**ω-Models in Analytical Hierarchy**

by

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**Summary.** We consider ω-models of the second order arithmetic $A_2$, which are the subsets of the class of analytical sets. It is proved that there exist ω-models of $A_2$, such that $\mathfrak{A} \cap (\Pi_1^1 \cup \Sigma_1^1) = \Delta_1^1$ and $\mathfrak{A} \cup (\Pi_1^1 \cup \Sigma_1^1) \not\subseteq A_2$, and that there are no ω-models which lie lower in the analytical hierarchy. It is shown that there exist ω-models of $A_2$ in which every set is analytical but no code is analytical. Finally, we prove some results concerning hyperdegrees, e.g. the existence of "many" uncountable antichains of hyperdegrees.

1. **Introduction.** We consider the second order arithmetic $A_2$ (with the set variables) Every ω-model is identified with its class of sets, and so is a subset of the power set of ω.

It seems sensible to look for such ω-models which are least complicated in their structure. We must look for such ω-models among those that are the subsets of the class of all analytical sets, because only in this case we can examine the complexity of the structure of the ω-model.

The complication of the structure of such ω-models we may understand in two ways: as a level of the analytical hierarchy, of which the $\mathfrak{A}_{\Delta_1^1}$ is already a subset, or as the least level of the analytical hierarchy to which a certain code of the ω-model belongs. We show that these possibilities are quite different.

By $N_X$ we denote the set $\{(X)_k : k < \omega\}$, where $(X)_k = \{y : 2^k \cdot y \in X\}$.

If $\mathfrak{A}$ is a countable ω-model, then every set $X$ such that $N_X = \mathfrak{A}$ is called the code of $\mathfrak{A}$.

Terminology and notations are well-known in the recursion theory. In particular $X \subseteq_h Y$ denotes that $X$ is hyperarithmetical in $Y$, $X$ denotes the hyperdegree of the set $X$.

By 0 we denote the hyperdegree of the hyperarithmetical sets, and by $0'$ its hyperjump.

2. **ω-models of $A_2$.** Every ω-model of $A_2$ contains all $\Delta_1^1$ sets. By a theorem of Kleene (see [1]) $\Delta_1^1$ does not satisfy the comprehension schema, i.e. it is not the ω-model of $A_2$.

The next level of the analytical hierarchy, which could contain an ω-model of $A_2$ is $\Pi_1^1 \cup \Sigma_1^1$. However we have the following simple...
THEOREM 1. If \( \mathcal{U} \subseteq \Pi^1_1 \cup \Sigma^1_1 \), then \( \mathcal{U} \) is not an \( \omega \)-model of \( A_2 \).

Proof. Suppose, on the contrary, that \( \mathcal{U} \models A_2 \). By the above remarks there exists the set \( X \in \Pi^1_1 - \Sigma^1_1 \), such that \( X \in \mathcal{U} \). By the comprehension schema the set \( Y = \{ 2k : k \in X \} \cup \{ 2k+1 : k \notin X \} \) belongs to \( \mathcal{U} \). Hence \( Y \in \Pi^1_1 \cup \Sigma^1_1 \), which implies that \( X \in \Delta^1_1 \), which is a contradiction.

Spector proved (see e.g. theorem XXX II p. 411 in [2]) that all \( \Pi^1_1 - \Sigma^1_1 \) sets have the same hyperdegree, namely \( \text{O} \). Every \( \omega \)-model of \( A_2 \) is hyperarithmetically closed (i.e. \( \mathcal{U} \models A_2, X \in \mathcal{U}, Y \in \mathcal{U} \) implies that \( Y \in \mathcal{U} \)) and therefore we have the following.

Remark. If an \( \omega \)-model of \( A_2 \) contains a set from \( \Pi^1_1 - \Sigma^1_1 \) (or from \( \Sigma^1_1 - \Pi^1_1 \)), then it contains every set from \( \Pi^1_1 \cup \Sigma^1_1 \).

The next level of the analytical hierarchy is \( \Delta^1_1 \). In this case we have already got the positive result. Namely we prove

THEOREM 2. There exists an \( \omega \)-model of \( A_2 \) \( \mathcal{U} \) such that \( \mathcal{U} \) has \( \Delta^1_2 \) code and

\[
\mathcal{U} \cap (\Pi^1_1 \cup \Sigma^1_1) = \Delta^1_2 \quad \text{and} \quad \mathcal{U} \cup (\Pi^1_1 \cup \Sigma^1_1) \neq \Delta^1_2.
\]

Proof. It follows from Mostowski [3], that the relation \( \text{Mod} \omega (X) \leftrightarrow N_x \models A_2 \) is a \( \Delta^1_2 \) one. By the Skolem-Löwenheim theorem \( \exists X \text{ Mod} \omega (X) \), hence by Gandy's basis theorem (see [4]) there exists a set \( X \), such that \( X <_h 0^\prime \) and \( \text{Mod} \omega (X) \). \( X \) is \( \Delta^1_1 \) in \( \Pi^1_1 \) sets, and so is \( \Delta^1_2 \) set (see e.g. the exercise 18 b, p. 198 in [5]). If \( Y \in N_x \), then \( Y <_h 0^\prime \), i.e. \( N_x \) does not contain the sets from \( 0^\prime \). This means, by the above remark, that \( N_x \cap (\Pi^1_1 \cup \Sigma^1_1) = \Delta^1_1 \).

It suffices now to show that \( \mathcal{U} \models A_2, \mathcal{U} \subseteq \Delta^1_2 \) and

\[ \mathcal{U} \cap (\Pi^1_1 \cup \Sigma^1_1) = \Delta^1_2 \quad \text{implies that} \quad \mathcal{U} \cup (\Pi^1_1 \cup \Sigma^1_1) \neq \Delta^1_2. \]

Let \( Y \in 0^\prime \). Then

\[ Y \in \Delta^1_2 - (\Pi^1_1 \cup \Sigma^1_1) \quad \text{and} \quad \mathcal{U} \neq Y. \]

The \( \omega \)-model obtained in the above Theorem has the least complicated structure in every sense. Its code lies as low as possible in the analytical hierarchy and in the structure of the hyperdegrees (in the sense of the levels determined by \( 0, 0^\prime, 0^\prime\prime, \ldots \)).

The following two theorems show that both notions of the complication of the structure of the \( \omega \)-model are quite different.

THEOREM 3. There exists an \( \omega \)-model of \( A_2 \) \( \mathcal{U} \) which satisfies the conditions (*) from the Theorem 2, but has no \( \Delta^1_2 \) code.

Proof. Let \( \text{AC} \) denote the choice schema for \( A_3 \) (i.e. the following one:

\[ \forall x \exists y \phi (x, y) \rightarrow \exists y \forall x \phi (x, y). \]

By [3] the relation \( \text{Mod}^\omega (X) \leftrightarrow N_x \models A_2 + \text{AC} \) is \( \Delta^1_1 \) one. From the proof of Theorem 2 there exists \( X \in \Delta^1_2 \), such that \( N_x \models A_2 + \text{AC} \) and \( N_x \cap (\Pi^1_1 \cup \Sigma^1_1) = \Delta^1_1 \).

The relation \( \text{EL} (X, Y) \leftrightarrow N_x <_h N_y \) is \( \Delta^1_1 \) one (see Mostowski [6]), hence the relation \( \text{ELM} (X, Y) \leftrightarrow N_x \models A_2 + \text{AC} \) and \( \text{EL} (X, Y) \) and \( N_y \cap (\Pi^1_1 \cup \Sigma^1_1) = \Delta^1_1 \) is \( \Delta^1_1 \) one. Indeed, provided \( N_y \models A_3 \), we have on the base of Remark:

\[ N_y \cap (\Pi^1_1 \cup \Sigma^1_1) = \Delta^1_1 \leftrightarrow \forall k ((Y)_k \neq Z), \]

where \( Z \in \Pi^1_1 - \Sigma^1_1 \), hence it is easily seen that it is \( \Delta^1_2 \) relation with respect to \( Y \).
Keisler proved in [7] (p. 155) that (we give a weaker version of the theorem) if \( \mathfrak{A} \) is a countable \( \omega \)-model of \( A_2 + AC \) and \( \mathcal{C} \subseteq 2^\omega \) is a countable set, such that \( \mathfrak{A} \cap \mathcal{C} = \emptyset \), then there exists a \( \omega \)-model \( \mathfrak{B} \), such that \( \mathfrak{A} \not\subseteq \mathfrak{B} \) and \( \mathfrak{B} \cap \mathcal{C} = \emptyset \). This implies that \( \exists Y \) ELM \((X_0, Y)\). Using the relativized form of the Kondo—Addison basis theorem (see [2]) we obtain the set \( X_1 \) which is \( A_1^1 \) in \( X_0 \) (i.e. really \( A_1^1 \) one) and such that ELM \((X_0, X_1)\). Iterating this procedure we get the sequence \( X_0, X_1, \ldots \) of \( A_1^1 \) sets such that \( \text{ELM} (X_n, X_{n+1}) \) for \( n \geq 0 \). By Tarski’s Lemma, \( \bigcup_{n<\omega} N_{X_n} \models A_2 + AC \).

Suppose that the thesis is false. Because \( \bigcup_{n<\omega} N_{X_n} \) satisfies the conditions (\( * \)) (see the proof of Theorem 2), there exists the set \( X_{\omega} \in A_1^1 \) such that \( N_{X_0} = \bigcup_{n<\omega} N_{X_n} \).

Continuing this procedure we obtain a strictly increasing elementary chain of \( \omega \)-models of the length \( \omega_1 \), whose sum is the subset of \( A_1^1 \), which gives a contradiction.

**Theorem 4.** There exists an \( \omega \)-model of \( A_2 \) which has no analytical code, but is a subset of the class of analytical sets.

**Proof.** Suppose that the thesis is false. We show, that then exists a strictly increasing elementary chain of \( \omega \)-models \( A_2 + AC \) of the length \( \omega_1 \), such that the code of each of these \( \omega \)-models is analytical, which of course is impossible. Let \( N_{X_1} \models A_2 + AC \) for some \( X_1 \in A_1^1 \), where \( k \geq 2 (\tau < \omega_1) \). On the ground of the theorems of Keisler and Kondo—Addison there exists the set \( X_{\omega+1} \) which is \( A_1^1 \) in \( A_1^1 \) sets, (i.e. is \( A_1^1 \) set) such that \( \text{EL} (X_0, X_{\omega+1}) \).

On the limit steps the sum of \( \omega \)-models constructed till now has (by our assumption) the analytical code, because on the base of Tarski’s Lemma it is a model of \( A_2 + AC \).

It seems that the following conjecture is true.

**Conjecture.** There exists an \( \omega \)-model of \( A_2 \) \( \mathfrak{A} \) such that for some \( k \) \( \mathfrak{A} \subseteq A_1^1 \), but \( \mathfrak{A} \) has no analytical code. However the author was unable to prove it.

No level of the analytical hierarchy is \( \omega \)-model of \( A_2 \). Namely, we have the following

**Theorem 5.** a). \( \Pi_4^1 \cup \Sigma_4^1 \) is not the \( \omega \)-model of \( A_2 \) (\( n \geq 1 \)).

b). \( A_1^1 \) is not the \( \omega \)-model of \( A_2 \).

c). If every non-empty \( \Sigma_{n+1}^1 \) relational has \( A_{n+1}^1 \) set as its element, then \( A_{n+1}^1 \) is not the \( \omega \)-model of \( A_2 \).

d). Boolean closure of \( \Pi_4^1 \cup \Sigma_4^1 \) \((\Pi_4^1 \cup \Sigma_4^1)_{\text{B}}\) is not the \( \omega \)-model of \( A_2 \) (\( n \geq 1 \)).

The condition in c) follows from the axiom of constructibility, (see [8]).

**Proof.** a) The proof is analogous to the proof of Theorem 1, b) and c) see [9]. d) Suppose that \((\Pi_4^1 \cup \Sigma_4^1)\) is the \( \omega \)-model of \( A_2 \). Then it is hyperarithmetically closed. If \( X \in (\Pi_4^1 \cup \Sigma_4^1)_{\text{B}} \) then \( X \) is recursive in a \( \Pi_4^1 \) complete set, say \( P_n \). Hence \((\Pi_4^1 \cup \Sigma_4^1)_{\text{B}}\) would be just the collection of all \( A_1^1 \) sets in \( P_n \), what would contradict the relativized form of Kleene’s theorem, that \( A_1^1 \) is not the \( \omega \)-model of \( A_2 \).
3. Hyperdegrees. Up to now we have used the connection between models and hyperdegrees only in one direction. Now we show how by using the theorems about \( \omega \)-models we can examine hyperdegrees.

THEOREM 6. Let \( \mathbb{C} \) be the countable set of nonhyperarithmetic sets. Then there exists a set \( A \) which is hyperarithmetically incomparable with every set from \( \mathbb{C} \).

Proof. By a theorem of Mostowski (see [10]) there exists the countable \( \omega \)-model of \( A_2 \mathcal{M} \), such that \( \mathcal{M} \cap \mathbb{C} = \emptyset \). From the already cited theorem of Keisler there exists \( \omega \)-model \( \mathcal{B} \) of power \( \omega \), such that \( \mathcal{M} \subseteq \mathcal{B} \) and \( \mathcal{B} \cap \mathbb{C} = \emptyset \). The set \( D = \{ Z : Z \leq_n C \text{ for some } C \in \mathbb{C} \} \) is the countable one. Hence there exists \( A \in \mathcal{B} - D \). If \( Z \leq_n A \) then \( Z \in \mathcal{B} \), what shows that \( A \) is the required set.

THEOREM 7. If \( X \neq \emptyset \), then there exists the antichain of hyperdegrees of power \( \omega \), which has \( X \) as its element.

Proof. By Zorn's lemma there exists a maximal antichain of hyperdegrees which has \( X \) as its element. This antichain cannot be countable on the ground of Theorem 6.

REFERENCES