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NOTES ON (THE PHILOSOPHY OF) LINEARIZATION

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This note, originally a handout for the special session on linearization at the 1983 CDC in San Antonio consists of a discussion and initial guide to the literature on linearization, in various parts of mathematics. Special attention is paid to linearization of control systems.

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These notes, intended mainly as background information for the special session on "Linearization and Geometric Methods" at the 1983 CDC, San Antonio, Wednesday morning Dec. 14, are organized around the references I happen to know about. Quite a number of papers containing linearization results quoted in one of the papers mentioned below are not listed explicitly. I will be grateful for comments, remarks and additional references. A few lines about the special session itself are contained in Section 5 below.

1. The Simplest Cases. Basically the situation here is that one has a certain class of objects, e.g. input-output dynamical systems; a corresponding notion of isomorphism e.g. (local) state space equivalence, or feedback equivalence, or feedback equivalence; and a subclass of systems called linear. The general problem is to characterize those objects which are isomorphic to the linear ones and to find effective ways of constructing the isomorphism.

1.1 n-tuples of Differential Equations (locally). One of the best known and most studied linearization problems is concerned with when a system (1.2)

$$(1.2) \qquad \dot{x} = f(x), \quad f(0) = 0, \quad x \in \mathbb{R}^n$$

or, more generally an m-tuple (or a whole Lie algebra of such things)

$$\dot{x} = f_i(x), \quad f_i(0) = 0, \quad i = 1,\ldots,m$$

is equivalent to its linear part $\dot{x} = Ax$, A the Jacobian matrix of $f(x)$ at $x = 0$, resp. the n-tuple of linear parts $\dot{x} = A_i x$. This can be studied in a formal setting (when does there exist a formal power series substitution $y = \phi(x)$ which linearizes (1.2)); a real analytic setting; a C^r- or C^∞-infinity setting; a C^0-(i.e. homeomorphism) setting. Some references are respectively Sternberg [20], Hermann [10,11,33]; Poincare [22], Guillemin-Sternberg [34], Chen [72], Sedwick-Elliott [35], Basart [58], Livingston-Elliott [70]; Sternberg [23,24,25]; Hartmann [21,55]. Things may go wrong at all levels. I.e. there may even fail to be a formal substitution which does the job; if there is a formal linearizing substitution it may fail to converge so as to give a real analytic one, but there may be generalized "convergence" in sectors (Francoix [54]). It is true though that Poincare's formal condition also suffices for C^∞ linearizability. For the various differentiable cases Lie algebras of vector fields play an important role.

If (1.2) derives from a Hamiltonian setting $\dot{p} = \frac{\partial H}{\partial q}$, $\dot{q} = -\frac{\partial H}{\partial p}$ one is of course interested in linearizing transformations which preserve this feature (canonical transformations). Both in this case and the usual case (1.2) it is natural at first sight to concentrate at first on points where $dH = 0$ (if $dH \neq 0$, there exist a canonical transformation such that in the new coordinates $\dot{q}_1 = 1, \dot{q}_2 = \ldots = \dot{q}_n = 0 = \dot{p}_1 = \ldots = \dot{p}_n$, cf. e.g. Abraham [47;page 112]) respectively $f(x) = 0$ (if $f(x) \neq 0$ there is obviously a similar result). These (zero'th order) "linearized" versions carry very little information about the local structure of the flow near the point under consideration. This is essentially the same level of information as that which says that if $f(x) = 0$, x is an equilibrium point. To study the flow up to first order near a non-equilibrium point Perrizo [14,26] considered the flow derivative at the point under consideration carried back along the flow. This is obviously the level of information one will need for global linearization results obtained via (i) local results and (ii) local, global theorems.

1.3 Conjugacy Problems. Consider a differentiable manifold M and two differentiable maps $S:M \to M, T:M \to M$. The general question is, when does there exist a diffeomorphism $\psi:M \to M$ such that $T = \psi^{-1}S\psi$. For instance one has the following result [Sternberg,21]. Let S,T be two orientation preserving homeomorphisms of some neighborhood of $0 \in \mathbb{R}^n$ into itself such that $||Sx|| < ||x||$, $||Tx|| < ||x||$, there then exists a homeomorphism ψ such that $T = \psi^{-1}S\psi$. This does not mean that this problem is easy and completely settled. Even the easy to state problem of desiding when two linear endomorphisms $S,T:\mathbb{R}^n \to \mathbb{R}^n$ are topologically conjugate gives rise to very difficult topological questions (Kuiper, Cappell-Shaneson).

1.4. Dynamical Control Systems. Now consider more generally a system of differential equations like (1.2) but with extra control parameters

(1.5) $\dot{x} = f(x,u)$, $f(0,0) = 0$

and the question of when and in what sense this is equivalent to a linear system $\dot{x} = Ax + Bu$ locally near 0. Two natural groups of allowable transformations are diffeomorphisms of state space $y = \psi(x)$ as in 1.1 above (Krener [36]) and feedback equivalence (Brockett [41]), which can also be viewed as block triangular diffeomorphisms in state plust control space in that one allows transformations of the form $y = \psi(x)$, $v = \psi(x,u)$. There are now nice sufficient conditions known for feedback linearizability cf. Hunt-Su [37], Jakubczyk-Respondek [40], Su [9], Isidori-Krener [38], Krener-Isidori [46]. It is perhaps interesting to note that both the Hunt-Su method [37] and Poincare's method [22] for (1.2) proceed by means of writing down partial differential equations for y as a function of x. The theory of linearization by feedback also is significant for actual applications [Meyer-Cicolani [39], Meyer [1].

As in the cases described in 1.1 above the question of global linearization has hardly been touched. And perhaps there is less need to do so in the feedback case as further stabilizing feedback applied to a controllable linearization will help to keep the system near the 0 state (its operating point).

Let me also remark that - at least for applications - whether the allowable transformations form a (transformation) group or not is not so important; what is important is that one can pass freely from the system to the linearized version and back.

1.6. Normal Forms, Moduli, Structural Stability. It is clear that as a rule only few objects will be linearizable. More generally one then wants to know a complete set of inequivalent normal forms of objects and one wants to know whether these normal forms (canonical forms) form discrete or continuous families (moduli). The last question is especially important in view of robustness and structural stability: do objects which are close to each other have the same normal forms or, if not, normal forms of the same kind, e.g. both linear?

1.7. Nonlinear Representations. A dynamical system on a manifold $\dot{x} = f(x)$ gives rise to an action of the group $\mathbb{R}$ on M (assuming that solutions exist uniquely and globally). This is a simple case of a topological transformation group on a nonlinear representation of a lie group. One can ask whether every such representation is equivalent to a linear one. (The associated local question whether this is the case for the associated lie algebra plays - as I already remarked - a big role in the local linearizability of m-tuples of differentiable equations discussed in 1.1 above.)

Bochner [13] showed that compact groups of differentiable transformations near a fix point are linearizable (the analytic case goes back to Cartan [75]). Other linearizability results for representations are contained in Flato a.o. [43] and Simon [67].

2. Weaker Notions of Linearizability. Often it will be the case that not all or not enough objects will be linearizable. It will then still be possible and be of interest to compare them to the class of linear ones. This gives rise to several groups of questions which might be viewed as weaker notions of linearizability. Examples are: "is every object a subobject, quotient or sub-quotient of a linear one and questions of partial and approximate linearization.

2.1 Embeddings. Can a nonlinear object always be seen as a subobject of a linear one. In differential topology e.g., Whitney's theorem (cf. e.g. Poenaru [48]) that every differentiable m-manifold can be embedded in an $\mathbb{R}^{2m}$

has been and is of enormous importance. And so are nonembedding results such as that the projective plane cannot be embedded in R^3.

For manifolds together with a flow on them there are (local) embedding theorems of McCann [17] and Janos [16].

The question has also been considered for control systems and input-output systems by Monaco, Normand-Cyrot, Claude, Fliess and Isidori [29, 30,31,32]. Cf. also Isidori [78] for a related approach by means of which he can solve the matching problem.

Similarly one can ask when a group of differentiable transformations is linearizable in the sense that it is a subobject of a linear one. I.e. we are now interested in the embedding version of the matter discussed in 1.6 above. Here there are results of Mostow [76,77] and in the topological case there are the remarkable results of Baayen-de Groot [12] and de Vries [27,28] which essentially say that for every reasonably nice (such as locally compact) group G there exists a universal linear G-space in which every G-space of smaller weight can be embedded. On obvious question with applications in control and system theory is whether something similar could be true for semigroups instead of groups.

2.2. Quotients. Dually one can ask when a manifold or a manifold with a flow on it (i.e. (locally) an n-tuple of differentiable equations) is a quotient of a linear one. In this category this appears to be a rather stronger property than being a subobject of a linear one. So strong indeed that such objects are often called linearizable. For instance the matrix Riccati equation $\dot{K} = -Q-A^TK-KA+KBB^TK$ is said to be linearizable. More precisely the situation is that it can be completed to an equation on a suitable Grassmann manifold. This Grassmann manifold is a quotient of a space of matrices of full rank and on this space there exists a linear flow which descends to the Grassmann manifold and induces the Riccati flow.

Much the same kind of picture is presented by the Sabat-Zaharov socalled "dressing method" cf. e.g. Zaharov-Mahankov [74] of solving the completely integrable equations of mathematical physics such as the KdV-equation, the sine-Gordon and the nonlinear Schrodinger equation. There the "covering linearization" is a suitable Riemann-Hilbert boundary value problem.

2.3. Other Kinds of Linearization. Consider by way of example the KdV equation $u_t + uu_x + u_{xxx} = 0$ with initial data $u(0,x)$. The inverse scattering transform method of solving the KdV associates to the initial potential $u(0,x)$ certain asymptotic scattering data, these evolve linearly if u evolves according to the KdV and via the Gelfand-Levitan equation or Marčenko equation (as the case may be) the potential $u(t,x)$ may be recovered from the scattering data at time t, cf. e.g. Drazin [73]. Whether this can be viewed as a linearization as in 1.1 above, with $\mathbb{R}^n$ replaced by a suitable function space, is not clear to me. Apparently all the socalled "completely integrable equations" of mathematical physics are linearizable in some such sense, cf. also Adler-v Moerbeke [44], Krishnaprasad [63]. A generalization to "noncommutative complete integrability" is discussed by Marle [45]. The (not completely integrable) Yang-Mills equations are also linearizable, at least formally, Flato-Simon [42].

2.4. Partial Linearization, Approximate Linearization. If e.g. a control system or set of differential equations is not linearizable one can ask whether it can be presented (more or less) as a fibre product of two systems one of which is linear and maximally large (in dimension) with respect to that property. This can take various forms. A fibre linear system (cf. Hazewinkel [71], Respondek [7]) is one of the form $\dot{y} = g(y,u)$, $\dot{x} = A(y)x + B(y)u$ at least locally, where $A(y)$, $B(y)$ are matrices depending on the possibly nonlinearly evolving vector y. This relates to linear extensions of a flow and linearization around an invariant submanifold, cf. e.g. Samoilenko [53], Osipenko [56]. The extended Kalman filter can also be viewed in this light. Alternatively and more or less dually one may look for equivalent systems of the form $\dot{y} = Ay + Bv$, $\dot{x} = f(x,y,v)$. This type of partial linearization occurs in Krener-Isidori-Respondek [4]. Stochastic linearization (Beaman [18], Taylor [64,65]) seems to be of a somewhat different nature and more related to the idea of a more global linear approximation, a topic which I shall not discuss here. Though it is clear that on occasion related ideas like "averaging" e.g. may well result in a linear quotient of a nonlinear system and thus a fibre structure of the second type indicated above, Balbi [19].

3. Obstructions. As a rule one expects that such properties as being linearizable, being a subobject of a linear one, a quotient of a linear one, etc. will be of a cohomological nature. That is whether such a property holds or not is determined by whether certain cohomology classes (obstructions) vanish or not. Such obstructions may arise at various levels, e.g. in the case of vector fields (cf. 1.1 above) there are formal obstructions (cf. e.g. Hermann [10,11]) which are definitely of a cohomological nature (for the related case of systems, cf. Hermann [52]), there are local obstructions and there are no doubt local global obstructions which may prevent a system that is everywhere locally linearizable from being globally isomorphic to a linear one.

4. Uses of Linearization. Obviously, assuming that we know more about linear objects than nonlinear ones, linearization is an exceedingly valuable tool in all sorts of applications. This makes it important to have algorithms for deciding when a given object is linearizable. In many cases effective recognition procedures do not exist (not even in principle), e.g. in the case of the completely integrable equations of mathematical physics.

The use of repeated infinitesimal linearization in numerical procedures is well known [59-62]. (Newton methods, gradient methods). Less known are such methods as Razumihin's [69] for dealing with e.g. quadratic (more generally polynomial) programming problems. Consider e.g. max $\Sigma b_{ij}x_ix_j$ subject to linear constraints. Replace this by max $\Sigma b_{ij}x_iy_j$ with additional constraints $x_i=y_i$. Now fix a $y^{(0)}$, solve the resulting (inconsistent as a rule, i.e. nonfeasible) LP problem to find $x^{(0)}$; now fix $x^{(0)}$ to find $y^{(1)}$; fix $y^{(1)}$ to find $x^{(1)}$; etc. This converges and constitutes a repeated linearization process which is not of the infinitesimal type but more of the embedding sort.

Perturbation calculus is of course linearization around a solution i.e. linearization around an invariant subobject and will not be further discussed here. Nor interpolation and the question of approximating objects by piecewise linear ones, all valuable methods based on a philosophy of linearization.

Let me remark though that presenting or obtaining an object as a quotient of a linear one is especially useful for getting hold of its symmetry properties especially more global ones. This is the most promising approach to showing that the symmetries of the integrable equations of mathematical physics are, as they shoud be, Kac-Moody Lie algebras and, leads e.g. to super position principles. Cf. e.g. Shayman [8] for symmetries of the matrix Riccati equation (via linearization) and Harnad a.o. [51] for super position principles for these equations.

5. On the Special Session "Linearization and Geometric Methods".
The topic of the session is somewhat wider than linearization itself. More accurately it can be described as the use of nonlinear feedback control to achieve certain goals such as (partial) linearization. But linearization may be the wrong thing to do (Nymeyer [5]). Lie-algebraic ideas and techniques tend to play a central role.

In [1] G. Meyer discusses the idea and philosophy of feedback linearization as applied to flight control, both the problems and successes of this approach in th

In [2] R. Hermann is concerned with feedback linearization and especially the algebraic invariants of the associated Pfaffian systems. He turns the problem into a problem of isomorphism for an associated Cartan-Vessoit algebra (related to deformation theoretic ideas) and shows the local obstructions to linearizability to be cohomological in nature.

In [3] Hunt and Su discuss linear approximation if one is away from an equilibrium point of the drift term of a nonlinear system. It could be interesting to compare this to the work of Perrizo [26,14] and to study the resulting family of linear systems parametrized by state space.

In [4] Krener, Isidori and Respondek are concerned with partial linearization and robustness of the process, cf. also 1.6 above. The linearization conditions are not robust and this has implications for the technique of linearization, certainly in applications. A much related preprint is [7].

In [5] Nymeyer is not concerned with linearization but with the generalization of techniques (of decoupling) which work in the linear case to a nonlinear solution. Here linearization might be the wrong thing to do.

Finally in [6] Gilbert and Ha present a theory of nonlinear (feedback) control for tracking problems which yields a unified framework for a number of problems in the control of mechanical manipulators.

Some Papers Re Linearization

1. G. Meyer, Applications of Linearization to Flight Control.

2. R. Hermann, Pfaffian Systems and Feedback Linearization/Obstruction.

3. L. R. Hunt, R. Su, Linear Approximations of Nonlinear Systems.

4. A. Isidori, A. Krener, W. Respondek, Partial and Robust Linearization by Feedback.

5. H. Nymeyer, Noninteracting Control for Nonlinear Systems.

6. E. G. Gilbert, T. J. Ha, An Approach to Nonlinear Feedback Control With Applications to Robotics.

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