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ON DECENTRALISATION, SYMMETRY AND SPECIAL STRUCTURE IN LINEAR SYSTEMS

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ON DECENTRALISATION, SYMMETRY AND SPECIAL STRUCTURE IN LINEAR SYSTEMS [1)]

by

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Abstract. In this paper we discuss linear dynamical systems "with special structure" such as networks of identical smaller systems connected in various ways and e.g. two helicopters connected by means of a rigid lifting beam. (The twin lift problem). In this paper we develop a tool for recognizing special structure and making effective use of the presence of special structure. This tool is the symmetry algebra of a system with special structure. As we shall see it is a useful concept for instance when discussing such matters as the stabilization of a system with special structure in such a way that the special structure is preserved.

Contents.

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1. INTRODUCTION.

Many systems in nature and engineering (linear dynamical input/output systems in state space form) have a good deal of special structure. They may consist e.g. of a collection of identical units connected together in various ways, or there may be large systems consisting of many subunits which fall into a small number of types. Think of electrical or neural networks. Another example consists of twin helicopters connected with a rigid beam, a system which is of considerable importance for practical applications and as we shall see which poses many unsolved problems. Other examples arises from the discretization (in space) of partial differential equations ([Brockett-Willems, 1974])

As still another example one may for quantization purposes be interested in systems which can be put into Lagrangian form ([Tarn-Huang-Clark, 1980]), or one may for identification purposes be exclusily interested in systems which have a certain number of perscribed zero's and ones in their Hankel matrices or in their state space representation. In this last case one of the more immediate questions is: What is special structure? For control systems $\dot{x} = Ax + Bu$ there is e.g. usually nothing special about the class for which the upper righthand corner element of A is zero. All ≥ 3 dimensional completely reachable systems have an equivalent representation for which this is the case. Ideally special structure should be defined in an invariant way.

A system with special structure should be grosso modo easier to analyze than an equally large system with no particular properties and the question arises how to take advantage of the special structure. Also in cases, one would e.g. like to stabilize a given system with special structure in such a way that the new system still has the same special structure. Indeed often the mechanics of the situation will be such that this is the only reasonable thing to do.

In this paper we develop a tool: the symmetry algebra of a class of systems with special structure. This provides an invariant description of special structure and if the symmetry algebra is nontrivial there is definitely special structure present, i.e. it can not be an artefact of a special state space representation. Also as we shall see it can be a most effective tool in reducing the complexity of a problem and is solving e.g. stabilization problems while remaining in the same class of systems with special structure.

2. EXAMPLES OF DYNAMICAL SYSTEMS WITH SPECIAL STRUCTURE.

The classes of systems we wish to consider in this paper often arise when several identical systems are interconnected so that there is dynamic interaction. Due to physical and/or economic contraints feedback controls are restricted to less than the full state. Such systems have been considered in the socalled "decentralized linear system" literature.

Our primary example comes from aeronautics. Helicopters are routinely used to transport materials in the construction trade. However, load size limitations restrict their usefulness. An attractive solution is to use several helicopters to simultaneously transport very large masses. We consider the case when two helicopters are used, the mass is connected to the center of a rigid bar via a cable and the helicopters are connected by cable to the ends of the bar. Velocities are generally low and linear models seem particularly suited. Let the helicopter dynamics be modeled as a linear system $\dot{x} = Ax + Bu$. Typically A will be about a 27 x 27 matrix and B will be about 27 x 3. The coupling between the two ships is a function of the length of the bar and the length of the two cables. We ignore the pendulum effect of swung load and consider the mass as a point mass in the center of the bar. This results in a system model of the form

$$\begin{pmatrix} \dot{x}_1 \\ \dot{x}_2 \end{pmatrix} = \begin{pmatrix} A & H \\ -H & A \end{pmatrix} \begin{pmatrix} x_1 \\ x_2 \end{pmatrix} + \begin{pmatrix} B & 0 \\ 0 & B \end{pmatrix} \begin{pmatrix} u_1 \\ u_2 \end{pmatrix} . \tag{2.1}$$

The coupling is through the matrix H.

The pilot workload is very high in normal helicopter tasks and to expect the pilot to monitor the state of the other helicopter is not realistic. Thus the feedback control has to be of the form $u_i = K_i x_i$. On the other hand highly trained pilots should react very similarly to perturbations and so we should expect that the K_i's are almost equal. Thus if we assume equality we have the system 2.1 and an allowable set of control laws

$$\begin{pmatrix} u_1 \\ u_2 \end{pmatrix} = \begin{pmatrix} K & 0 \\ 0 & K \end{pmatrix} \begin{pmatrix} x_1 \\ x_2 \end{pmatrix} + \begin{pmatrix} u_1' \\ u_2' \end{pmatrix} . \qquad (2.2)$$

Similar examples arise when trying to control a file of moving vehicles. Typically the feedback has been restricted to the adjacent vehicles or to some small number of adjacent vehicles. More complicated examples occur when one tries to control a two dimensional array of moving objects. The problem is the same as the helicopter control problem in that there are several identical units with restricted communication and by symmetry one would expect that identical feedback control laws would be used. The question we address in this paper is whether as not the resulting structured models can be exploited in the analysis and control of such systems.

3. THE SYMMETRY ALGEBRA OF SYSTEMS WITH SPECIAL STRUCTURE.

For the moment let $\dot{x} = Fx + Gu$ be a class of systems with special structure as provisorily defined above. I.e. we consider all such systems for which the entries of F and G satisfy a number of given linear equations. E.g. F and G could be required to be of the form

$$F = \begin{pmatrix} A & H & H \\ H & A & H \\ H & H & A \end{pmatrix} , \quad G = \begin{pmatrix} B & 0 & 0 \\ 0 & B & 0 \\ 0 & 0 & B \end{pmatrix} \qquad (3.1)$$

In this section the symmetry algebra of such a class of systems is defined and this will also yield a better definition of "class of systems with special structure".

3.2. Definition of the symmetry algebra of a class of systems. Let $M_n(\mathbb{R})$ denote the $\mathbb{R}$-algebra of real n x n matrices. Let $\underline{C}$ be a class of systems of dimension n and with m inputs. Then the symmetry algebra of $\underline{C}$ is defined by

$$R(\underline{C}) = \{(S,T) \in M_n(\mathbb{R}) \times M_m(\mathbb{R}) : SF = FS,\ SG = GT \text{ all } (F,G) \in \underline{C}\} \tag{3.3}$$

3.4. Remarks. Often $R(\underline{C})$ is uniquely determined by its image in $M_n(\mathbb{R})$ under the projection $M_n(\mathbb{R}) \times M_m(\mathbb{R}) \to M_n(\mathbb{R})$. Indeed this will happen in all cases that the class $\underline{C}$ contains a system F,G with G full rank (because then SB = BT uniquely determines T). However, this does not mean that $R(\underline{C}) = \{S \in M_n(\mathbb{R}) : SF = FS\}$. The restriction that there must exist a T for a given S such that SG = GT for all input matrices G occurring in the class may cause extra restrictions. Cf. e.g. example 3.11 below. Usually we shall describe $R(\underline{C})$ by means of its image in $M_n(\mathbb{R})$.

3.5. Example. The following class of systems (exemplified by two helicopters arranged as indicated )

$$\begin{pmatrix} A & H \\ -H & A \end{pmatrix}, \begin{pmatrix} B & 0 \\ 0 & B \end{pmatrix}$$

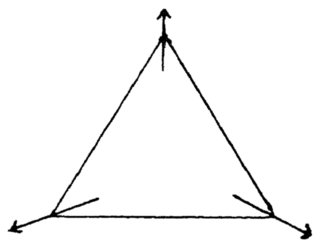
has as symmetry algebra $\mathbb{R}[i]$, $i^2 = -1$; i.e. the symmetry algebra is $\mathbb{C}$, the field of complex numbers. As a subalgebra of $M_{2n}(\mathbb{R}) \times M_{2m}(\mathbb{R})$ an $\mathbb{R}$-basis for this symmetry algebra is

$$\left(\begin{pmatrix} I_n & 0 \\ 0 & I_n \end{pmatrix}, \begin{pmatrix} I_m & 0 \\ 0 & I_m \end{pmatrix}\right), \left(\begin{pmatrix} 0 & I_n \\ -I_n & 0 \end{pmatrix}, \begin{pmatrix} 0 & I_m \\ -I_m & 0 \end{pmatrix}\right)$$

and inversely if a system admits $\mathbb{R}[i] \subset M_{2n}(\mathbb{R}) \times M_{2m}(\mathbb{R})$ as a symmetry algebra then the "A-matrix" is of the form indicated though

the "B-matrix" may be of the more general form $\begin{pmatrix} B & B' \\ -B' & B \end{pmatrix}$. This is generally the case in the examples considered below.

3.6. Example. Consider the class of systems (E.g. helicopters arranged as indicated)



$$\begin{pmatrix} A & H & -H \\ -H & A & H \\ H & -H & A \end{pmatrix}, \quad \begin{pmatrix} B & 0 & 0 \\ 0 & B & 0 \\ 0 & 0 & B \end{pmatrix}$$

The symmetry algebra is $R[\xi_3]$, $\xi_3^3 = 1$. A basis (for the image in $M_{3n}(\mathbb{R})$ is

$$\begin{pmatrix} I & 0 & 0 \\ 0 & I & 0 \\ 0 & 0 & I \end{pmatrix}, \quad \begin{pmatrix} 0 & I & 0 \\ 0 & 0 & I \\ I & 0 & 0 \end{pmatrix}, \quad \begin{pmatrix} 0 & 0 & I \\ I & 0 & 0 \\ 0 & I & 0 \end{pmatrix}$$

3.7. Example. Consider the class of systems (E.g. two helicopters arranged as indicated )

$$\begin{pmatrix} A & H \\ H & A \end{pmatrix}, \quad \begin{pmatrix} B & 0 \\ 0 & B \end{pmatrix}$$

The symmetry algebra is $\mathbb{R}[x]$, $x^2 = 1$. A basis is given by

$$\begin{pmatrix} I & 0 \\ 0 & I \end{pmatrix}, \quad \begin{pmatrix} 0 & I \\ I & 0 \end{pmatrix}$$

3.8. <u>Example</u>. Consider the class of systems given by

$$\begin{pmatrix} A & H_1 & 0 & 0 \\ H_1 & A & 0 & 0 \\ H_2 & H_2 & A-H_1 & 0 \\ H_3 & H_3 & 0 & A+H_1 \end{pmatrix}, \quad \begin{pmatrix} B & 0 & 0 & 0 \\ 0 & B & 0 & 0 \\ 0 & 0 & B & 0 \\ 0 & 0 & 0 & B \end{pmatrix}$$

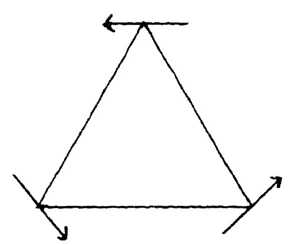
The symmetry algebra is four dimensional with basis 1, a, b, c and the multiplication defind by $a^2 = 1$, $ab = b$, $ba = -b$, $ac = ca = c$, $b^2 = c^2 = bc = cb = 0$. As a subalgebra of $M_{4n}(\mathbb{R})$ it manifests itself as the subalgebra spanned by

$$\begin{pmatrix} I & 0 & 0 & 0 \\ 0 & I & 0 & 0 \\ 0 & 0 & I & 0 \\ 0 & 0 & 0 & I \end{pmatrix}, \quad \begin{pmatrix} 0 & I & 0 & 0 \\ I & 0 & 0 & 0 \\ 0 & 0 & I & 0 \\ 0 & 0 & 0 & I \end{pmatrix}, \quad \begin{pmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ I & -I & 0 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix},$$

$$\begin{pmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ I & I & 0 & 0 \end{pmatrix}.$$

3.9. <u>Example</u>. Consider the class of systems (e.g. three helicopters)

$$\begin{pmatrix} A & H & H \\ H & A & H \\ H & H & A \end{pmatrix}, \quad \begin{pmatrix} B & 0 & 0 \\ 0 & B & 0 \\ 0 & 0 & B \end{pmatrix}$$



i.e. the class of systems consisting of three identical units interconnected in a completely symmetrical way. In such a case one expects of course that the symmetries will be generated by permutations of the subunits. And, indeed the symmetry algebra is the group ring $\mathbb{R}[S_3]$, where S_3 is the group of permutations on three letters. As a subalgebra of $M_{3n}(\mathbb{R})$ the symmetry algebra has a a basis

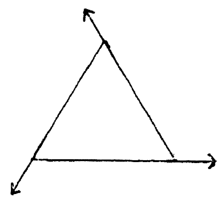
$$\begin{pmatrix} I & 0 & 0 \\ 0 & I & 0 \\ 0 & 0 & I \end{pmatrix}, \quad \begin{pmatrix} 0 & I & 0 \\ I & 0 & 0 \\ 0 & 0 & I \end{pmatrix}, \quad \begin{pmatrix} I & 0 & 0 \\ 0 & 0 & I \\ 0 & I & 0 \end{pmatrix},$$

$$\begin{pmatrix} 0 & 0 & I \\ 0 & I & 0 \\ I & 0 & 0 \end{pmatrix}, \quad \begin{pmatrix} 0 & 0 & I \\ I & 0 & 0 \\ 0 & I & 0 \end{pmatrix}, \quad \begin{pmatrix} 0 & I & 0 \\ 0 & 0 & I \\ I & 0 & 0 \end{pmatrix}.$$

Examples 3.7 and 3.9 of course generalize to give examples with symmetry algebra $\mathbb{R}[S_\ell]$, for all $\ell \in \mathbb{N}$

3.10 Example. Consider a system consisting of three helicopters symmetrically arranged as indicated. Then 1 acts on 2 as 2 on 3 and 3 on 1, and the 1 acts on 3 as minus the action of 1 on 2 but with 3 and 2 oriented in a 60% angle towards each other. This can be represented by the class of systems

$$\begin{pmatrix} A & H & \omega H \\ \omega H & A & H \\ H & \omega H & A \end{pmatrix}, \quad \begin{pmatrix} B & 0 & 0 \\ 0 & B & 0 \\ 0 & 0 & B \end{pmatrix}$$



where ωH stands for the interaction H twisted through 120^o. (One can of course write this in terms of real matrix by doubling the size of the matrices). The symmetry algebra of this also turns out to be $\mathbb{C}[S_3]$. It manifests itself as the algebra spanned by

$$\begin{pmatrix} I & 0 & 0 \\ 0 & I & 0 \\ 0 & 0 & I \end{pmatrix}, \quad \begin{pmatrix} 0 & \omega I & 0 \\ \omega^2 I & 0 & 0 \\ 0 & 0 & I \end{pmatrix}, \quad \begin{pmatrix} 0 & 0 & I \\ I & 0 & 0 \\ 0 & I & 0 \end{pmatrix},$$

$$\begin{pmatrix} 0 & I & 0 \\ 0 & 0 & I \\ 0 & I & 0 \end{pmatrix}, \begin{pmatrix} 0 & 0 & \omega^2 I \\ 0 & I & 0 \\ \omega I & 0 & 0 \end{pmatrix}, \begin{pmatrix} I & 0 & 0 \\ 0 & 0 & \omega I \\ 0 & \omega^2 I & 0 \end{pmatrix}.$$

Considered as a 6n-dimensional real system the symmetry algebra still is $\mathbb{C}[S_3]$.

3.11.Example. Consider two identical systems with one feeding into the other. This might be represented by the class

$$\begin{pmatrix} A & H \\ 0 & A \end{pmatrix}, \begin{pmatrix} B & 0 \\ 0 & B \end{pmatrix}$$

The symmetry algebra is 2 dimensional with basis 1, a, and multiplication given by $a^2 = 0$. As a subalgebra of $M_{2n}(\mathbb{R})$ it manifests itself as the algebra spanned by

$$\begin{pmatrix} I & 0 \\ 0 & I \end{pmatrix}, \begin{pmatrix} 0 & I \\ 0 & 0 \end{pmatrix}$$

However, if we insist that the two constituting units be fed the same inputs we get the class

$$\begin{pmatrix} A & H \\ 0 & A \end{pmatrix}, \begin{pmatrix} B \\ B \end{pmatrix} \tag{3.12}$$

The equation

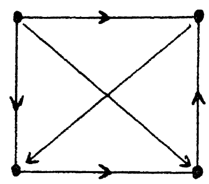
$$\begin{pmatrix} 0 & I \\ 0 & 0 \end{pmatrix} \begin{pmatrix} B \\ B \end{pmatrix} = \begin{pmatrix} B \\ B \end{pmatrix} T$$

has no solutions (if B is nonzero) and hence the symmetry algebra of the class (3.12) is $\mathbb{R}$, i.e. no larger than that of the class of systems with no special structure at all. Thus there appear to be aspects of "systems with special structure" which are not captured by the symmetry algebra point of view.

3.13.Example. Consider the class of systems given by

$$\begin{pmatrix} A & H_1 & H_2 & H_3 \\ -H_1 & A & -H_3 & H_2 \\ -H_2 & H_3 & A & -H_1 \\ -H_3 & -H_2 & H_1 & A \end{pmatrix} \quad \begin{pmatrix} B & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & B & 0 \\ 0 & 0 & 0 & B \end{pmatrix}$$

This could well represent four units interconnected in a square as shown. The symmetry algebra of this system is the algebra of the quaternions with basis 1, i, j, k and multiplication table $i^2 = j^2 = k^2 = -1$, $ij = k$, $ji = -k$, $jk = i$, $kj = -i$, $ki = j$, $ik = -j$.



3.14. Example. Circulant systems. Consider the class of systems given by

$$\begin{pmatrix} A_1 & A_2 & \cdots & A_{r-1} & A_r \\ A_2 & A_3 & \cdots & A_r & A_1 \\ \cdot & & \cdot & & \cdot \\ A_r & A_1 & \cdots & A_{r-2} & A_{r-1} \end{pmatrix}, \quad \begin{pmatrix} B_1 & B_2 & \cdots & B_r \\ B_2 & B_3 & \cdots & B_1 \\ \cdot & & \cdot & \cdot \\ B_r & B_1 & \cdots & B_{r-1} \end{pmatrix}$$

The symmetry algebra of this class of systems is $\mathbb{R}[\mathbb{Z}/n)]$, where $\mathbb{Z}/n)$ is the cyclic group of order n. These systems naturally occur as models arising from spatial discretizations of linear constant coefficient PDE's, cf. [Brockett-Willems, 1974]. Inversely if a system has this symmetry algebra then it is of the form indicated.

3.15. Definition of "systems with special structure". Let R be any associative finite dimensional algebra over $\mathbb{R}$. For every system $\Sigma = (A,B)$ consider its symmetry algebra

$$R(\Sigma) = \{(S,T) \in M_n(\mathbb{R}) \times M_m(\mathbb{R}) : SA = AS,\ SB = BT\} \qquad (3.16)$$

A system with special structure R is now a system Σ together with an injective ring homomorphism $R \to R(\Sigma)$. Note that this notion is state-space base change invariant. (Because if $\Sigma' = (gAg^{-1}, gBh^{-1})$, $g \in GL_n(\mathbb{R})$, $h \in GL_m(\mathbb{R})$ then the symmetry algebra $R(\Sigma')$ is equal to $\{(gSg^{-1}, hTh^{-1}) : (S,T) \in R(\Sigma)\}$.

More informally let R be a subalgebra of $M_n(\mathbb{R}) \times M_m(\mathbb{R})$. Then the class of systems with special structure $\underline{C}(R)$ consists of all $\Sigma = (A,B)$ such that $R \subset R(\Sigma)$, i.e. such that $SA = AS$, $SB = BT$ for all $(S,T) \in R$.

We can embed $M_n(\mathbb{R}) \times M_m(\mathbb{R})$ in $M_{tn}(\mathbb{R}) \times M_{sm}(\mathbb{R})$ by mapping $S \in M_n(\mathbb{R})$ to $S \otimes I_t$ in $M_{nt}(\mathbb{R})$ and $T \in M_m(\mathbb{R})$ to $T \otimes I_s$. Thus a single subalgebra R gives rise to collection of systems with special structure of varying dimensions. Just as is the case in all of the examples above.

3.17. Dimension reduction. Let $R \subset M_n(\mathbb{R}) \times M_m(\mathbb{R})$ and consider the class of systems with special structure R, state space $\mathbb{R}^n$ and input space $\mathbb{R}^m$. Via the embedding $R \subset M_m(\mathbb{R}) \times M_m(\mathbb{R})$ these become (left) R-modules. Now let $\Sigma = (A,B) \in \underline{C}(R)$. Then $AS = SA$, $SB = BT$ for all $(S,T) \in R$ which precisely means that $A: \mathbb{R}^n \to \mathbb{R}^n$ and
$B: \mathbb{R}^m \to \mathbb{R}^n$ are R-module homomorphisms. Thus we can consider $\Sigma = (A,B) \in \underline{C}(R)$ as a system over the ring R. Of course there is no guarantee that the R-modules $\mathbb{R}^n$ and $\mathbb{R}^m$ will be free R-modules.

However, as we shall see especially if the ring R is semi-simple, it may be advantageous to consider a system in $\underline{C}(R)$ as a system over R. In particular if we are dealing with systems over $\mathbb{C}$ with special structure and R is semisimple then the theory of systems with special structure R is naturally equivalent to the theory of the usual linear systems over $\mathbb{C}$. If we are dealing with real systems and R is semi-simple then the theory of systems with special structure R boils down to the union of the theory of ordinary real systems, ordinary complex systems and linear systems over the (noncommutative) field of the quaternions.

3.18. Every algebra can occur as a symmetry algebra of a class of systems. Let R be any finite dimensional associative algebra over $\mathbb{R}$. Consider the opposite algebra R^{opp} (same underlying vectorspace as R; the new multiplication, denoted by $*$, is defined by $a * b = ba$, $a,b \in R^{opp}$). Choose a basis $x_1, \ldots, x_n$ for R^{opp} and write down the matrices of multiplication with x_i, $i = 1, \ldots, r$.

$$A(i) = \begin{pmatrix} a_{11}(i) & \cdots & a_{1r}(i) \\ \vdots & & \vdots \\ a_{r1}(i) & \cdots & a_{rr}(i) \end{pmatrix}, \quad a_{k\ell}(i) \in \mathbb{R}$$

So that e.g.

$$x_s x_i = \sum_t a_{ts}(i) x_t$$

Now consider r matrices $H_1, \ldots, H_r$ and the class of systems

$$\begin{pmatrix} A_{11} & \cdots & A_{1r} \\ & \ddots & \\ A_{r1} & \cdots & A_{rr} \end{pmatrix}, \quad \begin{pmatrix} B & & 0 \\ & \ddots & \\ 0 & & B \end{pmatrix}$$

where the $A_{11}, \ldots, A_{rr}$ are given by

$$A_{ij} = \sum_{s=1}^{r} a_{ij}(s) H_s$$

Then the class of systems thus defined (for varying $H_1, \ldots, H_r, B$) has the symmetry algebra R. (NB the algebras R and R^{opp} need not be isomorphic; this happens e.g. for the symmetry algebra of example 3.8). Thus, in a sense, the theory of systems over (not necessarily commutative) finite dimensional $\mathbb{R}$-algebras is equivalent to the theory of linear systems with special structure.

Inversely if Σ is a system over the $\mathbb{R}$-algebra R, then considering the state module and input module of Σ as $\mathbb{R}$-vector spaces and writing out the matrices of the state transition and input morphism we find a system with special structure R.

3.19. Remark. For systems like (3.1) which are of the form $A = \Sigma M_i \otimes H_i$, $B = I_r \otimes B'$ for certain fixed r x r matrices M_i, just like the classes constructed in 3.18 above, it may be useful to calculate the algebra generated by the M_i [Martin, 1982]. The centre of this algebra is necessarily part of the symmetry algebra of the class of systems under consideration. Usually, however, the centre is $\mathbb{R}$ itself so that no information is gained. E.g. in the case of example 3.13 the algebra generated by the M_i is (isomorphic to) the quaternion algebra which has centre $\mathbb{R}$. In the case of the systems of example 3.9 the algebra generated by the M_i is 2 dimensional and gives one nontrivial element of the symmetry algebra. In the case of the circulant systems of example 3.14 the full symmetry algebra is found in this way.

4. ON THE THEORY OF SYSTEMS WITH SEMI-SIMPLE SYMMETRY ALGEBRA.

4.1. Preliminary General Remarks. Consider a (class of) system(s) with special structure as defined in (3.15), with symmetry algebra $R \subset M_n(\mathbb{R}) \times M_m(\mathbb{R})$. Then of course one meets the full array of problems of usual linear system theory, viz. "stabilization by state (and output) feedback, disturbance decoupling in various forms, ... and of course: what are the natural invariants? Of course when doing all these things one would like to preserve the special structure which means e.g. that the feedback matrix $L: \mathbb{R}^m \to \mathbb{R}^n$ must be an R-module homomorphism. The natural state space base changes in this setting are of course the elements of the subgroup $GL(\mathbb{R}^n, R)$ consisting of those $g \in GL_n(\mathbb{R})$ which are R-module endomorphisms of $\mathbb{R}^n$, and the natural invariants of the systems in $\underline{C}(R)$ are the invariants under this group action.

4.2. Recapitulation of some definitions of representation theory. Let R be a finite dimensional associative algebra with unit element over $\mathbb{R}$. A representation of R in a vectorspace V over $\mathbb{R}$ is a homomorphism of associative $\mathbb{R}$-algebras with unit element

$\rho: R \to End(V)$, where End(V) is the $\mathbb{R}$-algebra of endomorphisms of V (isomorphic to $M_n(\mathbb{R})$ if dim V = n). A subspace $W \subset V$ is a subrepresentation if $rw \in W$ for all $w \in W$, $r \in R$. Here *rw* is short for $\rho(r)(w)$. A representation V is irreducible iff every subrepresentation is either 0 or V itself. An algebra R is called semisimple if for every representation V of R and every subrepresentation $W \subset V$ there is a complimentary subrepresentation W', i.e. a subrepresentation W' such that $V = W \oplus W'$. In matrix terms this means that if $\rho: R \to M_n(\mathbb{R})$ is such that for certain S, $S^{-1}\rho(r)S$ is in block upper trangular form for all $r \in R$ then there exists an S_1 such that $S_1^{-1}\rho(r)S_1$ is in block diagonal form for all $r \in R$ with the same upper left corner blocks as the $S^{-1}\rho(r)S$. Instead of using real numbers $\mathbb{R}$ as a base field one can also use the complex numbers $\mathbb{C}$ is all of the above.

Examples of semisimple algebras are e.g.,

(i) The $\mathbb{R}$-algebra $\mathbb{C}$ of the complex numbers

(ii) The full matrix algebras $M_n(\mathbb{R})$, $M_n(\mathbb{C})$

(iii) The algebra $\mathbb{H}$ of the quaternions. This is the four dimensional algebra over $\mathbb{R}$ with basis 1, i, j, k and multiplication rules $ij = k$, $ji = -k$, $jk = i$, $kj = -i$, $ki = j$, $ik = -j$ (and 1 acts of course as the unit element). This algebra can be realized as the subalgebra of $M_4(\mathbb{R})$ consisting of all matrices of the form

$$\begin{pmatrix} a & b & c & -d \\ -b & a & -d & -c \\ -c & d & a & b \\ d & c & -b & a \end{pmatrix} \tag{4.3}$$

(iv) Let G be a finite group. The group algebra $\mathbb{R}[G]$ consists of all sums $\sum_{g \in G} a_j g$ with multiplication rule

$(\Sigma\, a_g g)(\Sigma b_g g) = \sum_g (\sum_h a_{gh^{-1}} b_h)g$. I.e. it is the algebra with as basis the elements of G and the multiplication on this basis defined by the group multiplication of G. E.g. let $G = \mathbb{Z}/(n)$ be the cyclic group of order n, then $\mathbb{R}[\mathbb{Z}/(n)] = \mathbb{R}[X]/(X^n-1)$. It is a wellknown and easy to prove fact from representation theory that $\mathbb{R}[G]$ is semisimple for all finite groups G.

4.4. Schur's lemma. We shall need two special cases of Schur's lemma. Let V be a representation of R. Then the endomorphism algebra $\mathrm{End}_R(V)$ consists of all vectorspace

endomorphisms
ϕ: $V \to V$ such that $\phi(rv) = r\phi(v)$ for all $r \in R$. Schur's lemma for real and complex representations now implies

(i) Let R be an algebra over $\mathbb{C}$ and V an irreducible complex representation of R. Then $\mathrm{End}_R(V) = \mathbb{C}$. I.e. the only R-endomorphisms of V are the multiplications with the elements of $\mathbb{C}$.

(ii) Let R be an algebra over $\mathbb{R}$ and V an irreducible representation of R (real of course). Then $\mathrm{End}_R(V) = \mathbb{R}$, $\mathbb{C}$ or $\mathbb{H}$.

(iii) If V and W are nonisomorphic representations of R then $\mathrm{Hom}_R(V,W) = 0$, where $\mathrm{Hom}_R(V,W)$ of course stands for the vectorspace of vectorspace homomorphisms ϕ: $V \to W$ such that $\phi(rv) = r\phi(v)$ all $r \in R$, $v \in V$. (This holds both for the real and complex case).

In general it is not easy to decide even for group algebras $\mathbb{R}[G]$ whether $\mathbb{H}$ occurs as an endomorphism algebra of an irreducible representation or not. However, if G is abelian or a symmetric group S_n (the group of all permutations on n letters) this does not happen. (In the case of S_n because all irreducible real representations of S_n are absolutely irreducible, i.e. if ρ: $S_n \to M_n(\mathbb{R})$ is irreducible then $\rho_{\mathbb{C}}$: $S_n \to M_n(\mathbb{C})$ given by the same matrices is still irreducible). On the other hand $\mathbb{H}$ can definitely occur. E.G. let H be the group formed by the elements $\pm 1, \pm i, \pm j, \pm k$ of the quaternion algebra defined in (4.2) (ii) above. The matrices (4.3) define an irreducible representation of H and the endomorphism algebra of this representation (which is the algebra of all matrices commuting with all matrices of the form (4.3)) is isomorphic to $\mathbb{H}$.

4.5. <u>On the theory of systems with special structure over $\mathbb{C}$ with semisimple symmetry algebra</u>. Let $R \subset M_n(\mathbb{C}) \times M_m(\mathbb{C})$ be a semisimple subalgebra and let (F,G) be a system with special structure R. Then the state space $\mathbb{C}^n$ and input space $\mathbb{C}^m$ can be considered as R-modules via $R \to M_n(\mathbb{C}) \times M_m(\mathbb{C}) \to M_n(\mathbb{C}) = \mathrm{End}(\mathbb{C}^n)$ and $R \to M_n(\mathbb{C}) \times M_m(\mathbb{C}) \to M_m(\mathbb{C}) = \mathrm{End}(\mathbb{C}^m)$ and the fact that (F,G) has special structure R precisely means that F and G are homomorphisms of R-modules. Because R is semisimple $\mathbb{C}^n$ and $\mathbb{C}^m$ can be written as a direct sum of irreducible modules, say

$$\mathbb{C}^n = \overbrace{V_1 \oplus \cdots \oplus V_1}^{n_1} \oplus \overbrace{V_2 \oplus \cdots \oplus V_2}^{n_2} \oplus \cdots \oplus \overbrace{V_r \oplus \cdots \oplus V_r}^{n_r}$$

$$(4.6) \qquad = V_1^{n_1} \oplus V_2^{n_2} \oplus \cdots \oplus V_r^{n_r}, \quad n_i \geq 0$$

$$\mathbb{C}^m = V_1^{m_1} \oplus V_2^{m_2} \oplus \cdots \oplus V_r^{m_r}, \quad m_i \geq 0$$

where the V_i are nonisomorphic irreducible R-modules. Let V and V' be two of the irreducible modules occurring in $\mathbb{C}^n$. Let $F_{V,V'}$ be the composite $V \to \mathbb{C}^n \xrightarrow{F} \mathbb{C}^n \to V'$ when the last map is the canonical projection coming from the direct sum decomposition. Then by Schurs lemma (4.4) (i) and (iii) we know that $F_{V,V'} = 0$ if V and V' are not isomorphic and $F_{V,V'}$ is multiplication with a complex scalar if V and V' are isomorphic. Let $F_i(j,k)$ be the scalar corresponding to $F_{V,V'}$ if V is the j-th component of type i occurring in (4.6) and V' the k-th component of type i. Similarly define the $G_i(j,k)$. Now associate to (F,G) the set of r ordinary complex linear systems (F_1,G_1), ..., (F_r,G_r). This collection of ordinary systems describes (F,G) completely and this construction reduces the study of systems with special structure R to the study of ordinary linear systems, as we shall now see. To see this it helps to note the following. Let $d_i = \dim_{\mathbb{C}}(V_i)$ and let $F_i \otimes I_{d_i}$ be the Kronecker product matrix

$$\begin{pmatrix} F_i(1,1)I_{d_i} & F_i(1,2)I_{d_i} & \cdots & F_i(1,n_i)I_{d_i} \\ \cdot & \cdot & & \cdot \\ F_i(n_i,1)I_{d_i} & & \cdots & F_i(n_i,n_i)I_{d_i} \end{pmatrix}$$

here I_{d_i} is the $d_i \times d_i$ unit matrix. Then with respect to bases dapted to the direct sum decompositions (4.6), F and G look like

$$(4.7)\quad \left(\begin{pmatrix} F_1 \otimes I_{d_1} & 0 & & 0 \\ 0 & F_2 \otimes I_{d_2} & & \\ & & \ddots & 0 \\ 0 & & 0 & F_r \otimes I_{d_r} \end{pmatrix}, \begin{pmatrix} G_1 \otimes I_{d_1} & 0 & & 0 \\ 0 & G_2 \otimes I_{d_2} & & \\ & & \ddots & 0 \\ 0 & & 0 & G_r \otimes I_{d_r} \end{pmatrix}\right)$$

and from this it immediately follows e.g. that

4.8. Theorem. Let R be a semisimple subalgebra of $M_n(\mathbb{C}) \times M_m(\mathbb{C})$ and (F,G) a system with special structure R. Let (F_1,G_1), ..., (F_r,G_r) be the associated ordinary systems and $d_i = \dim(V_i)$ as above. Then

(i) (F,G) is completely reachable iff the (F_i,G_i) are completely reachable for all i = 1, ..., r.

(ii) c(F), the characteristic polynomial of F is equal to

$$c(F) = c(F_1)^{d_1} \cdots c(F_r)^{d_r}$$

(iii) (F,G) is stable iff the (F_i,G_i) are stable for all i = 1,2, ..., r.

Similarly the theory of invariants, moduli etc. of systems with special structure R, a semisimple algebra over $\mathbb{C}$, is completely determined, by the corresponding theory of the associated (much smaller) ordinary linear systems. A short description how this works follows. The natural state space base changes respect the special structure R, i.e. they are R-module automorphisms of $\mathbb{C}^n$. Using Schur's lemma again we see as before that $GL(\mathbb{C}^n,R)$ identifies naturally with $GL_{n_1}(\mathbb{C}) \times \ldots \times \ldots \times GL_{n_r}(\mathbb{C})$. Thus the situation as regards invariants, moduli, continuous canonical forms is as follows. Let $V_1, \ldots, V_r$, $\dim(V_i) = d_i$ be a complete list of the irreducible R-modules. (There are of course only finitely many of them which all occur as submodules of R considered as a module over itself).

Let now $\Sigma = (F,G)$ be a system over $\mathbb{C}$ with special structure R. The first invariants of $\Sigma = (F,G)$ are the multiplicities $n_i(\Sigma)$, $m_i(\Sigma)$ with which V_i occurs in $\mathbb{C}^n$ (as a sub-R-module) and in $\mathbb{C}^m$. Let $L(\underline{n},\underline{m},R)$, $\underline{n} = (n_1,\ldots,n_r)$, $\underline{m} = (m_1,\ldots,m_r)$ denote the space of all systems Σ with special structure R and $n_i(\Sigma) = n_i, m_i(\Sigma) = m_i$, $i = 1, \ldots, r$, and $L^{cr}(\underline{n},\underline{m},R)$ be the open dense subspace of systems from $L(\underline{n},\underline{m},R)$ which are completely reachable. Let $L^{cr}_{n,m}(\mathbb{C})$ denote the space of all cr complex systems of dimension n and with m inputs and $L^{cr}(n,m,R)$ the union of all $L^{cr}(\underline{n},\underline{m},R)$ with $\Sigma n_i = n$, $\Sigma m_i = m$.

4.9. Theorem. $L(\underline{n},\underline{m},R)$ is stable under $GL(\mathbb{C}^n,R)$ and the bijective correspondence $(F,G) \to ((F_1,G_1); \ldots ; (F_r,G_r))$ is compatible with the action of $GL(\mathbb{C}^n,R)$ on $L(\underline{n},\underline{m},R)$ and the diagonal action of $GL_{n_1}(\mathbb{C}) \times\ldots\times GL_{n_r}(\mathbb{C})$ on $L_{n_1,m_1}(\mathbb{C}) \times\ldots\times L_{n_r,m_r}(\mathbb{C})$. In particular $M^{cr}(\underline{n},\underline{m},R) = L^{cr}(\underline{n},\underline{m},R)/GL(\mathbb{C}^n,R) = M^{cr}_{n_1,m_1}(\mathbb{C}) \times \ldots \times M^{cr}_{n_r,m_r}(\mathbb{C})$ is a smooth manifold of dimension $\sum_{i=1}^{r} n_i m_i$, and $M^{cr}(n,m,R) = L^{cr}(n,m,R)/GL(\mathbb{C}^n,R)$ is the disjoint union of the $M^{cr}(\underline{n},\underline{m},R)$ with $\Sigma n_i = n$, $\Sigma m_i = m$. Moreover there is a universal family over $M^{cr}(\underline{n},\underline{m},R)$ making $M^{cr}(\underline{n},\underline{m},R)$ a fine moduli space for continuous families of systems with special structure R. Finally there is a continuous canonical form on $L^{cr}(\underline{n},\underline{m},R)$ iff $m_i \leq 1$ for all $i = 1, \ldots, r$.

Let's continue by discussing feedback for complex systems with special structure R, where R is semisimple. If the feedback matrix L is to guarantee to preserve the special structure R it has to be a homomorphism of R-modules $L:\mathbb{C}^n \to \mathbb{C}^m$. Reasoning as before we see that L with respect to bases respecting the decomposition into irreducible R-modules of $\mathbb{C}^n$ and $\mathbb{C}^m$ must be of the form

$$(4.10)\quad \begin{pmatrix} K_1 \otimes I_{d_1} & 0 & & 0 \\ 0 & K_2 \otimes I_{d_2} & & \\ & & \ddots & 0 \\ 0 & & 0 & K_r \otimes I_{d_r} \end{pmatrix}$$

just as F and G in (4.7). Let F(R) denote the R-special structure preserving feedback group, i.e. F(R) is generated by $GL(\mathbb{C}^n,R)$ (base change in input space), $GL(\mathbb{C}^n,R)$ (base change in state space) and feedback laws $L: \mathbb{C}^n \to \mathbb{C}^m$ as above.

Suppose that there are also disturbances affecting (F,G). Then it is natural to assume that these enter through a matrix G' which is also compatible with the special structure R; i.e. through a matrix G' which is an R-module homomorphism $\mathbb{C}^{m'} \to \mathbb{C}^n$, (More precisely this means that we assume that there is a symmetry algebra $R' \subset M_n(\mathbb{C}) \times M_m(\mathbb{C}) \times M_{m'}(\mathbb{C})$ such that G' is an R'-module homomorphism $\mathbb{C}^{m'} \to \mathbb{C}^n$ and such that the projection $M_n(\mathbb{C}) \times M_m(\mathbb{C}) \times M_{m'}(\mathbb{C}) \to M_n(\mathbb{C}) \times M_m(\mathbb{C})$ maps R' isomorphically onto R. (Recall also that often $R \subset M_n(\mathbb{C}) \times M_m(\mathbb{C})$ is completely specified by its image in $M_n(\mathbb{C})$). Repeating the by now tedious arguments involving Schur's lemma we observe that $\mathbb{C}^{m'}$ decomposes as a direct sum $V_1^{m'_1} \oplus \dots \oplus V_r^{m'_r}$ and that the presence of such disturbances furnishes us with a finite collection of ordinary systems with disturbances

$$(F_1,G_1,G'_1), \dots, (F_r,G_r,G'_r).$$

Concerning feedback, stabilization, pole-placement coefficient assignability, disturbance decoupling we now have the theorem

4.11 <u>Theorem</u>. Let $\Sigma = (F,G)$ be a system with special structure R and $\Sigma' = (F,G,G')$ a system with disturbances of special structure R. Let (F_1,G_1), ..., (F_r,G_r) and (F_1,G_1,G_1'), ..., (F_r,G_r,G_r') be the associated ordinary systems over $\mathbb{C}$

(i) The invariants of the feedback group F(R) acting on $L^{cr}(\underline{n},\underline{m},R)$ are the Kronecker indices of the associated ordinary linear systems

(ii) (F,G) is stabilizable by special structure preserving feedback iff all the (F_i,G_i) are stabilizable. In particular if (F,G) is completely reachable then it is stabilizable by special structure preserving feedback.

(iii) If (F,G) is completely reachable then the coefficients of its characteristic polynomial can be assigned arbitrary by special structure preserving feedback subject to the sole condition that the characteristic polynomial must be of the form

$$p_1(\lambda)^{d_1} \cdots p_r(\lambda)^{d_r}, \text{ degree}(p_i) = n_i$$

and in particular the poles of (F,G) can be placed arbitarily subject to the condition that n_1 (not necessarily distinct) poles occur with multiplicity d_1, n_2 poles with multiplicity d_2, ..., n_r poles with multiplicity d_r.

(iv) The disturbance decoupling problem for Σ' can be solved by special structure preserving feedback if and only if the disturbance decoupling problem can be solved for each of the associated ordinary systems (F_i,G_i,G_i'), $i = 1,2, \ldots, r$.

This concludes our outline of how the theory of complex systems with special structure R, R semisimple, reduces to the theory of ordinary linear systems over $\mathbb{C}$. In case of real systems similar things apply but there are additional complications.

4.12. On the theory of real systems with special structure with a semi-simple symmetry algebra. Now let R be a semi-simple subalgebra of $M_n(\mathbb{R}) \times M_m(\mathbb{R})$ and let (F,G) be a system of special structure R. Then as in the complex case the state space $\mathbb{R}$ and input space $\mathbb{R}^m$ acquire an R-module structure and F and G are R-module morphisms. The irreducible R-modules are of three distinct types:

(i) Irreducible modules V such that $End_R(V) = \mathbb{R}$
(ii) Irreducible modules W such that $End_R(W) = \mathbb{C}$
(iii) Irreducible modules U such that $End_R(U) = \mathbb{H}$

Let $V_1, \ldots, V_r; W_1, \ldots, W_s; U_1, \ldots, U_t$ be a complete list of (isomorphism classes of) irreducible R-modules of the three types. Then the state space R-module $\mathbb{R}^n$ and input-space R-module decompose as direct sums

$$\mathbb{R}^n = V_1^{n_1} \oplus \ldots \oplus V_r^{n_r} \oplus W_1^{n_{r+1}} \oplus \ldots \oplus W_s^{n_{r+s}} \oplus U_1^{n_{r+s+1}} \oplus \ldots \oplus U_t^{n_{r+s+t}}$$

(4.13)

$$\mathbb{R}^m = V_1^{m_1} \oplus \ldots \oplus V_r^{m_r} \oplus W_1^{m_{r+1}} \oplus \ldots \oplus W_s^{m_{r+s}} \oplus U_1^{m_{r+s+1}} \oplus \ldots \oplus U_t^{m_{r+s+t}}$$

These multiplicities $n_1, \ldots, n_{r+s+t}; m_1, \ldots, m_{r+s+t}$ are a first invariant of (F,G) (even under the special structure preserving feedbackgroup). Now

$$Hom_R(V_i,V_i) = \mathbb{R},\ Hom_R(W_i,W_i) = \mathbb{C},\ Hom_R(U_i,U_i) = \mathbb{H}$$

and all other vectorspaces of homomorphisms are zero. Reasoning as in section 4.5 above one now finds from (F,G) a finite collection of linear systems

(F_i,G_i), $i = 1, \ldots, r$, real systems of dimensions n_i and m_i inputs
(F_j,G_j), $j = r+1, \ldots, r+s$, complex systems of complex dimension n_j and m_j inputs
(F_k,G_k), $k = r+s+1, \ldots, r+s+t$, quaternion systems of (quaternion) dimension n_k and with m_k inputs.

In this way the theory of real systems with special structure R, R semisimple, reduces to the theory of ordinary real linear systems, ordinary complex linear systems and "ordinary" quaternion linear systems. There is of course a mild snag here in that the area of inquiry of linear systems over the quaternions is still virgin territory and virtually no theory exists. This remains to be developed. In this paper we shall from now on limit ourselves to systems (F,G) with special structure such that no "quaternion irreducibles" occur, i.e. systems with special structure such that $n_{r+s+1} = \dots = n_{r+s+t} = m_{r+s+1} = \dots = m_{r+s+t} = 0$. For certain symmetry algebras R, e.g. $R = \mathbb{C}$, $R = \mathbb{R}[G]$, G abelian, $R = R[S_n]$, S_n the symmetric group on n-letters, all systems with special structure R have this property (because those algebras have no quaternion irreducibles).

Let $\dim V_i = d_i$, $\dim W_i = 2e_i$, $\dim U_i = 4f_i$ (as real vectorspaces). We have $\mathbb{C} = \mathrm{End}_R(W_i,W_i) \subset \mathrm{End}(W_i,W_i) = M_{2e_i}(R)$ and this defines a complex vectorspace structure on W_i so that $\dim_{\mathbb{R}}(W_i)$ is necessarily even and $e_i \in \mathbb{N} \cup \{0\}$. Similarly $\mathbb{H} = \mathrm{End}_R(U_i)$ makes the U_i vectorspaces over the quaternions so that the real dimensions of the U_i are multiples of four.

4.14. Theorem. Let R be a semi-simple subalgebra of $M_n(\mathbb{R}) \times M_m(\mathbb{R})$ and (F,G) a real system with special structure R. Assume that

$n_{r+s+k} = m_{r+s+k} = 0$, $k = 1, \dots, t$ and let

$(F_1,G_1), \dots, (F_r,G_r); (F_{r+1},G_{r+1}), \dots, (F_{r+s},G_{r+s})$ be the real and complex systems associated to (F,G). Then

(i) (F,G) is completely reachable if and only if the real systems (F_i,G_i), $i = 1, \dots, r$ and the complex systems (F_{r+j},G_{r+j}), $j = 1, \dots, s$ are all completely reachable.

(ii) the characteristic polynomial of F is equal to

$$c(F) = c(F_1)^{d_1} \dots c(f_r)^{d_r} c(F_{r+1})^{e_1} \overline{c(F_{r+1})}^{e_1} \dots$$
$$\dots c(F_{r+s})^{e_s} \overline{c(F_{r+s})}^{e_s}$$

where the upper bar denotes complex conjugates.

(iii) (F,G) is stable if and only if the (F_i,G_i), $i = 1, \dots, r$ and (F_{r+j},G_{r+j}), $j = 1, \dots, s$ are all stable.

4.15 Remarks on the case that there are quaternion systems present. Assume that some of the U_i do occur with non-zero multiplicity in $\mathbb{R}^n$. Assume that $\mathbb{H} = \mathrm{Hom}_R(U_i,U_i) \subset \mathrm{End}_{\mathbb{R}}(U_i)$ then gives U_i the structure of an $\mathbb{H}$-vectorspace and it follows that $\dim_{\mathbb{R}}(U_i) = 4f_i$ for some $f_i \in \mathbb{N}$. With respect to a suitable basis (over $\mathbb{R}$) the endomorphism F restricted to $U_i^{n_{r+s+i}}$ is given by a matrix of the form

$$(4.16) \qquad \begin{pmatrix} \alpha'_{11} & \alpha'_{12} & \cdots & \alpha'_{1k} \\ & & \ddots & \\ \alpha'_{k1} & \alpha'_{k2} & \cdots & \alpha'_{kk} \end{pmatrix}, \quad k = n_{r+s+i}$$

where the α'_{rs} are of the form $\alpha'_{rs} = \alpha_{rs} \otimes I_{f_i}$, $4f_i = \dim(U_i)$, where the α_{rs} are quaternions, i.e. 4 x 4 real matrices of the form

$$(4.17) \qquad \begin{pmatrix} a & b & c & d \\ -b & a & d & -c \\ -c & -d & a & b \\ -d & c & -b & a \end{pmatrix}$$

Now every quaternion matrix (4.16) is similar to a quaternion matrix consisting of diagonal blocks of the form

$$\begin{pmatrix} \alpha & & & 0 \\ 1 & \alpha & & \\ & \ddots & \ddots & \\ 0 & & 1 & \alpha \end{pmatrix}$$

and zero's elsewhere (cf. [Jacobson, 1943], Ch.3, section 12, page 51). The determinant of (4.17) is equal to $(a^2+b^2+c^2+d^2)^2$ and it follows that the characteristic polynomial of (4.16) is a product of factors of the form

$$\{(\lambda+a+\tfrac{1+i}{\sqrt{2}}\,t)(\lambda+a-\tfrac{1+i}{\sqrt{2}}\,t)(\lambda+a+\tfrac{1-i}{\sqrt{2}}\,t)(\lambda+a-\tfrac{1-i}{\sqrt{2}}\,t)\}^{f_i}, \quad t \geq 0, \; a \in \mathbb{R}$$

so that if there are irreducibles of quaternion type present in the state space $\mathbb{R}^n$ the corresponding poles occur in groups of four of the form $a \pm t\sqrt{\pm i}$ (each with multiplicity f_i). It is now easy to write down the analogue of statements (ii) and (iii) in theorem 4.14 in the general case (where the n_{r+s+k}, m_{r+s+k}, $k = 1, \ldots, t$ are not necessarily zero).

Concerning invariants, moduli etc. one has the following analogue of theorem 4.9.

4.18. Theorem. Let $\underline{n} = (n_1, \ldots, n_r, n_{r+1}, \ldots, n_{r+s}, 0, \ldots, 0)$, $\underline{m} = (m_1, \ldots, m_r, m_{r+2}, \ldots, m_{r+s}, 0, \ldots, 0)$ and $L(\underline{n},\underline{m},R)$ be the space of all real systems (F,G) with special structure R (R semi-simple). Then $L(\underline{n},\underline{m},R)$ is stable under the group $GL(\mathbb{R}^n,R)$ of special structure preserving state space base changes and the action of $GL(\mathbb{R}^n,R)$ on $L(\underline{n},\underline{m},R)$ is compatible with the diagonal action of $GL_{n_1}(\mathbb{R}) \times \ldots \times GL_{n_r}(\mathbb{R}) \times GL_{n_{r+1}}(\mathbb{C}) \times \ldots \times GL_{n_{r+s}}(\mathbb{C})$

on $L_{n_1,m_1}(\mathbb{R}) \times \ldots \times L_{n_1,m_1}(\mathbb{R}) \times L_{n_{r+1},m_{r+1}}(\mathbb{C}) \times \ldots \times L_{n_{r+s},m_{r+s}}(\mathbb{C})$

under the bijective correspondence $(F,G) \to ((F_1,G_1), \ldots, (F_r,G_r); (F_{r+1},G_{r+1}), \ldots, (F_{r+s},G_{r+s}))$ In particular the quotient space

$L^{cr}(\underline{n},\underline{m},R)/GL(\mathbb{R}^n,R) = M^{cr}(\underline{n},\underline{m},R)$ is equal to

$$M^{cr}_{n_1,m_1}(\mathbb{R}) \times \ldots \times M^{cr}_{n_1,m_1}(\mathbb{R}) \times M^{cr}_{n_{r+1},m_{r+1}}(\mathbb{C}) \times \ldots \times M^{cr}_{n_{r+s},m_{r+s}}(\mathbb{C})$$

and it is a smooth manifold of (real) dimension

$\sum_{i=1}^{r} n_i m_i + 2 \sum_{j=1}^{s} n_{r+j} m_{r+j}$. Moreover there is a universal family over $M^{cr}(\underline{n},\underline{m},R)$ making $M^{cr}(\underline{n},\underline{m},R)$ a fine moduli space for continuous families of systems with special structure R. Finally there is a continuous canonical form on $M^{cr}(\underline{n},\underline{m},R)$ iff $m_i \leq 1$ for all $i = 1, \ldots, r+s$.

As in the case of complex systems if a feedback matrix $L: \mathbb{R}^n \to \mathbb{R}^m$ is a homomorphism of R-modules it preserves the special structure R. Such a feedback homomorphism breaks up into a direct sum of feedback matrices corresponding to feedback for the associated real, complex (and quaternion) ordinary systems, again exactly as in the complex case. Thus concerning stabilization, disturbance decoupling etc. by means of structure preserving feedback we have

4.19 Theorem. Let R and (F,G), (F_1,G_1), ..., (F_{r+s},G_{r+s}), $\underline{n},\underline{m}$ be as in theorem 4.14. Then
(i) The invariants of the structure preserving feedback group acting on $L^{cr}(\underline{n},\underline{m},R)$ are the Kronecker indices of the associated real and complex systems (F_1,G_1), ..., (F_r,G_r); (F_{r+1},G_{r+1}), ..., (F_{r+s},G_{r+s}).
(ii) (F,G) is stabilizable by special structure preserving feedback iff all the real systems (F_i,G_i), $i = 1, \ldots, r$ are stabilizable by real feedback and all the complex systems (F_{r+i},G_{r+i}), $i = 1, \ldots, s$ are stabilizable by complex feedback.
(iii) If (F,G) is completely reachable then the coefficients of its characteristic polynomial can be assigned arbitrarily by special structure preserving feedback subject to the sole condition that the characteristic polynomial must be of the form

$$p_1(\lambda)^{d_1} \ldots p_r(\lambda)^{d_r}(q_1(\lambda)\overline{q}_1(\lambda))^{e_1} \ldots (q_s(\lambda)\overline{q}_s(\lambda))^{e_s},$$

$$\text{degree } (p_i) = n_i, \ \text{degree } (q_i) = n_{r+i}$$

(iv) If (F,G,G') is a system with disturbances with special structure R then the disturbance decoupling problem can be solved by special structure preserving feedback iff the disturbance decoupling problem for the real systems (F_i,G_i,G'_i), $i = 1, \ldots, r$ can be solved (by real feedback) for all $i = 1, \ldots, r$ and the disturbance decoupling problem for the complex systems (F_i,G_i,G'_i), $i = r+1, \ldots, r+s$ can also be solved (by complex feedback) for all $i = 1, \ldots, s$.

4.20. Remark. If $R = \mathbb{R}[\mathbb{Z}/(n)]$ the irreducible real representations of R (or equivalently $\mathbb{Z}/(n)$) are of course very well known. They are of dimension 1 or 2 and are given by mapping the generator g of $\mathbb{Z}/(n)$ to an n-th root of unity (interpreted as a rotation through an angle $2\pi n^{-1}$). The corresponding decomposition of a circulant matrix readily follows and using the results above all the results of [Brockett-Willems, 1974] concerning block circulant systems readily follow.

5. RESTRICTED STATE FEEDBACKPROBLEMS.

In many examples the matrix G has even more special structure then forced by the symmetry algebra R and it may be desirable to do e.g. stabilization by a feedback law which has a similar amount of extra special structure. In this section we discuss how to analyse such requirements in terms of symmetry algebras.

5.1. <u>Extra special structure on the G-matrix</u>. For almost all of the examples of section 3 above (the example (3.12) is the sole exception) the symmetry algebra R forces the F-matrix to have the form indicated, but often this symmetry alone admits more general G-matrices than the ones indicated. This does not affect the analysis and applications we have discussed so far. These techniques simply require (F,G) to have enough symmetry and it does not matter of course if G is even more special.

It is useful though, as we shall see, that extra special structure of G can also be described by symmetry ideas. To do this one considers two symmetry algebras

$$R \subset R' \subset M_n(\mathbb{R}) \times M_m(\mathbb{R}) \tag{5.2}$$

and the requirement that (F,G) has special structure $R \subset R'$ is that F: $\mathbb{R}^n \to \mathbb{R}^n$ be a homomorphism of R-modules and G: $\mathbb{R}^m \to \mathbb{R}^n$ be a homomorphism of R'-modules.

5.3. <u>Example</u>. We consider again the twin helicopter lift example of 3.5 above

$$F = \begin{pmatrix} A & H \\ -H & A \end{pmatrix}, \quad G = \begin{pmatrix} B & 0 \\ 0 & B \end{pmatrix}$$

Let R be as in example (3.5) and R' the algebra generated by R and the element

$$\left(\begin{pmatrix} 0 & I_n \\ I_n & 0 \end{pmatrix} , \begin{pmatrix} 0 & I_m \\ I_m & 0 \end{pmatrix} \right) \in M_{2n}(\mathbb{R}) \times M_{2m}(\mathbb{R}) .$$

Then the requirement that G be an R-module homomorphism results in a G-matrix of the form

$$\begin{pmatrix} B & B' \\ -B' & B \end{pmatrix}$$

and the requirement that it also be an R'-module homomorphism says that additionally we must have

$$\begin{pmatrix} B & B' \\ -B' & B \end{pmatrix} \begin{pmatrix} 0 & I_m \\ I_m & 0 \end{pmatrix} = \begin{pmatrix} 0 & I_n \\ I_n & 0 \end{pmatrix} \begin{pmatrix} B & B' \\ -B' & B \end{pmatrix}$$

which implies $B' = 0$.

In this example R is isomorphic to $\mathbb{C}$, and the theory of section 4 above associates to the system one complex system viz. the system

$$(5.4) \qquad (A+iH,B)$$

The extra symmetry in this formulation now manifests itself in the form of the property that the input matrix is real.

In this example there is special interest in the question: Can (F,G) be stabilized by means of a feedback matrix of the form

$$(5.5) \qquad L = \begin{pmatrix} K & 0 \\ 0 & K \end{pmatrix}$$

The reasons are as follows. The workload for the pilot is very high and there is virtually no time for him to pay attention to what the other pilot is doing. This makes it desirable that the off-diagonal blocks of the feedback matrix (5.5) are zero. Also pilots have similar training and if not absolutely necessary one would not like to have to teach different sets of responses depending on whether they are piloting the left or the right helicopter in a twin lift situation.

In terms of the symmetry algebras $(R \subset R') \simeq (\mathbb{C} \subset \mathbb{C}[\mathbb{Z}/(2)])$ the requirement that L be of the form (5.5) means that $L: \mathbb{R}^n \to \mathbb{R}^m$ must be a homomorphism of R'-modules (and not just of R-modules). In terms of the complex system, with real input matrix (5.4) the requirement that the feedback matrix have this extra symmetry property means that it must be real. Thus in this case we finish up with the problem:

> Given a system (F,G) with G real and F complex, when does there exist a real feedback matrix L such that F + GL is stable.

As we shall see there are examples which show that complete reachability of (F,G) (over $\mathbb{C}$ of course) is not sifficient; cf. section 6.1. In general this type of problem seems to be surprisingly hard.

5.6. Restricted feedback problems. The example above is just one of a class of problems which we shall call restricted feedback problems. Other examples can be readily imagined, e.g. decentralized systems with decentralized of partly decentralized feedback.

In terms of symmetry algebras, these problems can be described as follows. Let (F,G) be a system with special structure $(R \subset R')$ and (F,G,G') a system with disturbances with special structure $(R \subset R')$. (In this case $R \subset R' \subset M_n(\mathbb{R}) \times M_m(\mathbb{R}) \times M_{m'}(\mathbb{R})$ and both G and G' are required to be R'-module homomorphisms). The question is when does there exists a feedback matrix $L: \mathbb{R}^n \to \mathbb{R}^m$ which is an R'-module homomorphism and which stabilizes (F,G) (respectively disturbance decouples (F,G,G').

5.7. Restricted feedback and output feedback. Let us consider an output feedback stabilization problem. I.e. given

(F,G,H) it is required to find a matrix L: $\mathbb{R}^p \to \mathbb{R}^m$ such that F + GLH is stable. This is the same as finding a matrix L': $\mathbb{R}^n \to \mathbb{R}^m$ such that L'(KerH) = 0 and F + GL' is stable. Choose a different basis for $\mathbb{R}^n$ and write $\mathbb{R}^n = \mathbb{R}^s \oplus \mathbb{R}^t$, $\mathbb{R}^t$ = KerH. Consider the $\mathbb{R}$-algebra
$R'' \subset M_n(\mathbb{R}) \times M_m(\mathbb{R})$ generated by all pairs of the form

$$\left(\begin{pmatrix} I_s & 0 \\ S_{21} & S_{22} \end{pmatrix} , I_m \right)$$

Then if L' : $\mathbb{R}^n \to \mathbb{R}^m$ is an R''-module homomorphism we must have

$$(5.8) \qquad \begin{pmatrix} L_1' & L_2' \end{pmatrix} \begin{pmatrix} I_s & 0 \\ S_{21} & S_{22} \end{pmatrix} = I_m \begin{pmatrix} L_1' & L_2' \end{pmatrix}$$

for all S_{21}, S_{22}. This gives $L_1' + L_2' S_{21} = L_1'$, $L_2' S_{22} = L_2'$ so that $L_2' = 0$, i.e. L' is zero on KerH. And inversely if $L_2' = 0$, then (5.8) holds. Thus the output feedback problem can also be formulated as a restricted feedback problem in terms of symmetry algebras.

6. ON THE STABILIZATION OF 2-HELICOPTER SYSTEMS.

In this section we take a closer look at the restrated state feedback stabilization problem of example 5.3. (Twin-helicopter lift).

6.1. Example.

$$A = \begin{pmatrix} 0 & 0 \\ 0 & 0 \end{pmatrix}, \quad H = \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}, \quad F = \begin{pmatrix} 0 & i \\ -i & 0 \end{pmatrix}, \quad G = B = \begin{pmatrix} 0 \\ 1 \end{pmatrix}$$

Applying real feedback gives a matrix of the form

$$\begin{pmatrix} 0 & i \\ x-i & y \end{pmatrix}, \quad x,y \in \mathbb{R}$$

The two eigenvalues of this matrix are equal to $\lambda_1 = a + bi$, $\lambda_2 = c - bi$ (because their sum is real) for certain a,b,c which then must satisfy $\lambda_1 \lambda_2 = ac + b^2 + i(bc-ab) = -1 - ix$. So $ac + b^2 = -1$ which forces one of a,c to be negative and one to be positive.

Thus the assumption that (F,G) be completely reachable definitely does not suffice . It is reasonable (certainly in the twin-helicopter case) to assume that the single helicopters themselves are completely reachable as well, i.e.in addition to (A+iH,B) completely reachable we shall now also assume that (A,B) is completely reachable. This suffices for pairs of 2-dimensional helicopters as we shall see in 6.4 below

6.2. Reduction to the one input case. First we show how to reduce the problem to the single input case.

6.3. Proposition. Let (F,G) be a complex completely reachable system. Then there exist real feedback matrix K and a real vector y such that (F+GK,Gy) is completely reachable.

Proof. By Heyman's lemma there exist a complex K and a complex y such that $(F_1, g_1) = (F+GK, Gy)$ is completely reachable. Now write down the reachability matrix of (F_1, g_1)

$$(g_1 \quad F_1 g_1 \quad \cdots \quad F_1^{n-1} g_1)$$

The determinant D of this matrix is a nonzero complex polynomial in the entries of K and y. Let these complex variables be $z_1, \ldots, z_r$. Write D as a polynomial in z_r with coefficients from $\mathbb{C}[z_1, \ldots, z_{r-1}]$, $D = C_o + C_1 z_r + \ldots + C_s z_r^s$. A nonzero complex polynomial f(z) in one variable z has only finitely many roots so there are real values x of z so that $f(x) \neq 0$. With induction we can assume that for certain real $x_1, \ldots, x_{r-1}$ at least one of the $C_i(x_1, \ldots, x_{r-1})$ is nonzero, and then there is a real x_r such that $D(x_1, \ldots, x_1) \neq 0$, proving the proposition.

6.4. <u>Stabilization of pairs of 2-dimensional systems</u>. Let A and H be 2 x 2 real matrices and b an element of $\mathbb{R}^2$. Assume that (A,b) is controllable. It follows that (A+iH,b) is controllable by calculating the controllability matrix [b,(A+iH)b]. The proof though doesn't generalize to dimension greater than 2. With real change of basis and real feedback we may bring (A,b) into feedback canonical form, A is the matrix

$$\begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix} \quad \text{and} \quad b = \begin{pmatrix} 0 \\ 1 \end{pmatrix} . \text{ Then } A + iH = \begin{pmatrix} ih_{11} & 1+ih_{12} \\ ih_{21} & ih_{22} \end{pmatrix}$$

and the characteristic polynomial of A + iH + bk, k real, is

$$\lambda^2 - (i(h_{11}+h_{22})+k_2)\lambda + (h_{21}h_{12}-h_{11}h_{22}-k_1)+i(k_2h_{11}-k_1h_{12}-h_{21}),$$

By the Routh-Hurwitz theory the roots of this polynomial are in the left half plane iff the two determinants

$$\begin{vmatrix} 1 & -\mathrm{TrH} \\ 0 & -k_2 \end{vmatrix}$$

$$\begin{vmatrix} 1 & -\mathrm{TrH} & \mathrm{DetH}+k_1 & 0 \\ 0 & -k_2 & k_2h_{11}-k_1h_{12}-h_{21} & 0 \\ 0 & 1 & -\mathrm{TrH} & \mathrm{DetH}+k_1 \\ 0 & 0 & -k_2 & k_2h_{11}-k_{11}h_{12}-h_{21} \end{vmatrix}$$

are both positive.

The first determinant gives

$$k_2 < 0$$

as a necessary condition, and the second gives, upon letting $k_1 = k_2$

$$-k_2^3 + k_2^2(\mathrm{TrHh}_{11}-\mathrm{TrHh}_{12}-\mathrm{DetH}-(h_{11}-h_{12})^2) + \ \ldots > 0.$$

But letting k_2 be sufficiently negative we have this inequality satisfied. Thus we have shown that there is always a stabilizing feedback under the conditions imposed.

6.5. The canonical form case. There is one more class of 2-helicopter systems which are easily seen to be stabilizable by decentralized feedback. These are the (one input) complex systems which are in canonical form or which can be brought into canonical form by real base change.

We assume $(A+iH,b)$ is in control canonical form, so that

$$A + iH = \begin{pmatrix} 0 & 1 & & 0 \\ & & \ddots & \\ & & & 1 \\ \alpha_1 & & \cdots & \alpha_n \end{pmatrix}, \quad b = e_n \text{ with the } \alpha_i\text{'s}$$

complex numbers. To prove that $A + iH$ is stabilizable by real feedback we must show that there are real numbers $k_1, \ldots, k_n$ such that the polynomial

$p(\lambda) = \lambda^n - (\alpha_n+k_n)\lambda^{n-1} - (\alpha_{n-1}+k_{n-1})\lambda^{x-z} - \ldots -(\alpha_1+k_1)$ has all of its roots in the left half plane. We can without loss of generality assume that the α_i's are pure complex.

Let $k_r = -\binom{n}{n-r+1}t^{n-r+1}$. Then

$$p(\lambda,t) = \lambda^n-((\alpha_n - \binom{n}{1}t)\lambda^{n-1} + \ldots$$

By factoring out a factor t^n we can rewrite $p(\lambda,t)$ as

$$t^n((\tfrac{\lambda}{t}) - (\tfrac{\alpha_n}{t} - \binom{n}{1})(\tfrac{\lambda}{t})^{n-1} - \ldots - (\tfrac{\alpha_o}{t^n} - \binom{n}{o}))$$

Thus for t very large the roots of $p(\lambda,t)$ are approximately equal to the roots of

$$w^n + \binom{n}{1}w^{n-1} + \ldots + \binom{n}{n}w^o$$

$$= (w+1)^n$$

where $w = \lambda/t$. Thus for t very large and positive the roots of $p(\lambda,t)$ are equal to $-t + o(|t|)$ (Landau o symbol) and hence for sufficiently large t all of the roots are in the left half plane.

6.5. Higher Dimensional Systems. A combination of the techniques used in 6.4 and 6.3 can be used to prove the stabilizability of pairs of three dimensional systems. We first note that in the arguments of section 6.4 any polynomial with roots in the left plane could have been used instead of the polynomial $(\lambda+t)^n$.

Let $(A+iH,b)$ be such that $A = \begin{pmatrix} 0 & 1 & & 0 \\ & & \ddots & \\ & & & 1 \\ 0 & & & 0 \end{pmatrix}$ and $b = e_n$. Let T be a complex matrix such that $T^{-1}(A+iH)T = \begin{pmatrix} 0 & 1 & & 0 \\ & & \ddots & 1 \\ -\alpha_0 & & \dots & -\alpha_{n-1} \end{pmatrix}$

and $Te_n = e_n$. Then $(A+iH,b)$ is stabilized by a real gain iff $(T^{-1}(A+iH)T,e_n)$ is stabilized by a gain of the form kT. Applying the asymptotic technique of 6.4 we see that to stabilize $(A+iH,b)$ we must find a k such that $A - bkT$ is stable.

Let T_i denote the i'th column of T. The characteristic polynomial of $A - bkT$ is

$$p(\lambda) = \lambda^n + kT_n\lambda^{n-1} + \dots + kT_1.$$

Write $T = R + iS$. Note that since $Te_n = e_n, T_n = e_n$. Also since $(A+iH)e_n = Te_{n-1} - \alpha_{n-1}e_n$ and $\alpha_{n-1} = i \operatorname{tr} H$ we have that $R_{n-1} = e_{n-1}$. In general not much more can be said about the form of T. We can, however, for n = 3 apply Routh Hurwitz to the complex polynomial $p(\lambda)$ to obtain that a necessary and sufficient condition for the existence of a stabilizing k in that the following three determinants be positive.

$$\begin{vmatrix} 1 & 0 \\ 0 & k_3 \end{vmatrix}$$

$$\begin{vmatrix} 1 & 0 & -k_2 & -kS_1 \\ 0 & k_3 & kS_2 & kR_1 \\ 0 & 1 & 0 & -k_2 \\ 0 & 0 & k_3 & kS_2 \end{vmatrix}$$

$$\begin{vmatrix} 1 & 0 & -k_2 & -kS_1 & 0 & 0 \\ 0 & k_3 & kS_2 & kR_1 & 0 & 0 \\ 0 & 1 & 0 & -k_2 & -kS_1 & 0 \\ 0 & 0 & k_3 & kS_2 & kR_1 & 0 \\ 0 & 0 & 1 & 0 & -k_2 & -kS_1 \\ 0 & 0 & 0 & k_3 & kS_2 & kR_1 \end{vmatrix}$$

The first determinant yields simply that

$$k_3 > 0 .$$

The second gives

$$k_3 kR_1 + k_3^2 k_2 - (kS_2)^2 > 0$$

while the third gives

$$-k_3^2 k_2^2 kR_1 - k_3^3 (kS_1)^2 + k_3^2 kS_1 kS_2 k_2$$
$$-k_3 k_2 (kR_1)^2 + kR_1 kS_1 kS_2 k_3 + (kS_2)^2 kR_1 k_2 +$$
$$kS_2 kR_1 kS_1 k_3 - (kS_2)^3 kS_1 - (kR_1)^2 k_2 k_3 + kR_1 kS_2 kS_1 k_3$$
$$- (kR_1)^3 > 0.$$

Even in this simple low dimensional case it has proven infeasible to to find good necessary and sufficient condition for the stabilizability of the system (A+iH,b).

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