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Symmetry in physics and system theory. An introduction to past, present and future possibilities.

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Abstract.

Symmetry considerations and invariance principles play an extremely important role in physics. An outline is given of some of the ideas involved modivated by the expectation that such considerations will also be of importance in systems and control theory.

Introduction. General remarks on the role and importance of symmetry considerations.

Most scientists and engineers here probably know in a vague way that symmetry considerations and invariance principles and symmetry groups are important in physics, especially quantum physics and relativity theory and most especially elementary particle theory (irreducible representation → elementary particle). Yet most will be surprised to learn, as I was, that in a leading journal like J. of Math. Physics in 1983 some 36% of the papers dealt with aspects of group representations. symmetry or invariance (146 out of 406) and in Lett. Math. Physics the percentage is even higher: 54% (37 out of 69). By and large this is a fairly recent phenomenon: the ubiquity of symmetry considerations in physics and chemistry and their enormous success. To quote from [1]:

> "The importance of group theory and its utility in applications to various branches of physics and chemistry is now so well established and universally recognised that its explicit use needs neither apology nor justification".

It is my belief there is an equally impressive role awaiting symmetry in the engineering sciences, and in this introductory note I will try to indicate why by means of both general remarks and a relatively precise description of some specific applications of symmetry ideas and techniques in physics. Here is how in a field where symmetry considerations are quite recent, viz. critical phenomena (phase transitions, renormalization group, scaling invariance), the role of symmetry and groups is described by S.-K. MA [2]. He stresses two approaches to deal with *complex* physical problems:

(i) Direct solution approach. This means calculation of physical quantities of interest in terms of parameters given in the particular model - in other words, solving the model. The calculation may be done analytically or numerically, exactly or approximately.

(ii) Exploiting summetries. This approach does not attempt to solve the model. It considers how parameters change under certain symmetry transformations. From various symmetry properties one deduces some characteristics of physical quantities. These characteristics are generally independent of the quantitative values of the parameters.

Approach (ii) is not a substitute for approach (i). Experience tells us that one should try (ii) as far as one can before attempting (i), since (i) is often a very difficult task. Results of (ii) may simplify this task greatly. A great deal can be learned from (ii) without even attempting (i)."

Topics in physics and chemistry in which group theory and symmetry considerations play decisive roles include conservation laws, atomic and moleculer spectrocopy (including Raman scattering), collective models of the nucleus, chemical bond theory, elementary particle theory and grand unification (of the forces of nature), relativity theory [3], quantization theory [4], theory of phase transitions and critical phenomena [5], cosmology, selection rules and transition probabilities in quantum mechanics, theory of electron bands in solids, crystallography, soliton theory and its many applications, renormalization theory, theory of polymers, gauge fields, phonon dispersion relations, electronic and nuclear shell theory, theory of spin glasses.

The vigour of the subject is attested to by - among others - the yearly international colloquium on Group theoretical methods in physics [6]. A more or less random selection of books, besides the ones specifically quoted below or above, dealing with various aspects of symmetry and groups in physics and chemistry is [7]. Writing a complete survey of applications of symmetry and groups in physics would probably take 3000 plus pages. And that would be just the applications in physics. In mathematics itself groups and representations play an equally central role and many of the currently hot topics involve Lie groups in one way or another. Very possibly some sort of homogeneity, symmetry is needed to make a problem interesting or esthetically pleasing or, indeed, tractable [8].

That is useful to have some symmetry present when dealing with, say, differential equations is an old observation and indeed was the first impetus which lead to the "discovery" of Lie groups. In the words of Sophus Lie [9]:

> "Ich bemarkte, dass die meisten gewöhnlichen Differentialgleichungen, deren Integration durch die älteren Integrationsmethoden geleistet wird, bei gewissen leicht angebaren Schaaren von Transformationen invariant bleiben, und dass jene Integrationsmethoden in der Verwehrtung dieser Eigenschaft der betroffenden Differentialgleichung bestehen."

This aspect: symmetry inspired analysis of differential equations (both ordinary and partial) and also in the future difference equations), i.e. involving questions around the theme differential Galois theory, has been all but neglected for some 70 years after Lie, but has been the subject of a vigorous rivival in the last 20 especially the last 10 years or so [10]. It is a fast developing subject with a very large number of unsettled questions. Indeed in many contexts it is not yet al all clear what the right definition is of a symmetry. Cf. also below in section 12.

As I remarked above it would take a considerable number of pages even to give an indicative survey of the applications of symmetry ideas in physics. So here I will simply describe a very few of them in brief vignettes.

2. Groups, actions, representations.

Let G be a group, S a set. An *action* of G on S is a map $G \times S \rightarrow S$, $(g,s) \rightarrow gs$, such that $g(hs) = (gh)s$, $es = s$ for all $g,h \in G$, $s \in S$, where $e \in G$ is the identity element. Then S is called a $G-set$. Thus, if $gs = s$ all s implies $g = e$, the group G is realized as a group of transformations of the set S. The subset $Gs = \{gs : g \in G\}$ for a given $s \in S$ is called the *orbit* of G through s. A function f on S is *invariant* if $f(gs) = f(s)$ for all g,s.

Two G-sets S,T are *isomorphic* if there is a bijection $\phi: S \rightarrow T$ such that $\phi(gs) = g\phi(s)$ for all $g \in G$, $s \in S$. The union of all orbits in a G-set which are isomorphic to a given one (as G-sets) is called a *stratum.*

If a G-set V is a vectorspace and $s \rightarrow gs$ is a linear mapping for all $g \in G$ then V is called a *representation* of G. Such a representation is called *irreducible* if there are no subspaces W of V other than $\{0\}$ and V which are invariant.

A homomorphism of vector spaces $\phi V \rightarrow W$ between two representations is *covariant* if it is also a map of G-sets. Two representations are *isomorphic* or *equivalent* if there is a covariant isomorphism of vectorspaces between them.

Let V be a real or complex vectorspace with product $<,>$. If $<,>$ is *invariant*, i.e. $<gv, gw> = <v, w>$ for all $g \in G, v, w \in V$ the representation is said to be *unitary*. For many groups every class of equivalent representations contains unitary ones (Maschke's theorem).

Many applications of symmetry ideas in physics rest on (souped up versions) of one of the following three facts.

A. (Full reducibility). For many groups, e.g. finite and compact groups, every representation W decomposes into a direct sum of irreducible representations, $W = \bigoplus_\alpha V^{(\alpha)}$. Elements of W (usually functions or operators) belonging to nonisomorphic irreducible components behave differently (w.r.t. G; cf. e.g. C below) and thus the irreducible representations of G serve to label different types of objects, e.g. elementary particles. More generally one meets direct integral decompositions, generalizing spectral integral representations of operators.

B. (Schur's lemma). Let ϕ: $V \to W$ be a covariant homomorphism between the complex irreducible represenations V, W of G. Then $\phi = 0$ if V and W are inequivalent and $\phi = \lambda Id$ for some complex number λ if V and W are equivalent.

C. (Orthogonality relations). Let $V^{(\alpha)}$, $V^{(\beta)}$ be two irreducible unitary representations of a finite group G with $\#G$ elements. Choose bases $e_i^{(\alpha)}$ and $e_j^{(\beta)}$ for $V^{(\alpha)}$, $V^{(\beta)}$ and let $T_{ij}^{(\alpha)}(G)$, be the the (i,j)-element of the matrix w.r.t. to $e_i^{(\alpha)}$ of $g: V^{(\alpha)} \to V^{(\alpha)}$. Then

$$s_\alpha \sum_g T_{ij}^{(\alpha)}(g)\, T_{kl}^{(\beta)*}(g) = (\#G)\delta_{\alpha\beta}\delta_{ik}\delta_{jl} \tag{2.1}$$

where δ_{mn} denotes the Kronecker symbol and, $s_\alpha = dim V^{(\alpha)}$, and $T_{kl}^{(\beta)*}(g)$ is the (k,l) element of the adjoint matrix $T^{\beta}(g)^*$.

Let V be an unitary representation of the finite group G, $V^{(\alpha)}$ and $V^{(\beta)}$ two irreducible components and $e_i^{(\alpha)}, e_k^{(\beta)}$ bases for $V^{(\alpha)}$ and $V^{(\beta)}$. Then there are also orthogonality relations between these basis vectors

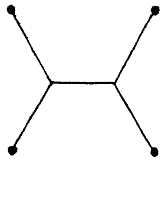
$$<e_i^{(\alpha)}, e_k^{(\beta)}> = s_\alpha^{-1}\delta_{\alpha\beta}\delta_{ik}\sum_j <e_j^{(\alpha)}, e_j^{(\alpha)}>. \tag{2.2}$$

So in particular if V is a space of functions and $<,>$ is given by a suitable integral (like $<f, g> = \int f(x)\bar{g}(x)dV$), as is often the case, and f belongs to an irreducible representation not isomorphic to the trivial one (i.e. not $g: V^{(\alpha)} \to V^{(\alpha)} = id$ for all g) then (taking $e_k^{(\beta)} = 1$) we find $\int f dV = 0$.

Let S be a G-set which is a single orbit (a transitive G-set). This looks like an irreducible situation, a primitive or elementary one incapable of further analysis. Consider however the space $\mathfrak{F}(S)$ of all complex valued functions on S. This becomes a G-set under $(gf)(s) = f(gs)$, indeed a representation of G, and it may very well be reducible, thus giving rise to a decomposition of every function on S into a sum of functions that are "nicely behaved" in a certain sense. For the case that $S = G$ is the circle group this leads to the statement that every function on the circle (or every 2π-periodic function) is representable as a Fourier series $\Sigma c_n e^{inx}$ and for the case of $G = \mathbb{R}$ this leads (via direct integrals instead of direct sums) to Fourier transform integral representations of functions. A large part of the theory of group representations as applied in analysis and physics is concerned with generalizations of Fourier expansion theorems in which $S \neq G$ and G is compact but not commutative.

3. The Purkiss principle.

Many problems, e.g. of the optimization kind, come with a natural symmetry built in, so to speak. This symmetry may for instance derive from a fact (axiom) like: "the physics is independent of the observer (or the coordinate system used to describe things). Nature also seems to like (more or less) symmetric solutions"), and it is up to the scientist to expain why. Designers, especially of large systems, also seem to like a good deal of homogeneity (symmetry). All this would be nicely understood if there were a principle like: symmetric solutions have symmetric solutions. This has been called the Purkiss principle by W.C. Waterhouse [12]. It does however certainly not hold in general.



My favourite counterexample asks for the shortest system of roads connecting four towns arranged in a square. The solution is something like depicted above. Here is also a positive result:

Theorem [12]. *Let M be a differentiable manifold, G a finite group of smooth maps from M to M, $m \in M$ a point fixed by G (i.e. $Gm = \{m\}$), and $f: M \to \mathbb{R}$ a differentiable function invariant under G. Assume that the induced action of G on the tangant space $T_m M$ at m is nontrivial and irreducible. Then m is a critical point of f and if m is nondegenerate it is a local extremum.*

There are obvious potential applications of such a theorem (and various conceivable generalizations) to all kinds of optimal control problems. These applications are different from such as occur in [13] which are based on the idea of passing to a "quotient space" M/G, or, more or less equivalently, a Noether theorem (reduction by means of symmetry, cf. also 9 below).

To the same circle of ideas belongs the following theorem of L. Michel.

Theorem. *Let G be a compact group acting smoothly on a differentiable manifold M and let f be an invariant function on M. Then if m is isolated in its stratum it is a critical point of f. And inversely if m is critical for all invariant functions on M then m is isolated in its stratum.*

Quite generally the presence of symmetry has enormous influence on critical points and singularities [14] and may also greatly help to overcome traditional difficulties associated with multiplicities and degeneracies, e.g. in bifurcation theory and spectral problems.

Quite often though an (optimal control) problem may itself be symmetric, but the boundary conditions not (initial and target set e.g.). In optimal growth theory for economies one has in this setting so-called "turnpike" theorems, which roughly say "go as quickly as you can to a ray of maximal balanced growth, then proceed along it for most of the time, near the end leave it to go to the target set" [29]. I know of no other theorems that say that solutions are necessarily symmetric execept for an adjusting boundary layer. This is a quite general problem that merits investigation.

4. Separation of variables, symmetry and special functions.

As an example consider the Helmholtz equation $Q\psi = 0$, where Q is the second order differential operator $Q = \partial^2/\partial x^2 + \partial^2/\partial y^2 + \omega^2 = \Delta + \omega^2$. A linear differential operator $L = \alpha\partial/\partial x + \beta\partial/\partial y + \gamma$, α, β, γ functions on $\mathbb{R}^2$, is a *symmetry* of the Helmholtz equation if $[L, Q] := LQ - QL = \delta Q$ for some function δ on $\mathbb{R}^2$. It turns out that the linear space of symmetry operators is four dimensional with basis $P_1 = \partial/\partial x$, $P_2 = \partial/\partial y$, $M = y\partial_x - x\partial_y$, $E = id$. The first three relate respectively to the obvious symmetry transformations: translations along x and y axes and rotations. The group of symmetry operations belonging to this Lie algebra is E_2, the group of rigid motions of the plane. A second order differential operator S is said to be a *symmetry* of the Helmholtz equation if $[S, Q] = UQ$ where U is a first order differential operator $U = \alpha\partial/\partial x + \beta\partial/\partial y + \gamma$. Symmetry operators (of first or second order) map solutions into solutions. Examples are the operators αQ which act trivially on the space of solutions. Killing off these the space of $\leqslant$ second order differential operators is 9 dimensional with basis E (zero order); P_1, P_2, M (purely first order); P_1^2, P_1P_2, M^2, $MP_1 + P_1M$, $MP_2 + P_2M$ (purely second order). The group E_2 acts in an order preserving way on these symmetry operators (adjoint action on the universal enveloping algebra of its Lie algebra) and it turns out that the purely second order operators fall into four orbits.

Now consider separation of variables for the Helmholtz equation. I.e. we are looking for (not necassarily globally defined) coordinates u, v on $\mathbb{R}^2$ and solutions which can be written in the form $\psi = \psi_1(u)\psi_2(v)$. E.g. if $(u, v) = (x, y)$, $\psi = \psi_1(x)\psi_2(y)$, the Helmholtz equation becomes $\psi_1''\psi_2 + \psi_1\psi_2'' + \omega^2\psi_1\psi_2 = 0$ or $\psi_1''\psi_1^{-1} = -\psi_2''\psi_2^{-1} - \omega^2$ which leads to $\psi_1'' + k^2\psi_1 = 0$, $\psi_2'' + (\omega^2 - k^2)\psi_2 = 0$ for some constant k^2. Similarly polar coordinates (x, θ) lead to solutions $\psi = \psi_1(\theta)\psi_2(r)$ with $\psi_1'' + k^2\psi_1 = 0$, $r^2\psi_2'' + r\psi_2' + (r^2\omega^2 - k^2)\psi_2 = 0$. This last equation is Bessel's equation. A coordinate transformation by means of an element of E_2 does not lead to really different coordinate system which admits separation of variables. Up to this equivalence it turns out that there are exactly four coordinate systems which admit separation of variables.

Moreover it turns out to each of these systems there is associated a purely second order symmetry S such that the corresponding separated solution is an eigen function of that operator in such an way that orbits correspond. In this case this sets up a bijective correspondence between orbits of separation of variables coordinate systems and orbits of symmetry operators according to the following scheme

Operator	Coordinates
P_2^2	Cartesian x,y
M^2	Polar, $x = r\cos V, y = \sin V$
MP_2+P_2M	Parabolic, $x = \frac{1}{2}(u^2-v^2), y = uv$
$M^2+d^2P_1^2$	Elliptic, $x = d\cosh u\cos v$ $y = d\sinh u\sin v$

Separated solutions
Product of exponential
Bessel times exponential
Product of parabolic cylinder functions
Product of Mathieu functions

In general things are not quite as beautifull, e.g. for the Klein-Gordon equations, in that more than one orbit of symmetry operators may correspond to one and the same orbit of coordinate systems and that some operators do not have a separation coordinate system attacted to them. All this comes from [17]. There is much more known and also a large number of open questions.

Still there clearly are important relations between "separation of variables", "symmetry" and "special functions". Very much is known about relations between the latter two [16] and these relations explain e.g. addition formulas for special functions. This also shows that equations like the Bessel one and their solutions, Bessel functions, aparently have symmetry aspects which are far from obvious, much like a vector belonging to $a > 1$ dimensional irreducible representation has certain transformation (symmetry) properties. Special functions have always played and still play a most important role in applied mathematics and engineering (e.g. cooling of a surface, design of wings, Redheffer scattering, edge effects, theory of vibrations, electro magnetic theory [28])). It could be fruitful and enlightening to try to trace the symmetry hidden in these special functions to the problems they are applied to.

5. Symmetry in physics I: conservation laws.

There is an intimate interrelation between conservation laws and symmetries for mechanical systems. For a finite dimensional mechanical system in the Lagrangian formulation this takes the following form. Let M be a differential manifold. TM its tangent vectorbundle, $L: TM \to \mathbb{R}$ a differentiable function, the Lagrangian. Classically $M = \mathbb{R}^N$ with coordinates q, configuration space, and $TM = \mathbb{R}^N \times \mathbb{R}^N$ with coordinates $(q,\dot{q})$, phase space. A map $\gamma:\mathbb{R} \to TM$ is a trajectory if it is extremal for the functional $\Phi(\gamma) = \int_{t_0}^{t_1} L(\gamma,\dot{\gamma})dt$, where $\dot{\gamma}$ is the velocity component of γ. This leads (in local coordinates) to the Lagrangian equations $\frac{d}{dt}\frac{\partial L}{\partial \dot{q}} = \frac{\partial L}{\partial q}$. A conservation law is a map $c: TM \to \mathbb{R}$ such that $c(\gamma) =$ constant for all trajections γ. A differentiable mapping $h: M \to M$ is a symmetry for the Lagrangian system (M,L) if $L \ o\ dh = L\ (dh: TM \to TM)$. The Noether theorem now says [17]:

Theorem. *Suppose that the Lagrangian system (M,L) admits a one parameter group of diffeomorphisms $h^s: M \to M$ as symmetries, $s \in \mathbb{R}$. Then the Lagrangian system has a corresponding conservation law (first integral) $I: TM \to \mathbb{R}$ which in local coordinates $(q,\dot{q})$ for TM is given by* $I(q,\dot{q}) = \frac{\partial L}{\partial \dot{q}} \frac{\partial h^s(q)}{\partial s}\Big|_{s=0}$.

This is by no means the most general statement along these lines and things are in full development. For more general systems both the ideas of conservation law and symmetry must be generalized completely and it is at the moment far from clear how things can be or ought to be generalized [18] Noether type theorems for control systems have been established [19].

The presence of a conservation law also rise to a reduction in dimension of the system and this also has its analogues in system theory [20].

6. Symmetry in physis II: degeneracy, ordering the phenomena, symmetry breaking.

Let H be a quantum mechanical Hamiltonian acting on a Hilbert space (of functions) $\mathcal{H}$. And let H be invariant under a group G acting on $\mathcal{H}$. For example one may have $H = -h^2(8\pi^2 m)^{-1}\Delta - V$, where Δ is the three-dimensional Laplacian and V is the spherically symmetric potential $V = -c^2(x^2+y^2+z^2)^{-\frac{1}{2}}$ (one electron around a nucleus, neglecting spin). In this case there is a symmetry group $SO(3)$. Then two things happen

(i) The eigenfunctions and eigenvalues of H can be labelled by the irreducible representation of G

(ii) The energy (=eigenvalue) belonging to an irreducible representation V is degenerate with a multiplicity which is a multiple of dim V.

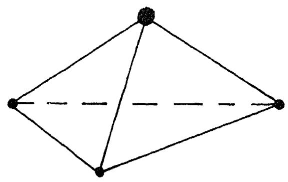
This is easy to see. Indeed invariance means $g^{-1}Hg = H$. So if $H\psi = E\psi$, then $Hg\psi = gH\psi = gE\psi = E(g\psi)$. So $g\psi$ is also an eigenfunction and it has the same eigenvalue as ψ. It also follows that knowing the representation theory of G and using it to break up $\mathcal{H}$ into a direct sum of irreducible representations helps in diagonalizing H.

Thus the presence of symmetry explains the occurence of degeneracy phenomena. Indeed usually the presence of a degeneracy (if not more or less clearly accidental) is taken as an indicator that some sort of (hidden) symmetry is present.

Given a symmetric situation as an electron in a spherically symmetric field one can try to destroy the symmetry, to break it to see in how many different levels a given degenerate energy level splits. In this case by a constant magnetic field (Zeeman effect). Turning to systems for a moment one notes that the description of a state-space linear finite dimensional description is highly degenerate with respect to the input-output description. In this case "symmetry breaking" can be accomplished by state-space feedback [21].

7. Symmetry in physics III. molecular vibrations.

Consider a molecule N atoms with symmetry. For instance the ammonia molecule NH_3 depicted below which has symmetry group C_{3v} consisting of 6 elements, viz. rotations of 120^o and 240^o around a vertical axis through the N-atom, the identity element and three reflections in three vertical planes containing the N-atom and one H atom.



Correspondingly the (classical) Hamiltonian is invariant under C_{3v}. As a result (the energies of) the possible classical vibrations of the molecule fall into degeneracy classes depending on what irreducible representations are involved. These are the representations of C_{3v} occuring in its (natural) representation on the $3N$-dimensional space of all possible displacements of the N-atoms under the motions of C_{3v}. The point is that this representation is easily calculated (without knowing anything about the Hamiltonian itself). More precisely it is its character which can be written down immediately.

Given a representation V of a finite group G its character is the function $\chi: G \to \mathbb{C}$, $\chi(g) = \Sigma\langle e_i, ge_i\rangle$, where $\{e_1,...,e_n\}$ is any orthonormal basis for V. Concerning characters one has

(i) two representations are equivalent iff their characters are equal

(ii) Let V be a representation of G with character χ_V and W an irreducible representation with character χ_W. Then the multiplicity with which W occurs in V is given by the formula $m = (\#G)^{-1} \sum_g \chi_V(g)\, \overline{\chi_W(g)}$.

In the case of molecular vibrations of NH_3 above one finds that (besides the 6 zero - frequency modes corresponding to displacements (translations and rotations) of the molecule as a whole) there are two non-degenerate vibrational modes and two two-fold degenerate modes. Similar considerations apply to the quantum case.

This is just the start of the applications of symmetry to atomic and vibrational spectroscopy [22].

8. Direct product representation. Clebsch-Gordon coefficients.

Let $V^{(\alpha)}$, $V^{(\beta)}$ with bases $e_i^{(\alpha)}, e_j^{(\beta)}$ be two representations of G. The tensor product of $V^{(\alpha)}$ and $V^{(\beta)}$ is the $n_\alpha n_\beta$-dimensional vectorspace with basis $e_i^{(\alpha)} \otimes e_j^{(\beta)}$, $i = j,...,n_\alpha$; $j = 1,...,n_\beta$. For $V = \Sigma a_i e_i^{(\alpha)}$, $w = \Sigma b_j e_j^{(\beta)}$, let $v \otimes w = \sum_{i,j} a_i b_j e_i^{(\alpha)} \otimes e_j^{(\beta)}$. The vectorspace $V^{(\alpha)} \otimes V^{(\beta)}$ then carries a representation of G defined by $g(v \otimes w) = gv \otimes gw$, called the direct product representation. Let $V^{(\alpha)}, V^{(\beta)}$ be irreducible. Then $V^{(\alpha)} \otimes V^{(\beta)}$ splits as a sum of irreducibles $V^{(\gamma)}$, possibly with multiplicities. So there must be new basis vectors $\psi_k^{(\gamma)t} = \sum_{i,j} C(\alpha\beta\gamma t, ijk) e_i^{(\alpha)} \otimes e_j^{(\beta)}$ which transform according to $V^{(\gamma)}$. Here t is a multiplicity label. Assuming orthonormal bases and unitary representations the transformation $e_i^{(\alpha)} \otimes e_j^{(\beta)} \to \psi_k^{(\gamma)t}$ is unitary.

The C's are called Clebsch-Gordan coefficients. (The names Wigner-coefficients, $3j$-symbols, vector coupling coefficients also occur.)

9. Symmetry in physics IV. Selection rules. Reduction and decomposition. Wigner-Eckart theorem.

Now consider again a quantum mechanical situation with symmetry. I.e. we have a Hamiltonian operator H, invariant under a symmetry group G acting on H. The group G acts also on operators 0 (by $(g,0) \to g0g^{-1}$). Let 0 be an operator with transforms according to a representation V (i.e. 0 is an element of a subrepresentation (of the representation of G in the space of all operators) which is equivalent to V). Consider a transition process governed by such an operator 0. The transition amplitudes of a state ϕ to a state ψ are given by the matrix elements $<\phi,0\psi>$. Now the eigenstates can be labelled according to the irreducible representations of G. Let ϕ_i transform according to the irreducible representation V_i, $i = 1,2$. Then $0\phi_2$ transforms according to the direct product representation $V \otimes V_2$ and from the orthogonality relations (2.2) it now follows that $<\phi_1,0\phi_2>$ is zero unless the representation V_1 occurs as an irreducible component in the direct product representation $V \otimes V_2$. This gives and explains selection rules and forbidden transitions.

We have already seen that in the presence of symmetry G for a Hamiltonian H the Hilbert space $\mathcal{H}$ breaks up into parts labelled by the irreducible representations of G, $\mathcal{H} = \Sigma V_{ij}$, $i = 1,...,d$, $j = 1,...,m_i$ where $V_1,...,V_d$ are the irreducibles of G, $V_{ij} \simeq V_i$, and $m_i \in \{1,2,...,\infty\}$ is the multiplicity with which V_i occurs in $\mathcal{H}$. Let e^t_{ij} be a basis for V_{ij}. Then with respect to this basis we have that $<e^t_{ij},He^s_{kl}> = \delta_{ik} a_{j,l} \delta_{ts}$ by Schur's lemma. So symmetry can give a very substantial reduction. Such reductions also occur in systems theory in the presence of symmetry [23].

A jazzed-up version of this result is the Wigner-Eckart theorem which deals with the case that an operator S is not necessarily invariant (which means that H transforms according to the trivial representation) but, according to an irreducible representation V. Let $\phi_i^{(\alpha)}$, $S_j^{(\beta)}$, $\phi_k^{(\gamma)}$ transform according to the irreducible representations $V^{(\alpha)}$, $V^{(\beta)}$, $V^{(\gamma)}$. Then $S_j^{(\beta)}\phi_k^{(\gamma)} = \Sigma\, C^*(\beta\gamma\epsilon t, jkl)\psi_l^{(\epsilon)t}$ (cf. section 8 above; $C^*(\cdots,\cdots)$ denotes the elements of the adjoint matrix) and hence $<\phi_i^{(\alpha)},S_j^{(\beta)}\phi_k^{(\gamma)}> = \sum_{\epsilon,t,l} C^*(\beta\gamma\epsilon t, jkl)\ <\phi_i^{(\alpha)},\psi_l^{(\epsilon)t}>$. But according to (2.2) $<\phi_i^{(\alpha)}, \psi_l^{(\epsilon)t}>$ is zero unless $\alpha = \epsilon$ and $i = l$ and is independent of the particular value of $i = l$. So we finally find an expression $<\phi_i^{(\alpha)},S_j^{(\beta)}\phi_k^{(\gamma)}> = \sum_k C^*(\beta\gamma\epsilon t, jkl) < \phi^{(\alpha)}\|S^{(\beta)}\|\phi^{(\gamma)}>_t$ where the so-called reduced matrix coefficients are given by $<\phi^{(\alpha)}\|S^{(\beta)}\|\phi^{(\gamma)}>_t = <\phi_i^{(\alpha)},\psi_i^{(\alpha)t}>$. Thus as the Clesch-Gordan coefficients are known from the representation theory of G alone (and have nothing to do with the particular operators $S_j^{(\beta)}$), there are lots of, so to speak, a priori, relations between the matrix coefficients of the $S_j^{(\beta)}$, and they are (universally) given in terms of a vastly smaller set of reduced coefficients. This Wigner-Eckart theorem has very many applications, e.g. in broken symmetry situations. For instance a Hamiltanian H_0 which is $S0(3) \times \cdots \times S0(3)$ symmetric (n-electrons around a nucleus, neglecting electron-electron interactions) is broken to an $S0(3)$ symmetric Hamiltonian $H_0 + H_1$ (by adding the electron-electron interaction terms). Other applications involve ligand field theory, coupling coefficients and transition probabilities (and selection rules, cf. above). Many of the books listed under [7] discuss such applications.

10. Symmetry and degeneration II. Bifurcation theory.

In bifurcation theory one deals with the problem of how the solution set of a set of (differential) equations $G(\lambda,u) = 0$ changes as the parameters λ changes. This is relatively easy to analyze in nondegenerate situation, e.g. if a single eigenvalue of the linearized operator $G_u(\lambda,u)$ crosses the imaginary axis or passes through zero. In more degenerate situations the bifurcation equations tend to become hopelessly complicated. Here again representation theory can help. If there is invariance under a group G, $KerG_u(\lambda,u)$ will be a G-space and this may very well turn out to be a irreducible representation. By now the reader will already suspect that this effectively reduces the matter to something like a nondegenerate case. For more details and an introduction to the vast literature, cf. [24].

11. Approximate symmetry. Broken symmetry.

Often of course a system will be not perfectly symmetric but only approximately so. It is then advantageous to treat it as a perturbation of the more symmetric situation. In quantum physics this has been extraordinary successful. Even when the symmetry is very badly broken (e.g. $SU(6)$ in high-energy particle physics). There are some questions here, as the mere fact that a given Hamiltonian can be embedded in a family H_t starting with arbitrarily highly symmetric one says nothing.

12. Highly symmetric versus elementary.

The elementary constituents we like to use in science to describe more complicated objects tend to be highly symmetric: lines, circles, ellipses, radially symmetric potentials, ...; from this point of view one might ask for all objects with a simple generating algorithm or (equivalently?) a highly symmetric structure. Besides lines, circles, spheres this leads also to fractals such as Koch islands, objects with scaling symmetries, and, if statistical symmetries are admitted, to Brownian motion (another popular building block) and fractional Brownian functions [25].

13. Concluding remarks.

(a) There are certainly other ways in which a model can be symmetric (homogeneous) without admitting a group of symmetries. For instance other algebras then group algebras may appear as symmetry algebras [23]; some very powerful results of Kostant dealing with adjoint orbits of Lie groups generalize to a symplectic geometry setting; some Lie groups (algebras) which should exist seem to be missing in the sense that most of the associated objects (a geometry, an automorphic function theory, a Macdonald type identity, ...) seem to be present but not the underlying explaining group (algebra) itself; special functions tend to belong to Lie groups except the grandfather of most of them, the hypergeometric function.

(b) How to define symmetries in various situations is often an open problem. E.g. in partial differential equations it seems necessary to admit symmetries of the types involving derivatives, i.e. automorphisms of phase space taking solutions into solutions are not general enough. This takes us into the realm of Bäcklund transformations, soliton theory and gauge theories (position dependent symmetries) [26]. Nonrigid molecules in chemistry pose problems of another kind.

(c) There are also "symmetric objects" (according to our intuition) which seem not to fit at all into a picture "symmetry is invariance under a subgroup of a natural group of transformations" cf. [11] for some examples.

(d) Finally there are not quite understood (for the moment) relations between linearization and symmetry. Symmetry relates to linear quotients [20]. On the other hand, the completely integrable systems attached to semisimple Lie groups e.g. are in a way quotients of linear flows. So is the Riccatti equation. And this accounts for superposition principles.

(e) In a situation where there is natural symmetry it seems natural to ask for procedures (of calculation), for identification of systems, say, which are equivariant with respect to the symmetries involved. To do the calculation one may have to break the symmetry; e.g. to lable the input and output channels in a certain way. Covariance then means that first permuting the input channels, then doing identification, then permuting the input channels back should give the same result as simply doing the identification according to the original labelling. This goes towards equivariant statistics [27]. It seems to me that system theory can greatly profit by paying more attention to such symmetry principles. I hope the, necessarily extremely incomplete, remarks made above concerning symmetry in physics will help.

Notes and references

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