Symmetry in physics and system theory. An introduction to past, present and future possibilities.

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Abstract.
Symmetry considerations and invariance principles play an extremely important role in physics. An outline is given of some of the ideas involved and the expectation that such considerations will also be of importance in systems and control theory.

Introduction. General remarks on the role and importance of symmetry considerations.
Most scientists and engineers here probably know in a vague way that symmetry considerations play an extremely important role in physics, especially quantum physics and relativity theory and most especially elementary particle theory (irreducible representation of the symmetry group). Yet most will be surprised to learn, as I was, that in a leading journal like J. of Math. Physics in 1983 some 36% of the papers dealt with aspects of group representations, symmetry or invariance (146 out of 406 and in Lett. Math. Physics the percentage is even higher: 54% (37 out of 69). By and large this is a fairly recent phenomenon: the ubiquity of symmetry considerations in physics and chemistry and their enormous success. To quote from 1:

"The importance of group theory and its utility in applications to various branches of physics and chemistry is now so well established and universally recognised that its explicit use needs neither apology nor justification".

It is my belief there is an equally impressive role awaiting symmetry in the engineering sciences, and in this introductory note I will try to indicate why by means of both general remarks and a relatively precise description of some specific applications of symmetry ideas and techniques in physics. Here is how in a field where symmetry considerations are quite recent, viz. the theory of phase transitions and critical phenomena (phase transitions, renormalization group, scaling invariance), the role of symmetry and groups is described by S.-K. MA. 2 He stresses two approaches to deal with complex physical problems:

(i) Direct solution approach. This means calculation of physical quantities of interest in terms of parameters given in the particular model - in other words, solving the model. The calculation may be done analytically or numerically, exactly or approximately.

(ii) Exploiting symmetries. This approach does not attempt to solve the model. It considers how parameters change under certain symmetry transformations. From various symmetry properties one deduces some characteristics of physical quantities. These characteristics are generally independent of the quantitative values of the parameters. Approach (ii) is not a substitute for approach (i). Experience tells us that one should try (ii) as far as one can before attempting (i), since (ii) is often a very difficult task. Results of (ii) may simplify this task greatly. A great deal can be learned from (ii) without even attempting (i)."

Topics in physics and chemistry in which group theory and symmetry considerations play decisive roles include conservation laws, atomic and molecular spectroscopy (including Raman scattering), collective models of the nucleus, chemical bond theory, elementary particle theory and grand unification (of the forces of nature), relativity theory 3, quantization theory 4, theory of phase transitions and critical phenomena 5, cosmology, selection rules and transition probabilities in quantum mechanics, theory of electron bands in solids, crystallography, soliton theory and its many applications, renormalization theory, theory of polymers, gauge fields, photon dispersion relations, electronic and nuclear shell theory, theory of spin g-states.

The vigour of the subject is attested to by - among others - the yearly international colloquium on Group theoretical methods in physics 6. A more or less random selection of books, besides the ones specifically quoted below or above, dealing with various aspects of symmetry and groups in physics and chemistry is 7. Writing a complete survey of applications of symmetry and groups in physics would probably take 3000 plus pages. And that would be just the applications in physics. In mathematics itself groups and representations play an equally central role and many of the currently hot topics involve Lie groups in one way or another. Very possibly some sort of homogeneity, symmetry is needed to make a problem interesting or esthetically pleasing or, indeed, tractable 8.

That is useful to have some symmetry present when dealing with, say, differential equations is an old observation and indeed was the first impetus which led to the "discovery" of Lie groups. In the words of Sophus Lie 9:

"Ich betrachte, dass die meisten gewohnlichen Differentialgleichungen, deren Integration durch die alten Integrationsmethoden gelöst wird, bei gewissen leicht angebaren Scharen von Transformationen invariant bleiben, und dass jene Integrationsmethoden in der Verwendung dieser Eigenschaft der betreffenden Differentialgleichung bestehen."

This aspect: symmetry inspired analysis of differential equations (both ordinary and partial) and also in the future difference equations, i.e. involving questions around the theme differential Galois theory, has been all but neglected for some 70 years after Lie, but has been the subject of a vigorous revival in the last 20 especially the last 10 years or so 10. It is a fast developing subject with a very large number of unsettled questions. Indeed in many contexts it is not yet at all clear what the right definition is of a symmetry. Cf. also below in section 12.

As I remarked above it would take a considerable number of pages even to give an indicative survey of the applications of symmetry ideas in physics. So here I will simply describe a very few of them in brief vignettes.

2. Groups, actions, representations.
Let G be a group, S a set. An action of G on S is a map G x S \rightarrow S, (g,s) \rightarrow gs, such that g(hs) = (gh)s, es = s for all g,h \in G, s \in S, where e \in G is the identity element. Then S is called a G-set. Thus, if gs = s all s implies g = e, the group G is realized as a group of transformations of the set S. The subset Gr = {gs \mid g \in G} for a given s in S is called the orbit of G through s. A function f on S is invariant if f(gs) = f(s) for all g,s.

Two G-sets S,T are isomorphic if there is a bijection \phi:S \rightarrow T such that \phi(gs) = g(\phi(s)) for all g \in G, s \in S. The union of all orbits in a G-set which are isomorphic to a given one (as G-sets) is called a stratum.

A homomorphism of vector spaces \phi:V \rightarrow W between two representations is covariant if it is also a map of G-sets. Two representations are isomorphic if there is a covariant isomorphism of vector spaces between them.
Let $V$ be a real or complex vector space with product $<.,.>$. If $<.,.>$ is invariant, i.e. $<g(u,v)> = <u,w>$ for all $g \in G, u, v, w \in V$, the representation is said to be unitary. For many groups every class of equivalent representations contains unitary ones (Maschke’s theorem).

Many applications of symmetry ideas in physics rest on (souped up versions) of one of the following three facts.

A. (Full reducibility). For many groups, e.g. finite and compact (i.e. not $g$: two
functions on $S$. This becomes a $G$-set under a decomposition of every function on $S$ into a sum of functions that are like an irreducible situation, a primitive or elementary one incapable of representation of $G$, serves to label different types of objects, e.g. elementary particles. More generally one meets direct integral decompositions, generalizing spectral integral representations of operators.

B. (Schur’s lemma). Let $\phi: V \rightarrow W$ be a covariant homomorphism between the complex irreducible representations $V, W$ of $G$. Then $\phi = 0$ if $V$ and $W$ are inequivalent and $\phi = \lambda I_d$ for some complex number $\lambda$ if $V$ and $W$ are equivalent.

C. (Orthogonality relations). Let $V^{(a)}, V^{(b)}$ be two irreducible unitary representations of a finite group $G$ with $\# G$ elements. Choose bases $e^{(a)}$ and $e^{(b)}$ for $V^{(a)}, V^{(b)}$ and let $T^{(a)}(g)$ be the the $(i,j)$-element of the matrix w.r.t. to $e^{(a)}$ of $g: V^{(a)} \rightarrow V^{(a)}$. Then

$$s_{a} \cdot T^{(a)}(g) \cdot T^{(b)}(g)^{*} = \delta^{ab} \cdot \delta_{ij}$$

where $\delta_{ab}$ denotes the Kronecker symbol and, $s_{a} = \dim(V^{(a)})$ and $T^{(a)}(g)$ is the $(k,l)$ element of the adjoint matrix $T^{(a)}$. Let $V$ be an irreducible representation of the finite group $G$, $V^{(a)}$ and $V^{(b)}$ two irreducible components and $e^{(a)}, e^{(b)}$ bases for $V^{(a)}$ and $V^{(b)}$. Then there also are orthogonality relations between these basis vectors

$$<e^{(a)}| e^{(b)}> = s_{a}^{-1} \delta_{ag} \delta_{bj} \sum_{c} c e^{(a)}| e^{(c)}>. (2.2)$$

So in particular if $V$ is a space of functions and $<.,.>$ is given by a suitable integral (like $<f,g> = \int f(x)g(x) dx$), as is often the case, and $f$ belongs to an irreducible representation not isomorphic to the trivial one (i.e. not $g$; $V^{(a)} \rightarrow V^{(a)} = id$ for all $g$) then (taking $s_{a} = 1$) we find $\int f dx = 0$.

Let $S$ be a $G$-set which is a single orbit (a transitive $G$-set). This looks like an irreducible situation, a primitive or elementary one incapable of further analysis. Consider however the space $\mathcal{S}(S)$ of all complex valued functions on $S$. This becomes a $G$-set under $(gf)(x) = f(gx)$, indeed a representation of $G$, and it may very well be reducible, thus giving rise to a decomposition of every function on $S$ into a sum of functions that are "nicely behaved" in a certain sense. For the case that $S = G$ is the circle group this leads to the statement that every function on the circle (or every $G$-representation of $G$, which leads to solutions of maximal balanced growth, then proceed along it for most of the time, near the end leave it to go to the target set". I know of no other theorems that say that solutions are necessarily symmetric except for an adjoining boundary layer. This is a quite general theorem that merits investigation.

3. The Purkiss principle.

Many problems, e.g. of the optimization kind, come with a natural symmetry built in, so to speak. This symmetry may for instance derive from a fact (axiom) like: "the physics is independent of the observer (or the coordinate system used to describe things). Nature also seems to like (more or less) symmetric solutions\), and it is up to the scientist to explain why. Designers, especially of large systems, also seem to like a good deal of homogeneity (symmetry). All this would be nicely understood if there were a principle like: symmetric solutions have symmetric solutions. This has been called the Purkiss principle by W.C. Waterhouse. It does however certainly not hold in general.

My favourite counterexample asks for the shortest system of roads connecting four towns arranged in a square. The solution is something like depicted above. Here is also a positive result:

**Theorem 1.** Let $M$ be a differentiable manifold, $G$ a finite group of smooth maps from $M$ to $M$, $m \in M$ a point fixed by $G$ (i.e. $Gm = (m)$), and $f: M \rightarrow R$ a differentiable function invariant under $G$. Assume that the induced action of $G$ on the tangent space $T_{m}M$ at $m$ is nontrivial and irreducible. Then $m$ is a critical point of $f$ and if $m$ is nondegenerate it is a local extremum.

There are obvious potential applications of such a theorem (and various conceivable generalizations) to all kinds of optimal control problems. These applications are different from such as $13$ which are based on the idea of passing to a "quotient space" $M / G$, or, more or less equivalently, a Noether theorem (reduction by means of symmetry, cf. also $9$ below).

To the same circle of ideas belongs the following theorem of L. Michel.

**Theorem.** Let $G$ be a compact group acting smoothly on a differentiable manifold $M$ and let $f$ be an invariant function on $M$. Then if $m$ is isolated in its stratum it is a critical point of $f$. And inversely if $m$ is critical for all invariant functions on $M$ then $m$ is isolated in its stratum.

Quite generally the presence of symmetry has enormous influence on critical points and singularities and may also greatly help to overcome traditional difficulties associated with multiplicities and degeneracies, e.g. in bifurcation theory and spectral problems.

Quite often though an (optimal control) problem may itself be symmetric, but the boundary conditions not (initial and target set e.g.). In optimal growth theory for economies one has in this setting so-called "turbine" theorems, which roughly say "go as quickly as you can to a ray of maximal balanced growth, then proceed along it for most of the time, near the end leave it to go to the target set". I know of no other theorems that say that solutions are necessarily symmetric except for an adjoining boundary layer. This is a quite general theorem that merits investigation.

4. Separation of variables, symmetry and special functions.

As an example consider the Helmholtz equation $Q \phi = 0$, where $Q$ is the second order differential operator $Q = \partial^{2}/\partial x^{2} + \partial^{2}/\partial y^{2} + \partial^{2}/\partial z^{2}$. A linear differential operator $L = \partial^{2}/\partial x^{2} + \partial^{2}/\partial y^{2} + \gamma \partial^{2}/\partial z^{2}$ functions on $R^{2}$, is a symmetry of the Helmholtz equation if $[L, Q] = LQ - QL = 0 \phi$ for some function $\phi$ on $R^{2}$. It turns out that the linear space of symmetry operators is four dimensional with basis $P_{x}, P_{y}, P_{z}, \partial_{x} \partial_{y} \partial_{z}$, and the first three relate respectively to the obvious symmetry transformations: translations along $x$ and $y$ axes and rotations. The group of symmetry operators belonging to this Lie algebra is $E_{3}$, the group of rigid motions of a plane. A second order differential operator $L$ is said to be a symmetry of the Helmholtz equation if $[L, Q] = LQ - QL = 0 \phi$ for spaces of solutions. Killing off these the space of $\approx 2$ second order differential operators is $9$ dimensional with basis $E$ (zero order); $P_{x}, P_{y}, P_{z}$ (purely first order); $P_{x}^{2}, P_{y}^{2}, P_{z}^{2}, MP_{x}, MP_{y}, MP_{z}; M^{2}P_{x}, MP_{y}, MP_{z}$ (purely second order). The group $E_{2}$ acts in an order preserving way on these symmetry operators (adjoint action on the universal enveloping algebra of its Lie algebra) and it turns out that the purely second order operators fall into four orbits.

Now consider separation of variables for the Helmholtz equation. I.e. we are looking for (not necessarily globally defined) coordinates $u, v, w$ on $R^{2}$ and solutions which can be written in the form $\psi = \phi(u, v, w)$. E.g. if $(u, v, w) = (r, \phi, \psi)$, the Helmholtz equation becomes $\phi_{rr} + \phi_{\phi \phi} + \phi_{\psi \psi} = 0$ or $\phi_{rr} = -\phi_{\phi \phi} = \phi_{\psi \psi}$ which leads to $\phi_{rr} + k^{2}\phi = 0, \phi_{rr} + k^{2}\phi_{\phi \phi} = 0$ for some constant $k^{2}$. Similarly polar coordinates $(\rho, \theta)$ lead to solutions $\psi = \phi(\rho, \theta)$ with $\phi_{\rho \rho} + k^{2}\phi = 0, \phi_{\theta \theta} + k^{2}\phi = 0$. This last equation is Bessel’s equation. A coordinate transformation by means of an element of $E_{2}$ does not lead to really different coordinate system which admits separation of variables. Up to this equivalence it turns out that there are exactly four coordinate systems which admit separation of variables.

Moreover it turns out to each of these systems there is associated a purely second order symmetry $S$ such that the corresponding separated solution is an eigen function of that operator in such an way that orbits correspond. In this case this sets up a bijective correspondence between orbits of separation of variables coordinate systems and orbits of symmetry operators according to the following scheme...
There is an intimate interrelation between conservation laws and symmetry. For instance the ammonia molecule $NH_3$ depicted below which has symmetry group $C_3v$ consisting of 6 elements, viz. rotations of $120^\circ$ and $240^\circ$ around a vertical axis through the $N$-atom, the identity element and three reflections in three vertical planes containing the $N$-atom and one $H$ atom.

Correspondingly the (classical) Hamiltonian is invariant under $C_{3v}$. As a result the (energies of) the possible classical vibrations of the molecule fall into degeneracy classes depending on what irreducible representations are involved. These are the representations of $C_{3v}$ occurring in its (natural) representation on the 3V-dimensional space of all possible displacements of the $N$-atoms under the motions of $C_{3v}$. The point is that this representation is easily calculated (without knowing anything about the Hamiltonian itself). More precisely it is its character which can be written down immediately.

Given a representation $V$ of a finite group $G$ its character is the function $\chi:G \rightarrow \mathbb{C}$ which is the sum of all eigenvalues with their respective multiplicity. The multiplicity of $V$ in $G$ is given by $\chi(g) = \sum_{\rho} m(\rho) \chi_{\rho}(g)$ where $\chi_\rho(g)$ is the $\rho$th character of $G$.

In the case of molecular vibrations of $NH_3$ above one finds that (besides the 6 zero - frequency modes corresponding to displacements (translations and rotations) of the molecule as a whole) there are two non-degenerate vibrational modes and two two-fold degenerate modes. Similar considerations apply to the electronic case.

This is just the start of the applications of symmetry to atomic and vibrational spectroscopy [22].


Let $\nu^m, \nu^n$ be functions with bases $e^m, e^n$ be two representations of $G$. The tensor product of $\nu^m$ and $\nu^n$ is the $|m,n\rangle$-dimensional vector space basis with $e^m \otimes e^n, i,j = 1,...,|m,n\rangle$. For example one may have $H = -\hbar^2/2m (x^2+y^2)$. Then $\nu^m \otimes \nu^n$ represents the $2|m,n\rangle=2\sum_{i,j} a_i b_j e^m \otimes e^n$. The vector space $\nu^m \otimes \nu^n$ then carries a representation of $G$ defined by $g(\nu^m \otimes \nu^n) = \nu^m \otimes \nu^n g$, called the direct product representation. Let $\nu^m \otimes \nu^n$ be irreducible. Then $\nu^m \otimes \nu^n$ splits as a sum of irreducibles $\nu^{ij}$, possibly with multiplicities. So there must be new basis vectors $\psi^{ij}$ such that $\sum_{i,j} a_i b_j \psi^{ij}(\rho) \otimes \psi^{ij}(\sigma) = \nu^{ij}$. This is the Clebsch-Gordan identity, assuming orthonormality and unitarity, etc.

The C's are called Clebsch-Gordan coefficients. (The names Wigner-coefficients, 3-j-symbols, vector coupling coefficients also occur.)

Now consider again a quantum mechanical situation with symmetry. We have a Hamiltonian operator \( H \) invariant under a symmetry group \( G \) acting on \( H \). The group \( G \) acts also on operators \( O \) (by \( (g \cdot O)(\psi) = gOg^{-1}\psi \)). Let 0 be an operator with transforms according to a representation \( \mathcal{V} \) (i.e. 0 is an element of a subrepresentation (of the representation of \( G \) in the space of all operators) which is equivalent to \( \mathcal{V} \)). Consider a transition process governed by such an operator 0. The transition amplitudes of a state \( \phi \) to a state \( \psi \) are given by the matrix elements \( \langle \psi | O | \phi \rangle \). Now the eigenstates can be labelled according to the irreducible representations of \( G \). Let \( \phi \) transform according to the irreducible representation \( \mathcal{V} \); \( i = 1,2 \). Then \( \mathcal{V}_0 \) transforms according to the direct product representation \( \mathcal{V} \otimes \mathcal{V}_0 \) and from the orthogonality relations (2.2) it now follows that \( \langle \psi_0 | O | \phi \rangle \) is zero unless the representation \( \mathcal{V}_0 \) occurs as an irreducible component in the direct product representation \( \mathcal{V} \otimes \mathcal{V}_0 \). This gives and explains selection rules and forbidden transitions.

We have already seen that in the presence of symmetry \( G \) for a Hamiltonian \( H \) the Hilbert space \( \mathcal{H} \) breaks up into parts labelled by the irreducible representations of \( G \), \( \mathcal{H} = \bigoplus \mathcal{H}_i \). Each \( \mathcal{H}_i \) is an element of a subrepresentation (of the full representation) but, unless \( \mathcal{V} \) is the trivial representation, \( \langle \psi | O | \phi \rangle \) is not defined for \( \psi \) in \( \mathcal{H}_i \) unless \( \phi \) is in \( \mathcal{H}_i \). This gives rise to the idea of projection operators. In quantum mechanics one now expects to find that the matrix coefficient of an operator \( O \) is zero unless \( \phi \) and \( \psi \) belong to the same irreducible representation of \( G \). This is the Wigner-Eckart theorem of selection rules and forbidden transitions, which appears to have been first stated by G. Eckart in 1926.


10. Symmetry and degeneration II. Bifurcation theory.

In bifurcation theory one deals with the problem of how the solution set of a set of (differential) equations \( G(\lambda) = 0 \) changes as the parameters \( \lambda \) change. This is relatively easy to analyze in nondegenerate situations, e.g. if \( \lambda \) is a single eigenvalue of the linearized operator \( G(\lambda) \). In quantum mechanics the eigenvalues of the Hamiltonian \( H \) are the energies of the system. In this case the change in the Hamiltonian \( H \) as \( \lambda \) changes is the change in the energy of the system. In general, however, the change in the Hamiltonian \( H \) as \( \lambda \) changes is not easy to analyze. However, there are some special cases where the change in the Hamiltonian \( H \) as \( \lambda \) changes can be analyzed. One such case is when the Hamiltonian \( H \) is a function of the energy \( E \) of the system. In this case the change in the Hamiltonian \( H \) as \( \lambda \) changes is the change in the energy of the system. This is the case when the Hamiltonian \( H \) is a function of the energy \( E \) of the system. In this case the change in the Hamiltonian \( H \) as \( \lambda \) changes is the change in the energy of the system. This is the case when the Hamiltonian \( H \) is a function of the energy \( E \) of the system.


Often of course a system will be perfectly symmetric but only approximately so. It is then advantageous to treat it as a perturbation of the more symmetric situation. In quantum physics this has been extraordinarily successful. Even when the symmetry is very badly broken (e.g. \( SU(6) \) in high-energy particle physics), there is some hope here, as the mere fact that a given Hamiltonian can be embedded in a family \( H_\tau \) starting with arbitrarily highly symmetric one says nothing.

12. Highly symmetric versus elementary.

The elementary constituents we like to use in science to describe more complicated objects tend to be highly symmetric: lines, circles, ellipses, radially symmetric potentials, ...; from this point of view one might ask for all objects with a simple generating algorithm or (equivalently?) a highly symmetric structure. Besides lines, circles, spheres this leads also to fractals such as Koch islands, objects with scaling symmetries, and, if statistical symmetries are admitted, to Brownian motion (another popular building block) and fractional Brownian functions.

13. Concluding remarks.

(a) There are certainly other ways in which a model can be symmetric (homogeneous) without admitting a group of symmetries. For instance other algebra then group algebras may appear as symmetry algebras 23; some of the powerful results of Kostant dealing with adjoint orbits of Lie groups generalize to a symplectic geometry setting; some Lie groups (algebra types) which should exist but not the underlying explaining group (algebra itself); special functions tend to belong to Lie groups except the fathergrandfather of most of them, the hypergeometric function.

(b) How to define symmetries in various situations is often an open problem. E.g. in partial differential equations it seems necessary to admit symmetries of the types of involving derivatives, i.e. automorphisms of phase space taking solutions into solutions are not general enough. This takes us into the realm of Bäcklund transformations, soliton theory and gauge theories (position dependent symmetries). None other molecules in chemistry pose problems of another kind.

(c) There are also "symmetric objects" (according to our intuition) which seem not to fit at all into a picture "symmetry is invariant under a symmetry group" of the kind for a simple Lie group, say, or a non-simple Lie group. For instance a Hamiltonian \( H \) is a G-space and this may very well turn out to be a fractal such as Koch islands, objects with scaling symmetries, and, if one adopts a geometric quantization approach to the whole,

Notes and references

3. Frank and Fromm (Ann. der Physik 24 (1911), 2) have already pointed out that the "relativistic equation" alone does not determine the whole structure of the relativistic theory and in particular forces the existence of a (numerically determined) fixed inelastic velocity; see also R.U. Seel, H.K. Urbanke, Relativität, Gruppen Theorien, Springer, 1979.
4. The development of quantum field theory itself can be based on symmetry considerations, cf. G.H. Dwyer, A development of quantum mechanics, Reidel, 1984. Supersymmetry is also a symmetry principle. Also (geometric) quantization and supersymmetry seems to be present but not the underlying symmetry principle. Also (geometric) quantization and supersymmetry seems to be present but not the underlying symmetry principle.

There is nothing particular about this list; this is simply my own collection of books on the topic. I have not included books on gauge theory, or soliton theory, nor books with titles like Differential geometric methods in physics or Mathematical problems of suitable Lie groups. (In the Bessel case the group SU(2) would have made the list twice as long.)

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