leftrightarrow \mathrm{Tr}_{n_0}(\ulcorner\phi\urcorner)$.

But $\mathfrak{A} \models$ "Y_{n_0} is f_R-n_0-complete", so in particular (point (c)) $\mathfrak{A} \models \ulcorner\phi\urcorner \in Y_{n_0} \leftrightarrow \mathrm{Tr}_{n_0}(\ulcorner\phi\urcorner)$. Hence $\mathfrak{A} \models \ulcorner\phi\urcorner \in \ulcorner Y_{n_0}\urcorner$, so $\phi \in T$, thus $\mathfrak{B} \models \phi$.

From this we obtain that

$$\forall n(n \in X_0 \leftrightarrow \mathfrak{A} \models \phi_0(\boldsymbol{n}) \leftrightarrow \mathfrak{B} \models \phi_0(\boldsymbol{n})),$$

so by comprehension $X_0 \in \mathfrak{B}$. Since $\mathfrak{A} \models \mathrm{Bord}[X_0]$ and $\mathfrak{B} \subseteq \mathfrak{A}$, $\mathfrak{B} \models \mathrm{Bord}[X_0]$. And so $\mathfrak{B}$ indeed is not a β-model.

PROOF OF THE FACT 2. It is easy to see that the different infinite branches in $\tilde{V}$ determine the elementarily nonequivalent models.

PROOF OF THE FACT 3. In the proof of Friedman's theorem instead of the notion of f_R-n-completeness there occurs ω-n-completeness. Also the condition (c) does not occur there. The proof of Fact 3 can be carried over without any change from the proof of Friedman's theorem, so we omit it here. The only particular facts used there are: the counterpart of the Fact 1, $\mathfrak{A} = \mathrm{Def}(\mathfrak{A})$, $\mathfrak{A} \models A_2^- \cup \{\forall X\,\mathrm{Constr}(X)\}$, $\mathfrak{A}$ is not a β-model. By assumption the model $\mathfrak{A}$ satisfies these conditions.

PROOF OF THEOREM 10. By assumption $P(\omega) \models \forall X\,\mathrm{Constr}(X)$, so (see the proof of Theorem 7)

$$((A_2^- \cup \{\forall X\,\mathrm{Constr}(X)\})_\beta \not\subseteq (A_2^- \cup \{\forall X\,\mathrm{Constr}(X)\})_K)$$

because $(A_2^- \cup \{\forall X\,\mathrm{Constr}(X)\})_{f_R} = (A_2^- \cup \{\forall X\,\mathrm{Constr}(X)\})_K$.

If $\mathfrak{A} \in K$ and $\mathfrak{A} \models A_2^- \cup \{\forall X\,\mathrm{Constr}(X)\}$ then $\mathrm{Def}(\mathfrak{A}) \prec \mathfrak{A}$ and by assumption $\mathrm{Def}(\mathfrak{A}) \in K$.

From these two above facts it follows that there exists $\mathfrak{A} = \mathrm{Def}(\mathfrak{A})$ such that $\mathfrak{A} \in K$, $\mathfrak{A} \models A_2^- \cup \{\forall X\, \mathrm{Constr}(X)\}$ and $\mathfrak{A}$ is not a β-model. Now, using Lemma 6 ω times we obtain the theorem.

As with Theorem 7 we are able to omit here the assumption $P(\omega) \subseteq L$ in the case when $R \in \Pi_3^1$. This assumption was needed only in order to have

$$((A_2^- \cup \{\forall X\, \mathrm{Constr}(X)\})_\beta \not\subset (A_2^- \cup \{\forall X\, \mathrm{Constr}(X)\})_{f_R}).$$

But, as we observed in the proof of the Theorem 9, for regular rules f_R such that $R \in \Pi_3^1$ the above statement is true also without the assumption $P(\omega) \subseteq L$.

From Lemma 6 there follows easy conclusions which concern the existence of minimal models in the ω-semantics.

COROLLARY. *Suppose that f_R is a regular rule with an ω-semantics K. Then*

(i) *If K has a $\subseteq$-minimal model of $A_2^- \cup \{\forall X\, \mathrm{Constr}(X)\}$ then it is a β-model.*

(ii) *If K has a $\subseteq$-minimal model of $A_2^- \cup \{\forall X\, \mathrm{Constr}(X)\}$ then every true Σ_1^1 sentence belongs to $(A_2^- \cup \{\forall X\, \mathrm{Constr}(X)\})_{f_R}$.*

PROOF. (i) From Lemma 6.

(ii) Let $\mathfrak{A}$ be this $\subseteq$-minimal model. By (i), $\mathfrak{A}$ is a β-model.

Let ϕ be a true Σ_1^1 sentence. Then $\mathfrak{A} \models \phi$. If $\mathfrak{B} \in K$ and $A_2^- \cup \{\forall X\, \mathrm{Constr}(X)\}$ then $\mathfrak{A} \subseteq \mathfrak{B}$, and so $\mathfrak{B} \models \phi$. Thus

$$\phi \in (A_2^- \cup \{\forall X\, \mathrm{Constr}(X)\})_K = (A_2^- \cup \{\forall X\, \mathrm{Constr}(X)\})_{f_R}.$$

Observe that the implication in Lemma 6 is still valid if we take for K the class of all ω-models in which the f_R-rule is valid and assume that, for every T such that $A_2^- \cup \{\forall X\, \mathrm{Constr}(X)\} \subset T \subset \mathrm{Sent}$, $(T)_\omega \subset (T)_{f_R}$. Indeed, the set T from the proof of the Fact 1 in the proof of Lemma 8 has then an ω-model belonging to K, because it is consistent, complete and f_R-complete, thus also ω-complete. Apart from this small change, the proof is exactly like that of Lemma 6.

The results of this paper reveal some essential properties of the regular rules. In order to get to know better the character of these rules one should first of all find the necessary and sufficient conditions under which a regular rule of proof satisfies the completeness theorem.

The only interesting rule of proof different from the regular rules of proof seems to us to be the one defined in Apt [1].

Def-rule. From the fact that $\vdash_{\mathrm{Def}} \exists X(\Psi(X) \wedge \forall x(x \in X \leftrightarrow \phi(x)))$ for every formula ϕ with one free variable infer that $\vdash_{\mathrm{Def}} \forall X \Psi(X)$.

The Def-rule satisfies the completeness theorem and the consistency theorem and the pointwise definable models for $L(A_2)$ form a semantics for it.

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