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Infinitistic Rules of Proof and Their Semantics

by

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Summary. We consider problems connected with the semantics of infinitary rules of proof formulated in the language of the second order arithmetics. We answer negatively two questions of Enderton concerning the semantics of his $\mathscr{A}$-rule. Finally, we discuss the problem of the existence of a satisfactory syntactical β-rule.

1. Introduction. We consider the problems connected with the semantics of certain infinitistic rules of proof formulated in the language of second-order arithmetic. Our results in this direction are far from being complete and we think that the subject deserves a further study.

We answer negatively for two questions posed in Enderton's paper [1] which concern the semantics of his $\mathscr{A}$-rule. Furthermore we try to throw some light on the problem of the existence of the satisfactory syntactical β-rule. We discuss some infinitistic rules of proof which could be useful in the search of a β-rule.

By (A) we mean the second-order arithmetic with function variables which is described in [2]. AC denotes the axiom of choice for (A), i.e. the following scheme: $\forall x \exists a \Phi(x, a) \to \exists a \forall x \Phi(x, (a)_x)$.

By A_2 we mean the second-order arithmetic with the choice scheme with set variables which is described in e.g. [3].

By ZFC$^-$ we mean ZF-set theory without the power-set axiom but with the following version of choice scheme: $(\forall x \in y) \exists z \Phi(x, z) \to \exists f \big(f$ is a function with domain $y \wedge (\forall x \in y) \Phi(x, f(x))\big)$.

Our notation is that of recursion theory carried out in the natural way into the language of second-order arithmetic. All the notions concerning the second-order arithmetic can be found in [4] or in [5].

Every ω-model for $L(A_2)$ we identify with its class of sets, i.e. with a subset of $P(\omega)$ (the power set of ω). If $\mathfrak{A}$ and $\mathfrak{B}$ are ω-models, we say that $\mathfrak{A} \prec_n^1 \mathfrak{B}$ iff $\mathfrak{A} \subseteq \mathfrak{B}$ and for every Σ_n^1 formula φ with parameters from $\mathfrak{A}$

$$\mathfrak{B} \models \varphi \Rightarrow \mathfrak{A} \models \varphi.$$

Let $\mathcal{L}(X)$ be Σ_2^1 formula of $L(A_2)$ which is satisfied exactly by the constructible subsets of ω [6]. This formula defines over every ω-model M, the ω-model $\mathcal{L}^M$.

If M is an ω-model for $L(A_2)$, by Def M we mean the class of all definable elements of M. If M is a model for L (A) and a is a unary function which is an element of M then we say that a is definable in M if for some formula $\Phi(x)$ with one free variable $M \models \forall x \left(a(x)=0 \leftrightarrow \Phi(x)\right)$. M is pointwise definable if every unary function from M is definable in M. If T is a set of formulas, by $Cn(T)$ we mean the least set of formulas containing T and axioms of logic and which is closed with respect to *modus ponens* (we assume that logic is so axiomatized, that the only rule of inference is *modus ponens*).

By $\ulcorner T \urcorner$ we mean the set of Gödel numbers of formulas from T.

Let *Form* be the set of all formulas of the language $L(A)$. In the most general way we may define an infinitistic rule of proof as a partial mapping f from 2^{Form} (the power set of *Form*) into *Form*, such that if $x \in \text{domain } f$ then x is infinite.

It is natural to impose the condition that f is definable in L (A).

We recall some definitions.

DEFINITION 1. Let T be a set of formulas of L (A)

(i) *T is closed under the rule f*, if $Cn(T)=T$ and for every x, $x \subseteq T$ and $x \in \text{domain} f$ implies $f(x) \in T$.

(ii) The closure of T under the rule f, $(T)_f$ is the least set of formulas which contains T and is closed under the rule f.

(iii) T is f-consistent if $Cn(T)=T$ and for every $x, x \subseteq T$ and $x \in \text{domain} f$ implies $\neg f(x) \notin T$.

DEFINITION 2. Let $\mathfrak{A}$ be a model for L (A). The rule f is sound for if Th ($\mathfrak{A}$) is closed under f.

DEFINITION 3. (i) The rule *f has a semantics* if there exists a class $\mathcal{K}$ of models for L (A) such that for every set T of sentences which contains the axioms of (A) $(T)_f=\{\varphi\colon \varphi$ is a sentence of L (A) and $\mathfrak{A} \models \varphi$ for each $\mathfrak{A} \in \mathcal{K}$ such that $\mathfrak{A} \models T\}$.

(ii) The rule f is *semantically consistent* if for each T which is f-consistent there exists a model of T for which f is sound.

Thus the rule f is syntactically consistent iff every f-consistent set of formulas can be extented to a complete one. Note that if f is semantically consistent then T is f-consistent implies $(T)_f$ is consistent.

Observe that for every set T of sentences, $(T)_f \subset \{\varphi\colon \varphi$ is a sentence of L(A) and $\mathfrak{A} \models \varphi$ for each $\mathfrak{A}$ for which f is sound and $\mathfrak{A} \models T\}$.

If f has a semantics and $\mathcal{K}$ is the appropriate class of structures then f is sound for $\mathfrak{A} \in \mathcal{K}$. Thus for f which has a semantics the above holds after replacing the inclusion by equality. It follows that in definition 3 (i) we may replace $\mathcal{K}$ by the set of all structures for which f is sound.

2. Negative solution of two problems of Enderton. Enderton introduced in [1] the following rule of proof:

$\mathcal{A}$ — rule: For any function a, from $\Phi\left(\bar{\alpha}(n)\right)$ for each n infer $\exists v \, \forall x \Phi\left(\bar{v}(x)\right)$.

According to this rule he introduced the notion of $d\beta$-model: a model M for $L(\mathrm{A})$ is called a $d\beta$-model if whenever

$$M \models \forall v \left(\forall x\, \forall y\, (v(x, y) = 0 \leftrightarrow \Phi(x, y)) \rightarrow \mathrm{Bord}(v)\right)$$

then the relation defined by Φ in M is really a well-ordering.

Thus M is a $d\beta$-model if the definable well-orderings of M are really well-orderings.

Every $d\beta$-model of (A) is an ω-model. Also $d\beta$-models of (A) are exactly the ω-models of (A) for which the $\mathscr{A}$-rule is sound. Enderton asked

1° If φ is true in every $d\beta$-model of (A) does $\varphi \in (\mathrm{A})_{\mathscr{A}}$?

2° If T is $\mathscr{A}$-consistent, does T have a $d\beta$-model?

If we replace in 1° (A) by (A)+AC then the answer to 1° is negative. The answer to 2° is also negative. We prove that the $\mathscr{A}$-rule has no semantics and is not semantically consistent.

In the same way we introduce the notion of a $d\beta$-model for $L(\mathrm{A}_2)$.

It is easy to see that (A)+AC is faithfully interpretable in A_2. This interpretation assigns for each $d\beta$-model for $L(\mathrm{A})$ a $d\beta$-model for $L(\mathrm{A}_2)$.

By theorem 3 of [1] $\ulcorner((\mathrm{A})+\mathrm{AC})_{\mathscr{A}}\urcorner \in \Sigma^1_2$. By theorem 8 of [1] $((\mathrm{A})+\mathrm{AC}))_{\mathscr{A}} \subset A_{d\beta}$, where $A_{d\beta}$ denotes the set of all sentences of $L(\mathrm{A})$ which are true in every $d\beta$-model of (A)+AC.

THEOREM 1. $((\mathrm{A})+\mathrm{AC})_{\mathscr{A}} \neq A_{d\beta}$.

Proof. By the above it suffices to show that $\ulcorner A_{d\beta}\urcorner \notin \Sigma^1_2$. Because of the facts connected with the interpretation of (A)+AC in A_2 it suffices to prove that $(\mathrm{A}_2)_{d\beta}$, the set of all sentences of $L(\mathrm{A}_2)$ which are true in every $d\beta$-model of A_2, is not a Σ^1_2 set.

Let M be a $d\beta$-model of A_2. Then

1° $\mathscr{L}^M \prec^1_2 M$,

2° $\mathscr{L}^M$ is a $d\beta$-model of A_2,

3° $\mathrm{Def}\, \mathscr{L}^M \prec \mathscr{L}^M$ and $\mathrm{Def}\, \mathscr{L}^M$ is a β-model of A_2*).

Proof of 1°. We prove this for arbitrary ω-models of A_2. Let $\exists X\, \forall Y \theta$ be a Σ^1_2 formula of $L(\mathrm{A}_2)$. By Shoenfield's absoluteness lemma (see [7])

$$\mathrm{ZFC} \vdash \forall Z \left(\mathscr{L}(Z) \rightarrow \left(\exists X\, \forall Y \theta(Z) \rightarrow \exists X\, (\mathscr{L}(X) \wedge \forall Y \theta(Z))\right)\right),$$

(we understand here that X, Y, Z range over 2^ω). The close inspection of the proofs from [6] and [7] shows that the above scheme is already provable in ZFC^-.

By a theorem of Kreisel (see [8] pp. 376—377, cf. also Zbierski [9]) ZFC^- is a conservative extension of A_2.

Hence the above scheme is provable in A_2 what easily settles the claim.

Proof of 2°. We prove at first that $\mathscr{L}^M \models \mathrm{A}_2$ for every ω-model M of A_2. The analysis of Gödel's proof shows that constructible sets form an interpretation of ZFC^- in ZFC^-.

*) We have to know the fact that M is a $d\beta$-model of A_2 implies $\mathrm{Def}\, \mathscr{L}^M$ is a β-model of A_2. This was already known to H. Friedman.

Let Φ be an axiom of A_2. Then $ZFC^- \vdash \Phi$ and $ZFC^- \vdash \Phi_L$, where Φ_L is the relativization of the sentence Φ to the constructible universe. But, as we already mentioned

$$ZFC^- \vdash (\forall x)\big(x \subseteq \omega \rightarrow (L(x) \leftrightarrow \mathcal{L}(x)\big).$$

which implies $ZFC^- \vdash \Phi_{\mathcal{L}}$ where $\Phi_{\mathcal{L}}$ is the formula which arises from Φ, by relativization of its set quantifiers to the formula $\mathcal{L}(X)$. By conservativeness $A_2 \vdash \Phi_{\mathcal{L}}$. Thus if M is an ω-model of A_2 then $\mathcal{L}^M \models \Phi$ for every Φ which is an axiom of A_2.

Let now M be a $d\beta$-model of A_2. Suppose that $\mathcal{L}^M \models \mathrm{Bord}(X)$, where $X \in \mathrm{Def}(\mathcal{L}^M)$. By 1° $M \models \mathrm{Bord}(X)$. Since $\mathcal{L}^M$ is definable over M, also $X \in \mathrm{Def}(M)$. Thus $\mathrm{Bord}(X)$, i.e. $\mathcal{L}^M$ is a $d\beta$-model.

Proof of 3°. If M is an ω-model of A_2, then in $\mathcal{L}^M$ there exists a definable (Δ^1_2) well-ordering of the universe (we use conservativeness), thus, by a well-known theorem of Montague-Vaught, $\mathrm{Def}\,\mathcal{L}^M \prec \mathcal{L}^M$.

Hence $\mathrm{Def}\,\mathcal{L}^M$ is a $d\beta$-model of A_2 whenever M is. But in $\mathrm{Def}\,\mathcal{L}^M$ every set is definable, thus $\mathrm{Def}\,\mathcal{L}^M$ is then a β-model.

Let now Φ be a π^1_2 sentence of $L(A_2)$ which is true in 2^ω. Let M be a $d\beta$-model of A_2.

By theorem 6.3. of [4] and 3°, $\mathrm{Def}\,\mathcal{L}^M \models \Phi$. Thus by 1° and 3° $M \models \Phi$.

Thus a π^1_2 sentence Φ is true in 2^ω iff $\Phi \in (A_2)_{d\beta}$. The set of Gödel numbers of all true π^1_2 sentences is a $\pi^1_2 - \Sigma^1_2$ set (see, e.g. [10], Corollary XIV § 16.2), thus $\ulcorner (A_2)_{d\beta} \urcorner \notin \Sigma^1_2$.

Corollary 1. *The $\mathcal{A}$-rule has no semantics.*

Proof. According to Theorem 1 $\big((A)+AC\big)_{\mathcal{A}} \subsetneq A_{d\beta}$. We show that $A_{d\beta} = \{\varphi : \mathfrak{A} \models \varphi$ for every $\mathfrak{A}$ such that $\mathfrak{A} \models (A)+AC$ and the $\mathcal{A}$-rule is sound for $\mathfrak{A}\}$. Indeed, let $\mathfrak{A}$ be a model of $(A)+AC$, such that the $\mathcal{A}$-rule is sound for $\mathfrak{A}$. The ω-rule is derivable in (A) from the $\mathcal{A}$-rule ([1], Theorem 1), thus $\mathrm{Th}(\mathfrak{A})$ is ω-closed. By Henkin—Orey theorem (see, e.g. [11] p. 231) $\mathrm{Th}(\mathfrak{A})$ has an ω-model, say M. Thus M is an ω-model for which the $\mathcal{A}$-rule is sound, i.e. M is a $d\beta$-model which is elementary equivalent to $\mathfrak{A}$. Thus the equality holds. Now our statement follows from the remark at the end of **1**.

Corollary 2. *There exists a set T of sentences of $L(A)$, which is $\mathcal{A}$-consistent but has no $d\beta$-model.*

Proof. Let $\Phi \in A_{d\beta} - \big((A)+AC\big)_{\mathcal{A}}$.
Then $Cn\big(((A)+AC)_{\mathcal{A}} \cup \{\neg\Phi\}\big)$ is a consistent $\mathcal{A}$-closed set and *a fortiori* it is $\mathcal{A}$-consistent. Indeed, suppose that for some a

$$\psi\big(\bar{\alpha}(n)\big) \in \mathrm{Cn}\big(((A)+AC)_{\mathcal{A}} \cup \{\neg\Phi\}\big) \text{ for every } n.$$

Then for every n $\neg\Phi \rightarrow \psi\big(\bar{\alpha}(n)\big) \in \big((A)+AC\big)_{\mathcal{A}}$, thus by $\mathcal{A}$-rule $\neg\Phi \rightarrow \exists v\, \forall x \psi\big(\bar{v}(x)\big) \in \big((A)+AC\big)_{\mathcal{A}}$, i.e. $\exists v\, \forall x \psi\big(\bar{v}(x)\big) \in Cn\big(((A)+AC)_{\mathcal{A}} \cup \{\neg\Phi\}\big)$.
On the other hand $Cn\big(((A)+AC)_{\mathcal{A}} \cup \{\neg\Phi\}\big)$ has no $d\beta$-model.

Corollary 3. *The $\mathcal{A}$-rule is not semantically consistent.*

Proof. In the proof of Corollary 1 we observed that every model of (A) for which the $\mathscr{A}$-rule is sound is elementarily equivalent to a $d\beta$-model. Hence the above set $Cn\big(((\mathrm{A})+\mathrm{AC})_{\mathscr{A}}\cup\{\neg\Phi\}\big)$ violates the semantic consistency of the $\mathscr{A}$-rule.

3. Some infinitistic rules of proof. We now discuss certain rules of inference from the point of view of their semantical properties. Aczel observed in [11] (p. 326) that the $\mathscr{A}$-rule is an example of the rule of inference which comes from the notion of the generalized quantifier. Let $\mathscr{F}(a)$ be an analytical relation on ω^ω. Let $\mathscr{F}x\Phi(x)$ denote the following formula of $L(\mathrm{A})$ $\exists v\,(\forall x\,(v(x)=0\leftrightarrow\Phi(x))\wedge\mathscr{F}(v))$ (here we treat $\mathscr{F}(a)$ as a formula of $L(\mathrm{A})$).

$\mathscr{F}(a)$ determines the following infinitistic rule of proof:

$\mathscr{F}$-rule: from the fact that $\exists a\,(\forall n\,(a(n)=0\leftrightarrow\vdash_{\mathscr{F}}\Phi(\mathbf{n}))\wedge\mathscr{F}(a)$ infer $\mathscr{F}x\Phi(x)$.*)

Thus a set T of formulas is closed under the $\mathscr{F}$-rule if for every formula $\Phi(x)$ with one free number variable

$$\exists a\,(\forall n\,(a(n)=0\leftrightarrow\Phi(\mathbf{n})\in T)\wedge\mathscr{F}(a))\ \text{implies}\ \mathscr{F}x\Phi(x)\in T.$$

For $\mathscr{F}(a)\leftrightarrow\forall n\,(a(n)=0)$ we get the ω-rule, for $\mathscr{F}(a)\leftrightarrow\exists\beta\,\forall n\,(a(\bar{\beta}(n))=0)$ we get the $\mathscr{A}$-rule.

DEFINITION 4. Let M be an ω-model for $L(\mathrm{A})$. The formula $\Phi(v)$ with one free variable is *M-downward absolute with definable parameters* if for every $a\in M$ which is definable in M

$$\Phi(a)\rightarrow M\models\Phi(a).$$

We have the following simple

THEOREM 2. *Let M be an ω-model for $L(\mathrm{A})$. The $\mathscr{F}$-rule is sound for M iff $\mathscr{F}(v)$ as a formula is M-downward absolute with definable parameters.*

Proof. Immediate. We do not know any $\mathscr{F}$-rule not equivalent to the ω-rule which has a semantics or is semantically consistent. For the $\mathscr{F}$-rules stronger that the ω-rule the above theorem describes the semantics if it exists.

THEOREM 3. *If the $\mathscr{F}$-rule is semantically consistent then it has as semantics.*

The proof is straightforward and uses the fact that if φ is a sentence and $\Phi(x)$ a formula, then

$$\mathscr{F}x\,(\varphi\rightarrow\Phi(x))\ \text{is logically equivalent to}\ \varphi\rightarrow\mathscr{F}x\Phi(x).$$

Observe that Corollary 3 follows from Corollary 1 as a special case of the above theorem.

The only infinitary rule of proof known to us, which is not equivalent to the ω-rule, has a semantics and is semantically consistent, is the following

*) Such a formulation of the $\mathscr{F}$-rule will be useful in 4. However it is more natural to define the $\mathscr{F}$-rule in the language of A_2. Namely, let $\mathscr{F}$ be an analytical relation on 2^ω, definable in 2^ω by a formula φ, i.e. $X\in\mathscr{F}\leftrightarrow 2^\omega\models\varphi(X)$. Then the $\mathscr{F}$-rule is a mapping which assigns to the set of sentences $\{\Phi(n)\}_{n\in A}$ where $A\in\mathscr{F}$, the sentence $(\exists X)(\varphi(X)\wedge\forall x\,(x\in X\rightarrow\Phi(x)))$.

Def-rule. From $\{\vdash_{\mathrm{Def}} \exists v\big(\forall x\big(v(x)=0 \leftrightarrow \Phi(x)\big) \wedge \psi(v)\big)$ for all Φ with one free variable infer $\forall v\psi v$.

THEOREM 4. (i) *The pointwise definable models of* (A) *form a semantics for the* Def-rule.

(ii) *Every* Def-*consistent set of sentences has a pointwise definable model.*

Proof. Let $\Gamma=\{\exists x\,[(v(x)=0 \wedge \neg\Phi(x)) \wedge (v(x) \neq 0 \wedge \Phi(x)]: \Phi$ is a formula with one free variable}. Observe that a model M for $L(A)$ is pointwise definable iff it omits the type Γ. We proceed further analogously as in the proof of Henkin—Orey theorem.

The Def-rule is sound in the standard model of (A) iff the analytical basis theorem (every non-empty analytical family of unary functions has an analytical element) holds, which is known to be independent from the axioms of set theory.

4. Searching a satisfactory syntactical β-rule. It seems that the question raised by Mostowski in [4] about the existence of a syntactical β-rule should be formulated in the following way: does there exist an infinitistic rule of proof f such that the class of all β-models of (A) forms a semantics of this rule and every f-consistent set of sentences has a β-model?

This rule should be of course in a certain sense natural (we are not able to fomulate criteria which could decide whether a rule of proof is "natural" or not).

We shall construct now a rule of proof such that the closure of (A) under this rule is equal to A_β, i.e. to the set of all sentences of L (A) true in all β-models of (A).

We think that this rule is a good example of an "unnatural" one, because the connection between premises and conclusion is very artificial.

Consider the following

$\mathcal{B}$-rule: if $\forall \alpha\, \exists \beta\, \forall n \vdash_{\mathcal{B}} \Phi(\langle \bar{\alpha}(n), \bar{\beta}(n)\rangle)$ then $\vdash_{\mathcal{B}} \forall v\, \exists u x \Phi(\langle \bar{v}(x), u(x)\rangle)$.

Thus the $\mathcal{B}$-rule is an $\mathcal{F}$-rule for

$$\mathcal{F}(a) \leftrightarrow \forall \beta\, \exists \gamma\, \forall x\big(a(\langle \bar{\beta}(x), \bar{\gamma}(x)\rangle)=0\big).$$

Let φ be a true π_2^1 sentence of L (A). Then φ is of the form $\forall v\, \exists u\, \forall x\, \Phi\big(\langle \bar{v}(x), \bar{u}(x)\rangle\big)$, where Φ is an open formula. Since true open formulas of $L(A)$ are theorems of (A), we get $\varphi \in (\mathrm{A})_{\mathcal{B}}$.

Recall that a set S of natural numbers is weakly representable in a set of sentences T if for some formula Φ with one free number variable

$$\forall n\big(n \in S \leftrightarrow \Phi(\boldsymbol{n}) \in T\big).$$

By the above, every π_2^1 set is weakly representable in $(\mathrm{A})_{\mathcal{B}}$. $(\mathrm{A})_{\mathcal{B}}$ can be defined as the intersection of all sets of sentences which are closed under the $\mathcal{B}$-rule which gives that $\ulcorner(\mathrm{A})_{\mathcal{B}}\urcorner$ is a π_2^1 set. This fact together with the former one implies $\ulcorner(\mathrm{A})_{\mathcal{B}}\urcorner$ is a complete π_2^1 set.

A_β is also a complete π_2^1 set ([4]). By Myhill's isomorphism theorem there exists a 1—1 and onto recursive function $h(x)$ such that

$$(*) \qquad \forall x \left(x \in \ulcorner (\mathrm{A})_{\mathcal{B}} \urcorner \leftrightarrow h(x) \in \ulcorner \mathrm{A}_\beta \urcorner \right).$$

The index of this function can be found, because the sets $\ulcorner (\mathrm{A})_{\mathcal{B}} \urcorner$ and A_β have explicit definitions, which implies that the indices of the functions which 1—1 reduce one set to the other are given. Now the analysis of the proof of Myhill's theorem shows that the function $h(x)$ is effectively dependent only on these two functions.

Define now a $\mathcal{C}$-rule of proof by the following closure condition: a set T of sentences is closed under the C-rule if

1° $\forall \alpha \exists \beta \forall n \left(h\left(\ulcorner \Phi(\langle \bar{\alpha}(n), \bar{\beta}(n)\rangle)\urcorner\right) \in \ulcorner T \urcorner\right.$ and $h\left(\ulcorner \forall v \exists u \forall x \Phi(\langle \bar{v}(x), \bar{u}(x)\rangle)\urcorner\right)$ is the Gödel number of a sentence implies $h\left(\ulcorner \forall v \exists u \forall x\, \Phi(\langle \bar{v}(x), \bar{u}(x)\rangle)\urcorner\right) \in \ulcorner T \urcorner$.

2° $h(\ulcorner \psi \urcorner) \in \ulcorner T \urcorner \wedge h(\psi \rightarrow \ulcorner \varphi \urcorner) \in \ulcorner T \urcorner$ implies $h(\ulcorner \varphi \urcorner) \in \ulcorner T \urcorner$.

Using $(*)$ and the induction on the levels of the construction of $(\mathrm{A})_{\mathcal{B}}$ and $(\mathrm{A})_{\mathcal{C}}$ it is easy to see that $(\mathrm{A})_{\mathcal{C}} = \mathrm{A}_\beta$. We do not know whether h is identity. Probably not.

The example of the $\mathcal{C}$-rule suggests that the existence of an artificial β-rule is very probable. The heart of the problem thus rather lies in the finding of a natural β-rule.

One thing is certain. No β-rule can be simple. It follows from the following restriction "from below". Every rule f determines an inductive definition of the set of f-consequences. If the graph of f is a Σ_2^1 relation then the set $(\mathrm{A})_f$ is inductively defined with respect to a Σ_2^1 relation, thus is a Σ_2^1 set. (This follows from an unpublished result of Gandy). Thus the graph of the β-rule cannot be a Σ_2^1 relation. It can be π_2^1 and we should search among such rules.

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Addendum. P. Aczel has recently proved that the function h from page 885 is not identity. We generalized his result as: for no $\mathcal{F}$-rule $A_\beta \subset (A)_{\mathcal{F}}$. On the other hand, we constructed a syntactical β-rule with a π_2^1 graph (see [12]).

К. Р. Арт, **Бесконечные правила вывода и их семантика**

Содержание. В настоящей работе рассматриваются проблемы, связанные с семантикой бесконечных правил вывода в доказательствах, сформулируемых на языке второго порядка арифметики. Отрицательно решаются два вопроса Эндертона, касающиеся семантики его $\mathscr{A}$-правила вывода. В казлючение обсуждается проблема существования удовлетворительного синтаксического β-правила вывода.