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SYSTEM IDENTIFICATION AND NONLINEAR FILTERING : LIE ALGEBRAS

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Abstract

This paper is continuation of our previous work ([1], [2], [3]) to understand the identification problem of linear system theory from the viewpoint of nonlinear filtering. The estimation algebra of the identification problem is a subalgebra of a current algebra. It therefore follows that the estimation algebra is embeddable as a Lie algebra of vector fields on a finite dimensional manifold. These features permit us to develop a Wei-Norman type procedure for the associated Cauchy problem and reveal a set of functionals of the observations that play the role of joint sufficient statistics for the identification problem.

1. Introduction

Consider the stochastic differential system:

$$d\theta = 0$$
$$dx_t = A(\theta)x_t dt + b(\theta)dw_t \qquad (1)$$
$$dy_t = \langle c(\theta),x_t\rangle dt + dv_t.$$

Here $\{w_t\}$ and $\{v_t\}$ are independent, scalar, standard, Wiener processes, and $\{x_t\}$ is an $\mathbb{R}^n$-valued process. Assume that θ takes values in a smooth manifold $\Theta \to \mathbb{R}^N$, and the map $\theta \to \Sigma(\theta)$: $= (A(\theta), b(\theta), c(\theta))$ in a smooth map taking values in minimal triples. By the identification problem we shall mean the nonlinear filtering problem associated with eqn. (1); i.e. the problem of recursively computing conditional expectations of the form $\pi_t(\phi) \triangleq E[\phi(x_t,\theta)|Y_t]$ where Y_t is the σ-algebra generated by the observations $\{y_s : 0 \le s \le t\}$ and ϕ belongs to a suitable class of functions on $\mathbb{R}^n \times \Theta$.

The joint unnormalized conditional density $\rho \triangleq \rho(t,x,\theta)$ of x_t and θ given Y_t satisfies the stochastic partial differential equation (Stratonovitch sense)

$$d\rho = A_o\rho dt + B_o\rho dy_t \qquad (2)$$

where the operators A_o and B_o are given by

$$A_o: = \frac{1}{2}\langle b(\theta), \frac{\partial}{\partial x}\rangle^2 - \langle\frac{\partial}{\partial x}, A(\theta)x\rangle - \langle c(\theta),x\rangle^2/2 \qquad (3)$$

$$B_o: = \langle c(\theta),x\rangle . \qquad (4)$$

(see [4] for background).

From the Bayes formula ([5]), it follows that

$$\pi_t(\phi) = \sigma_t(\phi)/\sigma_t(1) \qquad (5)$$

where

$$\sigma_t(\phi) = \int_\Theta \int_{\mathbb{R}^n} \phi(x,\theta)\rho(t,x,\theta)|dx|.|d\theta| \qquad (6)$$

where $|dx|$ and $|d\theta|$ are fixed volume elements on $\mathbb{R}^n$ and Θ respectively. Further if $Q(t,\theta)$ denotes the unnormalized posterior density of θ given $_t$, then it satisfies the Ito equation:

$$dQ = E[\langle c(\theta),x_t|\theta,Y_t].Q(t,\theta)dy_t. \qquad (7)$$

Recent work in nonlinear filtering theory (see the proceedings [6]) shows that it is natural to look at eqn. (2) formally as a deterministic partial differential equation,

$$\frac{\partial\rho}{\partial t} = A_o\rho + \dot{y}B_o\rho. \qquad (8)$$

By the Lie algebra of the identification problem, we shall mean the operator Lie algebra $\tilde{G}$ generated by A_o and B_o. For more general nonlinear filtering problems, estimation algebras analogous to $\tilde{G}$ have been emphasized by Brockett and Clark [7], Brockett ([8] - [11]), Mitter ([12], [13]), Hazewinkel and Marcus [14] and others (see [6]) as being objects of central interest. In the papers ([1], [2]) the Lie algebra $\tilde{G}$ is used to classify identification problems and to understand the role of certain sufficient statistics.

2. The Structure of the Estimation Algebra $\tilde{G}$:

To understand the structure of the estimation algebra $\tilde{G}$ it is well-worth considering an example.

Example 1:

Let $dx_t = \theta.dw_t$; $d\theta = 0$
$dy_t = x_t dt + dv_t$

Then $A_o = \frac{\theta^2}{2}\frac{\partial^2}{\partial x^2} - \frac{x^2}{2}$ and $B_o = x$, and $\tilde{G} = \{A_o,B_o\}_{L.A.}$ is spanned by the set of operators $(\frac{\theta^2}{2}\frac{\partial^2}{\partial x^2} - \frac{x^2}{2})$, $\{\theta^{2n}x\}_{n=0}^{\infty}$, $\{\theta^{2n}\frac{\partial}{\partial x}\}_{n=1}^{\infty}$ and

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$\{\theta^{2n}1\}_{n=1}^{\infty}$. We then notice that,

$$\tilde{G} \subseteq \mathbb{R}[\theta^2] \otimes \{\frac{\partial^2}{\partial x^2}, x\frac{\partial}{\partial x}, \frac{\partial}{\partial x}, x^2, x, 1\}\text{L.A.}$$

is a subalgebra of the Lie algebra obtained by tensoring the polynomial ring $\mathbb{R}[\theta^2]$ with a 6 dimensional Lie algebra.//

The general situation is very much as in this example. Consider the vector space (over the reals) of operators spanned by the set,

$$S:=\{\frac{\partial^2}{\partial x_i \partial x_j}, x_i \frac{\partial}{\partial x_j}, \frac{\partial}{\partial x_i}, x_i x_i, x_j, 1\} \qquad (7)$$

$$i = 1,2,..,n, \quad j=1,2,..,n$$

This space of operators has the structure of a Lie algebra henceforth denoted as $\tilde{G}_o$ (of dimension $3n^2+2n+1$) under operator commutation (the commutation rules being $[\frac{\partial^2}{\partial x_i \partial x_j}, x_k] = \delta_{jk}\frac{\partial}{\partial x_i} + \delta_{ik}\frac{\partial}{\partial x_j}$ etc., where δ_{jk} denotes the Kronecker symbol). For each choice $\theta\varepsilon\Theta$, A_o and B_o take values in $\tilde{G}_o$. It follows that in general A_o and B_o are smooth maps from Θ into $\tilde{G}_o$. So let us consider the space of smooth maps $C^\infty(\Theta;\tilde{G}_o)$. This space can be given the structure of a Lie algebra (over the reals) in the following way:

given $\phi,\psi\varepsilon\ C^\infty(\Theta;\tilde{G}_o)$,

define the Lie bracket $[.,.]_c$ on $C^\infty(\Theta;\tilde{G}_o)$ by

$$[\phi,\psi]_c(P) = [\phi(P),\psi(P)] \qquad (10)$$

for every $P\varepsilon\Theta$. Here the bracket on the right hand side of eqn. (10) is in $\tilde{G}_o$. We denote as $\tilde{G}_c$ the Lie algebra $(C^\infty(\Theta;\tilde{G}_o);[.,.]_c)$. Whenever the dimension of Θ is greater than zero, $\tilde{G}_o$ is infinite dimensional and is an example of a <u>current algebra</u>. Current algebras play a fundamental role in the physics of Yang-Mills fields where they occur as Lie algebras of gauge transformations [15]. Elsewhere in mathematics they are studied under the guise of local Lie algebras ([16] [18]). The following is immediate.

<u>Proposition 1</u>:

The Lie algebra $\tilde{G}$ of operators generated by

$$A_o = \frac{1}{2}\langle b(\theta),\frac{\partial}{\partial x}\rangle^2 - \langle\frac{\partial}{\partial x},A(\theta)x\rangle - \langle c(\theta),x\rangle^2/_2$$

and $B_o = \langle c(\theta),x\rangle$, is a subalgebra of the current algebra $C^\infty(\Theta;\tilde{G}_o)$.

3. Representation Questions:

In [3] we observe that $\tilde{G}$ admits a faithful representation as a Lie algebra of vector fields on a finite dimensional manifold. Specifically, consider the system of equations,

$$d\theta = 0$$

$$dz = [A(\theta)-Pc(\theta)c^T(\theta)]zdt + Pc(\theta)dy_t$$

$$\frac{dP}{dt} = A(\theta)P + PA^T(\theta) + b(\theta)b^T(\theta)-Pc(\theta)c^T(\theta)P$$

$$ds = \frac{1}{2}\langle c(\theta),z\rangle^2 dt - \langle c(\theta),z\rangle dy_t \qquad (11)$$

The system of equations (11) evolves on the product manifold $\Theta \times \mathbb{R}^{n(n+3)/2+1}$. Associate with eqn. (11) the pair of vector fields (first order differential operators),

$$a_0^* = \langle (A(\theta)-Pc(\theta)c^T(\theta))z,\partial/\partial z\rangle + tr((A(\theta)P+PA^T(\theta)+b(\theta)b^T(\theta)-Pc(\theta)c^T(\theta)P).\partial/\partial P) + 1/2\langle c(\theta),z\rangle^2\partial/\partial s$$

and

$$b_o^* = \langle P(\theta),\partial/\partial z\rangle - \langle c(\theta),z\rangle\ \partial/\partial s. \qquad (12)$$

(Here $\partial/\partial P = [\partial/\partial P_{ij}] = (\partial/\partial P)^T$ = nxn symmetric matrix of differential operators). Consider the Lie algebra of vector fields generated by a_0^* and b_0^*. Since a_0^* and b_0^* are vertical vector fields with respect to the fibering $\Theta\times\mathbb{R}^{n(n+3)/2+1}\to\Theta$, so is every vector field in this Lie algebra. One of the main results in [3] is the following:

<u>Theorem 1</u>: The map

$$\Phi_k:\tilde{G}_c \to U\Theta\times\mathbb{R}^{n(n+3)/2+1}$$

defined by

$$\Phi_k(A_o) = a_0^* \ ; \ \Phi_k(B_0) = b_0^*$$

is a faithful representation of the Lie algebra of the identification problem as a Lie algebra of (vertical) vector fields on a finite dimensional manifold fibered over Θ.

<u>Example 2</u>:

To illustrate Theorem 1, consider the Lie algebra of example 1. The embedding equations (11) take the form

$$d\theta = 0$$

$$dp = (\theta^2-p^2)dt$$

$$dz = -pzdt + pdy_t$$

$$ds = z^2/2dt - zdy_t.$$

Then

$$\Phi_k(B_0) = \Phi_k(x) = b_0^* = p\partial/\partial z + (-z)\partial/\partial s$$

The induced maps on Lie brackets are given by

$$\Phi_k(\theta^{2k}\partial/\partial x) = \theta^{2k}\partial/\partial z \quad k = 0,1,2,\ldots$$
$$\Phi_k(\theta^{2k}x) = \theta^{2k}(p\partial/\partial z - z\partial/\partial s) \quad k = 1,2,\ldots$$
$$\Phi_k(\theta^{2k}1) = \theta^{2k}\partial/\partial s \quad k = 1,2,\ldots //$$

The embedding equations have the following statistical interpretation. Assume that the initial condition for (12) is of the form,

$$\rho_0(x,\theta) = (2\pi\det\Sigma(\theta))^{-n/2}\exp(-\frac{\langle x-\mu(\theta),\Sigma^{-1}(\theta)}{2}\cdot$$
$$\cdot(x-\mu(\theta))\rangle)\cdot Q_0(\theta) \qquad (13)$$

where $\theta\rightarrow(\mu(\theta),\Sigma(\theta),Q_0(\theta))$ is a smooth map, $\Sigma(\theta)>0$ $\theta\in\Theta$ and $Q_0(\theta)>0$ for $\theta\in\Theta$. Suppose eqn. (11) is initialized at,

$$(\theta_0,z_0,P_0,s_0)=(\theta_0,\mu(\theta_0),\Sigma(\theta_0),-\log(Q_0(\theta)) \qquad (14)$$

Append to the system (11) an output equation,

$$\overline{Q}_t = e^{-s_t}. \qquad (15)$$

Now if (11) is solved with initial condition (14), one can show by differentiating $\overline{Q}_t$ that $\overline{Q}_t$ satisfies eqn. (7). In other words, the system (11)-(15) with initial condition (14) is a finite dimensional recursive estimation for the posterior density $Q(t,\theta_0)$. We have thus verified the homomorphism principle of Brockett [8]: that finite dimensional recursive estimators must involve Lie algebras of vector fields that are homomorphic images of the Lie algebra of operators associated with the unnormalized conditional density equation.

4. A Sobolev Lie Group Associated to $\tilde{G}$:

It has been remarked elsewhere ([8], [13], [21], [22] and [3]) that the Cauchy problem associated with (8) may be viewed as a problem of integrating a Lie algebra representation. In this connection one should be interested whether there is an appropriate topological group associated with $\tilde{G}$. We have the following general procedure.

Let M be a compact Riemannian manifold of dimension d. Let L be a Lie algebra of dimension $n<\infty$. We can always view L as a subalgebra of the general linear Lie algebra $g\ell(m;\mathbb{R})$, $m>n$ (Ado's theorem).

Assumption:

Let $G=\{\exp(L)\}_G\subseteq g\ell(m;\mathbb{R})$ be the smallest Lie group containing the exponentials of elements of L. We assume that G is a closed subset of $g\ell(m;\mathbb{R})$.

Define,

$$\mathcal{R} = C^\infty(M;g\ell(m,\mathbb{R}))$$
$$\mathcal{L} = C^\infty(M;L)$$
$$\mathcal{G} = C^\infty(M;G).$$

Clearly $\mathcal{R}$ is an algebra under pointwise multiplication and

$$\mathcal{L}\subset\mathcal{R},\ \mathcal{G}\subset\mathcal{R}.$$

Let $\{(U_\alpha,\varphi_\alpha)\}$ be a C^∞ atlas for M. Then for $f_1,f_2\in\mathcal{R}$, define

$$\|f_1-f_2\|_k=[\int_{\varphi_\alpha(U_\alpha)} d\,\mathrm{vol}\sum_{\ell=0}^{k}|D^\ell(f_1-f_2)\varphi_\alpha^{-1}|^2]^{\frac{1}{2}} \qquad (16)$$

where

$$|f|^2 = \mathrm{tr}\,(f'f). \qquad (17)$$

(Here $k=d/2+s$. $s>0$). Let $\mathcal{R}_k$ be the completion of $\mathcal{R}$ and $\mathcal{G}_k$ the completion of $\mathcal{G}$ in the norm $\|.\|_k$. ($\mathcal{G}_k$ is closed in $\mathcal{R}_k$). By the Sobolev theorem, $\mathcal{R}_k$ is a Banach algebra and the group operation

$$\cdot:\mathcal{G}_k\times\mathcal{G}_k\rightarrow\mathcal{G}_k$$
$$(f_1,f_2)\rightarrow f_1f_2$$

when $(f_1f_2)(m) = f_1(m)f_2(m)$ is continuous. Thus $\mathcal{G}_k$ is a topological group.

Proceeding as before, one can given a Sobolev completion of $\mathcal{L}$ to obtain $\mathcal{L}_k$ an infinite dimensional Lie algebra where once again by the Sobolev theorem the bracket operation

$$[.,.]:\mathcal{L}_k\times\mathcal{L}_k\rightarrow\mathcal{L}_k$$
$$(f_1,f_2)\rightarrow[f_1,f_2]$$

with $[f_1,f_2](m) = [f_1(m),f_2(m)]$ is continuous. Now for a small enough neighborhood $V(0)$ of $0\in\mathcal{L}_k$ one can define

$$\exp: V(0)\rightarrow\mathcal{G}_k$$
$$\xi\rightarrow\exp(\xi)$$

by pointwise exponentiation. This permits us to provide a Lie group structure on $\mathcal{G}_k$ with $\mathcal{L}_k$ canonically identified as the Lie algebra of $\mathcal{G}_k$.

The procedure outlined above appears to play a significant role in several contexts (the index theorem, Yang-Mills fields [24] [25] [26] [27].

For our purposes L will be identified with a faithful matrix representation of G_0. Thus we associate with the identification problem a Sobolev Lie group which is a subgroup of $\mathcal{G}_k$ corresponding to $\tilde{G}_0$.

Remark:

One of the important differences between the problem of filtering and the problems of Yang-Mills theories is that in the latter case there are natural norms for Sobolev completion. This follows from the fact that in Yang-Mills theories, the algebra L is compact (semi-simple) and one has the Killing form to work with. In filtering problems $\tilde{G}_0$ is never compact.

Remark:

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5. The Integration Problem & Sufficient Statistics

In [3] we look for a representation of the form,

$$\rho(t,x,\theta)=\exp(g,(t,\theta)A1)\ldots\exp(g_n(t,\theta)A^n)\rho_0 \qquad (18)$$

for the solution to the equation (8). In the case of example (1) this takes the form

$$\rho(t,x,\theta)=\exp(g_1(t,\theta)\cdot(\frac{\theta^2}{2}\frac{\theta^2}{\theta_x}-\frac{x^2}{2})).$$
$$.\exp(g_2(t,\theta)\cdot\theta^2\frac{\partial}{\partial x})$$
$$.\exp(g_3(t,\theta)x).\exp(g_4(t,\theta).1)\rho_0 \qquad (19)$$

Differentiating and substituting in (8) we get,

$$\frac{\partial g}{\partial t},(t,\theta) = 1$$

$$\frac{\partial g_2}{\partial t}(t,\theta) = \cosh(g_1.\theta)\dot{y} \qquad (20)$$

$$\frac{\partial g_3}{\partial t}(t,\theta) = -\frac{1}{\theta}\sinh(g_1.\theta)\dot{y}$$

$$\frac{\partial g_4}{\partial t}(t,\theta) = \frac{\partial g_3}{\partial t}(t,\theta)g_2(t,\theta).$$

and $g_i(0,\theta) = 0$ for $i = 1,2,3,4,\theta\varepsilon\Theta$. The above first-order partial differential equations may be easily solved by quadrature and one has the representation,

$$\rho(t,x,\theta) = \int_{-\infty}^{\infty}\sqrt{\frac{1}{2\pi\sinh(|\theta|t)}}\exp(-\frac{1}{2}\coth^2(\frac{|x|^2}{|\theta|}+z).$$
$$.t|\theta|)\ .\exp(\frac{xz}{\sqrt{|\theta|\sinh(|\theta|t)}}).\exp(g_4(t,\theta)\theta^2).$$
$$.\exp(g_2(t,\theta)\sqrt{|\theta|z)}.\rho_0(g_3(t,\theta)\theta^2\sqrt{|\theta|z},\theta)dz \qquad (21)$$

where $\rho_0(,\theta)\varepsilon L_2(\mathbb{R})$ for every $\theta\varepsilon\Theta$ and is smooth in θ. Further $\Theta\subset\mathbb{R}$ is a bounded set and $0\notin$ closure Θ.

In equation (21) the g_i's should be viewed as canonical coordinates of the second kind on the corresponding SobolevLie group. Now expand g_2 and g_3 to obtain

$$g_2(t,\theta) = \sum_{k=0}^{\infty}\theta^{2k}\int_0^t\frac{\sigma^{2k}}{(2k)!}\dot{y}_\sigma d\sigma \quad k=1,2,\ldots$$
$$g_3(t,\theta) = -\sum_{k=0}^{\infty}\theta^{2k}\int_0^t\frac{\sigma^{2k+1}}{(2k+1)!}\dot{y}_\sigma d\sigma \quad k=1,2,\ldots \qquad (22)$$

It follows that all the "information" contained by the observations $\{y_\sigma:0\leq\sigma\leq t\}$ about the joint unnormalized conditional density is contained in the sequence

$$T\underline{\Delta}\{\int\frac{\sigma^k}{k!}\dot{y}_\sigma d\sigma;\ k=0,1,2,\ldots\} \qquad (23)$$

Thus T is nothing but a joint sufficient statistic for the identification problem.

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