

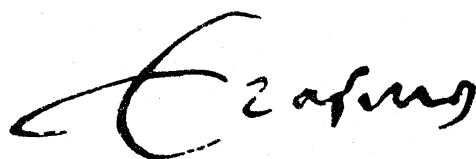
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SYSTEM IDENTIFICATION AND
NONLINEAR FILTERING: LIE ALGEBRAS
AND
A SUMMARY OF APPROXIMATION METHODS FOR
NONLINEAR FILTERING PROBLEMS ARISING IN
SYSTEM IDENTIFICATION

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SYSTEM IDENTIFICATION AND NONLINEAR FILTERING : LIE ALGEBRAS

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Abstract

This paper is continuation of our previous work ([1], [2], [3]) to understand the identification problem of linear system theory from the viewpoint of nonlinear filtering. The estimation algebra of the identification problem is a sub-algebra of a current algebra. It therefore follows that the estimation algebra is embeddable as a Lie algebra of vector fields on a finite dimensional manifold. These features permit us to develop a Wei-Norman type procedure for the associated Cauchy problem and reveal a set of functionals of the observations that play the role of joint sufficient statistics for the identification problem.

1. Introduction

Consider the stochastic differential system:

$$\begin{aligned} d\theta &= 0 \\ dx_t &= A(\theta)x_t dt + b(\theta)dw_t \\ dy_t &= \langle c(\theta), x_t \rangle dt + dv_t \end{aligned} \quad (1)$$

Here $\{w_t\}$ and $\{v_t\}$ are independent, scalar, standard, Wiener processes, and $\{x_t\}$ is an \mathbb{R}^n -valued

process. Assume that θ takes values in a smooth manifold $\Theta \rightarrow \mathbb{R}^N$, and the map $\theta \rightarrow \Sigma(\theta) := (A(\theta), b(\theta), c(\theta))$ in a smooth map taking values in minimal triples. By the identification problem we shall mean the nonlinear filtering problem associated with eqn. (1); i.e. the problem of recursively computing conditional expectations of the form $\pi_t(\phi) \triangleq E[\phi(x_t, \theta) | Y_t]$ where Y_t is the σ -algebra generated by the observations $\{y_s : 0 \leq s \leq t\}$ and ϕ belongs to a suitable class of functions on $\mathbb{R}^n \times \Theta$.

The joint unnormalized conditional density $\rho \Delta p(t, x, \theta)$ of x_t and θ given Y_t satisfies the stochastic partial differential equation (Stratonovich sense)

$$d\rho = A_0 \rho dt + B_0 \rho dy_t \quad (2)$$

where the operators A_0 and B_0 are given by

$$A_0 := \frac{1}{2} \langle b(\theta), \frac{\partial}{\partial x} \rangle^2 - \langle \frac{\partial}{\partial x}, A(\theta)x \rangle - \langle c(\theta), x \rangle^2 / 2 \quad (3)$$

$$B_0 := \langle c(\theta), x \rangle \quad (4)$$

(see [4] for background).

From the Bayes formula ([5]), it follows that

$$\pi_t(\phi) = \sigma_t(\phi) / \sigma_t(1) \quad (5)$$

where

$$\sigma_t(\phi) = \int_{\Theta} \int_{\mathbb{R}^n} \phi(x, \theta) \rho(t, x, \theta) |dx| \cdot |d\theta| \quad (6)$$

where $|dx|$ and $|d\theta|$ are fixed volume elements on \mathbb{R}^n and Θ respectively. Further if $Q(t, \theta)$ denotes the unnormalized posterior density of θ given Y_t , then it satisfies the Ito equation:

$$dQ = E[\langle c(\theta), x_t | \theta, Y_t \rangle \cdot Q(t, \theta) dy_t] \quad (7)$$

Recent work in nonlinear filtering theory (see the proceedings [6]) shows that it is natural to look at eqn. (2) formally as a deterministic partial differential equation,

$$\frac{\partial \rho}{\partial t} = A_0 \rho + \dot{y} B_0 \rho \quad (8)$$

By the Lie algebra of the identification problem, we shall mean the operator Lie algebra \mathcal{G} generated by A_0 and B_0 . For more general nonlinear filtering

problems, estimation algebras analogous to \mathcal{G} have been emphasized by Brockett and Clark [7], Brockett ([8] - [11]), Mitter ([12], [13]), Hazewinkel and Marcus [14] and others (see [6]) as being objects of central interest. In the papers ([1], [2]) the Lie algebra \mathcal{G} is used to classify identification problems and to understand the role of certain sufficient statistics.

2. The Structure of the Estimation Algebra \mathcal{G} :

To understand the structure of the estimation algebra \mathcal{G} it is well-worth considering an example.

Example 1:

Let $dx_t = \theta \cdot dw_t$; $d\theta = 0$

$$dy_t = x_t dt + dv_t$$

Then $A_0 = \frac{\theta^2}{2} \frac{\partial^2}{\partial x^2} - \frac{x^2}{2}$ and $B_0 = x$, and

$\mathcal{G} = \{A_0, B_0\}$ is spanned by the set of operators $(\frac{\theta^2}{2} \frac{\partial^2}{\partial x^2} - \frac{x^2}{2})$, $(\theta^{2n} x)_{n=0}^{\infty}$, $(\theta^{2n} \frac{\partial}{\partial x})_{n=1}^{\infty}$ and

$\{\theta^{2n} 1\}_{n=1}^{\infty}$. We then notice that,

$$\tilde{G} \subseteq \mathbb{R}[\theta^2] \otimes \left\{ \frac{\partial^2}{\partial x^2}, x \frac{\partial}{\partial x}, \frac{\partial}{\partial x}, x^2, x, 1 \right\} \text{L.A.}$$

is a subalgebra of the Lie algebra obtained by tensoring the polynomial ring $\mathbb{R}[\theta^2]$ with a 6 dimensional Lie algebra. //

The general situation is very much as in this example. Consider the vector space (over the reals) of operators spanned by the set,

$$S = \left\{ \frac{\partial^2}{\partial x_i \partial x_j}, x_i \frac{\partial}{\partial x_j}, \frac{\partial}{\partial x_i}, x_i x_j, 1 \right\} \quad (7)$$

$i = 1, 2, \dots, n, \quad j = 1, 2, \dots, n$

This space of operators has the structure of a Lie algebra henceforth denoted as \tilde{G}_0 (of dimension $3n^2 + 2n + 1$) under operator commutation (the commutation rules being $\left[\frac{\partial^2}{\partial x_i \partial x_j}, x_k \right] = \delta_{jk} \frac{\partial}{\partial x_i} + \delta_{ik} \frac{\partial}{\partial x_j}$ etc., where δ_{jk} denotes the

Kronecker symbol). For each choice $\theta \in \mathbb{R}$, A_0 and B_0 take values in \tilde{G}_0 . It follows that in general A_0 and B_0 are smooth maps from Θ into \tilde{G}_0 . So let us consider the space of smooth maps $C^\infty(\Theta; \tilde{G}_0)$. This space can be given the structure of a Lie algebra (over the reals) in the following way:

given $\phi, \psi \in C^\infty(\Theta; \tilde{G}_0)$,

define the Lie bracket $[\dots]_c$ on $C^\infty(\Theta; \tilde{G}_0)$ by

$$[\phi, \psi]_c(P) = [\phi(P), \psi(P)] \quad (10)$$

for every $P \in \Theta$. Here the bracket on the right hand side of eqn. (10) is in \tilde{G}_0 . We denote as \tilde{G}_c the Lie algebra $(C^\infty(\Theta; \tilde{G}_0); [\dots]_c)$. Whenever the dimension of Θ is greater than zero, \tilde{G}_c is

infinite dimensional and is an example of a current algebra. Current algebras play a fundamental role in the physics of Yang-Mills fields where they occur as Lie algebras of gauge transformations [15]. Elsewhere in mathematics they are studied under the guise of local Lie algebras ([16] [18]). The following is immediate.

Proposition 1:

The Lie algebra \tilde{G} of operators generated by

$$A_0 = \frac{1}{2} \langle b(\theta), \frac{\partial}{\partial x} \rangle^2 - \left\langle \frac{\partial}{\partial x}, A(\theta)x \right\rangle - \langle c(\theta), x \rangle^2 / 2$$

and $B_0 = \langle c(\theta), x \rangle$, is a subalgebra of the current algebra $C^\infty(\Theta; \tilde{G}_0)$.

3. Representation Questions:

In [3] we observe that \tilde{G} admits a faithful representation as a Lie algebra of vector fields on a finite dimensional manifold. Specifically, consider the system of equations,

$$\begin{aligned} d\theta &= 0 \\ dz &= [A(\theta) - Pc(\theta)c^T(\theta)]z dt + Pc(\theta)dy_t \\ \frac{dP}{dt} &= A(\theta)P + PA^T(\theta) + b(\theta)b^T(\theta) - Pc(\theta)c^T(\theta)P \\ ds &= \frac{1}{2} \langle c(\theta), z \rangle^2 dt - \langle c(\theta), z \rangle dy_t \end{aligned} \quad (11)$$

The system of equations (11) evolves on the product manifold $\Theta \times \mathbb{R}^{n(n+3)/2+1}$. Associate with eqn. (11) the pair of vector fields (first order differential operators),

$$\begin{aligned} a_0^* &= \langle A(\theta) - Pc(\theta)c^T(\theta)z, \partial/\partial z \rangle \\ &+ \text{tr}((A(\theta)P + PA^T(\theta) + b(\theta)b^T(\theta) - Pc(\theta)c^T(\theta)P) \cdot \partial/\partial P) \\ &+ 1/2 \langle c(\theta), z \rangle^2 \partial/\partial s \end{aligned}$$

and

$$b_0^* = \langle P(\theta), \partial/\partial z \rangle - \langle c(\theta), z \rangle \partial/\partial s. \quad (12)$$

(Here $\partial/\partial P = [\partial/\partial P_{ij}] = (\partial/\partial P)^T = n \times n$ symmetric matrix of differential operators). Consider the Lie algebra of vector fields generated by a_0^* and b_0^* . Since a_0^* and b_0^* are vertical vector fields with respect to the fibering $\Theta \times \mathbb{R}^{n(n+3)/2+1} \rightarrow \Theta$, so is every vector field in this Lie algebra. One of the main results in [3] is the following:

Theorem 1: The map

$$\phi_k: \tilde{G}_c \rightarrow \mathbb{R}^{n(n+3)/2+1}$$

defined by

$$\phi_k(A_0) = a_0^*; \quad \phi_k(B_0) = b_0^*$$

is a faithful representation of the Lie algebra of the identification problem as a Lie algebra of (vertical) vector fields on a finite dimensional manifold fibered over Θ .

Example 2:

To illustrate Theorem 1, consider the Lie algebra of example 1. The embedding equations (11) take the form

$$\begin{aligned} d\theta &= 0 \\ dp &= (\theta^2 - p^2) dt \\ dz &= -pzd t + pdy_t \\ ds &= z^2/2 dt - zdy_t. \end{aligned}$$

Then

$$\begin{aligned} \phi_k(B_0) &= \phi_k(x) \\ &= b_0^* \\ &= p\partial/\partial z + (-z)\partial/\partial s \end{aligned}$$

The induced maps on Lie brackets are given by

$$\begin{aligned} \phi_k(\theta^{2k}\partial/\partial x) &= \theta^{2k}\partial/\partial z \quad k = 0, 1, 2, \dots \\ \phi_k(\theta^{2k}x) &= \theta^{2k}(p\partial/\partial z - z\partial/\partial s) \quad k = 1, 2, \dots \\ \phi_k(\theta^{2k}1) &= \theta^{2k}\partial/\partial s \quad k = 1, 2, \dots // \end{aligned}$$

The embedding equations have the following statistical interpretation. Assume that the initial condition for (12) is of the form,

$$\rho_0(x, \theta) = (2\pi \det \Sigma(\theta))^{-n/2} \exp(-\frac{1}{2}(x - \mu(\theta)) \Sigma^{-1}(\theta) (x - \mu(\theta))) \cdot Q_0(\theta) \quad (13)$$

where $\theta \mapsto (\mu(\theta)\Sigma(\theta), Q_0(\theta))$ is a smooth map, $\Sigma(\theta) > 0$ $\theta \in \Theta$ and $Q_0(\theta) > 0$ for $\theta \in \Theta$. Suppose eqn. (11) is initialized at,

$$(\theta_0, z_0, p_0, s_0) = (\theta_0, \mu(\theta_0), \Sigma(\theta_0), -\log(Q_0(\theta_0))) \quad (14)$$

Append to the system (11) an output equation,

$$\bar{Q}_t = e^{-\frac{t}{\epsilon}} \bar{Q}_0 \quad (15)$$

Now if (11) is solved with initial condition (14), one can show by differentiating \bar{Q}_t that \bar{Q}_t satisfies eqn. (7). In other words, the system (11)-(15) with initial condition (14) is a finite dimensional recursive estimation for the posterior density $Q(t, \theta_0)$. We have thus verified the homomorphism principle of Brockett [8]: that finite dimensional recursive estimators must involve Lie algebras of vector fields that are homomorphic images of the Lie algebra of operators associated with the unnormalized conditional density equation.

4. A Sobolev Lie Group Associated to G:

It has been remarked elsewhere ([8], [13], [21], [22] and [3]) that the Cauchy problem associated with (8) may be viewed as a problem of integrating a Lie algebra representation. In this connection one should be interested whether there is an appropriate topological group associated with G. We have the following general procedure.

Let M be a compact Riemannian manifold of dimension d. Let L be a Lie algebra of dimension $n < \infty$. We can always view L as a subalgebra of the general linear Lie algebra $gl(m; \mathbb{R})$, $m > n$ (Ado's theorem).

Assumption:

Let $G = \{\exp(L)\}_G \subset gl(m; \mathbb{R})$ be the smallest Lie group containing the exponentials of elements of L. We assume that G is a closed subset of $gl(m; \mathbb{R})$.

Define,

$$\mathcal{R} = C^\infty(M; gl(m, \mathbb{R}))$$

$$\mathcal{L} = C^\infty(M; L)$$

$$\mathcal{G} = C^\infty(M; G).$$

Clearly \mathcal{R} is an algebra under pointwise multiplication and

$$\mathcal{L} \subset \mathcal{R}, \quad \mathcal{G} \subset \mathcal{R}.$$

Let $\{(U_\alpha, \varphi_\alpha)\}$ be a C^∞ atlas for M. Then for $f_1, f_2 \in \mathcal{R}$, define

$$\|f_1 - f_2\|_k = \left[\int_{\varphi_\alpha(U_\alpha)} d \text{vol} \sum_{l=0}^k |D^l (f_1 - f_2) \varphi_\alpha^{-1}|^2 \right]^{1/2} \quad (16)$$

where

$$|f|^2 = \text{tr}(f'f). \quad (17)$$

(Here $k = d/2 + s$, $s > 0$). Let \mathcal{R}_k be the completion of \mathcal{R} and \mathcal{G}_k the completion of \mathcal{G} in the norm $\|\cdot\|_k$. (\mathcal{G}_k is closed in \mathcal{R}_k). By the Sobolev theorem, \mathcal{R}_k is a Banach algebra and the group operation

$$\begin{aligned} \mathcal{G}_k \times \mathcal{G}_k &\rightarrow \mathcal{G}_k \\ (f_1, f_2) &\rightarrow f_1 f_2 \end{aligned}$$

when $(f_1 f_2)(m) = f_1(m) f_2(m)$ is continuous. Thus \mathcal{G}_k is a topological group.

Proceeding as before, one can give a Sobolev completion of \mathcal{L} to obtain \mathcal{L}_k an infinite dimensional Lie algebra where once again by the Sobolev theorem the bracket operation

$$\begin{aligned} [\dots] \mathcal{L}_k \times \mathcal{L}_k &\rightarrow \mathcal{L}_k \\ (f_1, f_2) &\rightarrow [f_1, f_2] \end{aligned}$$

with $[f_1, f_2](m) = [f_1(m), f_2(m)]$ is continuous. Now for a small enough neighborhood $V(0)$ of $0 \in \mathcal{L}_k$ one can define

$$\begin{aligned} \exp: V(0) &\rightarrow \mathcal{G}_k \\ \xi &\rightarrow \exp(\xi) \end{aligned}$$

by pointwise exponentiation. This permits us to provide a Lie group structure on \mathcal{G}_k with \mathcal{L}_k canonically identified as the Lie algebra of \mathcal{G}_k .

The procedure outlined above appears to play a significant role in several contexts (the index theorem, Yang-Mills fields [24] [25] [26] [27]).

For our purposes L will be identified with a faithful matrix representation of G_0 . Thus we associate with the identification problem a Sobolev Lie group which is a subgroup of \mathcal{G}_k corresponding to G_0 .

Remark:

One of the important differences between the problem of filtering and the problems of Yang-Mills theories is that in the latter case there are natural norms for Sobolev completion. This follows from the fact that in Yang-Mills theories, the algebra L is compact (semi-simple) and one has the Killing form to work with. In filtering problems G_0 is never compact.

Remark:

We would like to acknowledge here that Prof. Sanjoy Mitter was kind enough to acquaint one of us (P.S.K) with the work of P.K. Mitter.

5. The Integration Problem & Sufficient Statistics

In [3] we look for a representation of the form,

$$\rho(t, x, \theta) = \exp(g_1(t, \theta)A_1) \dots \exp(g_n(t, \theta)A^n) \rho_0 \quad (18)$$

for the solution to the equation (8). In the case of example (1) this takes the form

$$\begin{aligned} \rho(t, x, \theta) = & \exp(g_1(t, \theta) \cdot (\frac{\theta^2}{2} \frac{\partial^2}{\partial x^2} - \frac{x^2}{2})) \cdot \\ & \cdot \exp(g_2(t, \theta) \cdot \theta \frac{\partial}{\partial x}) \\ & \cdot \exp(g_3(t, \theta)x) \cdot \exp(g_4(t, \theta) \cdot 1) \rho_0 \quad (19) \end{aligned}$$

Differentiating and substituting in (8) we get,

$$\begin{aligned} \frac{\partial g_1}{\partial t}(t, \theta) &= 1 \\ \frac{\partial g_2}{\partial t}(t, \theta) &= \cosh(g_1 \cdot \theta) \dot{y} \quad (20) \\ \frac{\partial g_3}{\partial t}(t, \theta) &= -\frac{1}{\theta} \sinh(g_1 \cdot \theta) \dot{y} \\ \frac{\partial g_4}{\partial t}(t, \theta) &= \frac{\partial g_3}{\partial t}(t, \theta) g_2(t, \theta). \end{aligned}$$

and $g_i(0, \theta) = 0$ for $i = 1, 2, 3, 4, \theta \in \Theta$. The above first-order partial differential equations may be easily solved by quadrature and one has the representation,

$$\begin{aligned} \rho(t, x, \theta) = & \int_{-\infty}^{\infty} \frac{1}{\sqrt{2\pi \sinh(|\theta|t)}} \exp(-\frac{1}{2} \coth^2(|\theta|t) (\frac{|x|^2}{|\theta|} + z)) \cdot \\ & \cdot t|\theta| \cdot \exp(\frac{xz}{|\theta| \sinh(|\theta|t)}) \cdot \exp(g_4(t, \theta)\theta^2) \cdot \\ & \cdot \exp(g_2(t, \theta)\sqrt{|\theta|z}) \cdot \rho_0(g_3(t, \theta)\theta^2\sqrt{|\theta|z}, \theta) dz \quad (21) \end{aligned}$$

where $\rho_0(\cdot, \theta) \in L_2(\mathbb{R})$ for every $\theta \in \Theta$ and is smooth in θ . Further $\Theta \subset \mathbb{R}$ is a bounded set and Θ of closure Θ .

In equation (21) the g_i 's should be viewed as canonical coordinates of the second kind on the corresponding Sobolev Lie group. Now expand g_2 and g_3 to obtain

$$\begin{aligned} g_2(t, \theta) &= \sum_{k=0}^{\infty} \theta^{2k} \int_0^t \frac{\sigma^{2k}}{(2k)!} \dot{y}_\sigma d\sigma \quad k=1, 2, \dots \\ g_3(t, \theta) &= - \sum_{k=0}^{\infty} \theta^{2k} \int_0^t \frac{\sigma^{2k+1}}{(2k+1)!} \dot{y}_\sigma d\sigma \quad k=1, 2, \dots \quad (22) \end{aligned}$$

It follows that all the "information" contained by the observations $\{y_\sigma: 0 \leq \sigma \leq t\}$ about the joint unnormalized conditional density is contained in the sequence

$$T\Delta \left\{ \frac{\sigma^k}{k!} \dot{y}_\sigma d\sigma; k=0, 1, 2, \dots \right\} \quad (23)$$

Thus T is nothing but a joint sufficient statistic for the identification problem.

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APPROXIMATION METHODS FOR NONLINEAR FILTERING PROBLEMS
ARISING IN SYSTEM IDENTIFICATION

Summary

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Abstract: In this paper we investigate various approximate methods for computing the conditional density of a parameter. These techniques are related to the structure of certain Lie algebras of operators with the identification problem.

Summary

Consider the stochastic differential system:

$$\begin{aligned} d\theta &= 0 \\ dx_t &= A(\theta)x_t dt + b(\theta)dw_t \end{aligned} \quad (1)$$

$$dy_t = \langle c(\theta), x_t \rangle dt + dv_t.$$

Here $\{w_t\}$ and $\{v_t\}$ are independent, scalar, standard Wiener processes and $\{x_t\}$ is an \mathbb{R}^n -valued process. We let θ take values in a smooth manifold $\mathcal{U} \subset \mathbb{R}^n$. Assume that the map

$$\theta \rightarrow \mathcal{L}(\theta) := (A(\theta), b(\theta), c(\theta)) \quad (2)$$

is sufficiently smooth and takes values in the space of minimal triples.

Define two differential operators,

$$\begin{aligned} A_0 &:= \frac{1}{2} \langle b(\theta), \partial/\partial x \rangle^2 - \langle \partial/\partial x, A(\theta)x \rangle \\ &\quad - \langle c(\theta), x \rangle^2 / 2 \end{aligned} \quad (3)$$

$$B_0 := \langle c(\theta), x \rangle. \quad (4)$$

The problem is to devise approximate finite dimensional, recursive techniques for calculating the conditional density of the parameter θ given $Y_t = \sigma$ -algebra generated by the observations $\{y_s : 0 \leq s \leq t\}$. The general formulas are known:

$$Q(t, \theta) = \frac{\int \rho(t, x, \theta) |dx|}{\int \rho(t, x, \theta) |dx| \cdot |d\theta|} \quad (5)$$

where $|dx|$ and $|d\theta|$ are fixed Riemannian volume elements on \mathbb{R}^n and \mathcal{U} and

$$\rho(t, x, \theta) = e^{\langle c(\theta), x \rangle y_t} \psi(t, x, \theta) \quad (6)$$

and

$$\frac{\partial \psi}{\partial t} = \{z_0 + y_t z_1 + \frac{y_t^2}{2} z_2 + z_3\} \psi \quad (7)$$

where

$$\begin{aligned} z_0 &:= A_0 \\ z_1 &:= \langle c(\theta), b(\theta) \rangle \langle b(\theta), \partial/\partial x \rangle - \langle c(\theta), A(\theta)x \rangle \\ z_2 &:= \langle c(\theta), b(\theta) \rangle^2 \\ z_3 &= -\text{tr}(A(\theta)). \end{aligned} \quad (8)$$

Let $Q(t, \theta) = e^{-S(t, \theta)}$. In this pair we consider approximations related to

(a) local series approximations

$$S(t, \theta) = \sum_{i=0}^{\infty} a_i(t) \theta^{[i]}$$

(b) Gaussian initial conditions:

$\rho(0, \cdot, \theta)$ Gaussian for $\theta \in \mathcal{U}$

Both these approximations are connected to the following algebraic objects.

(a) A sequence of Lie algebras $\{\tilde{G}^{(k)}\}_{k=0}^{\infty}$

where $\tilde{G}^{(0)} := \{A_0, B_0\}$ L.A.

$$\tilde{G}^{(1)} := \left\{ \begin{bmatrix} A_0 & 0 \\ \frac{\partial A_0}{\partial \theta} & A_0 \end{bmatrix}, \begin{bmatrix} B_0 & 0 \\ \frac{\partial B_0}{\partial \theta} & B_0 \end{bmatrix} \right\} \text{ L.A.}$$

(b) Finite dimensional quotients of $\tilde{G}^{(0)}$ in one-to-one correspondence with rings that are quotients of $\mathbb{R}[\theta]$.

Our results use the fact that $\tilde{G}^{(0)}$ is a subalgebra of a current algebra ([1], [2]).

References

- [1] P.S. Krishnaprasad and S.I. Marcus: "On the Lie Algebra of the identification problem", IFAC Symposium on Digital Control, New Delhi, Jan. 1982.
- [2] P.S. Krishnaprasad, S.I. Marcus and M. Hazewinkel, "System identification and nonlinear filtering: Lie Algebras", Proc. 20th IEEE Conference on Decision and Control. San Diego, 1981.