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MATHEMATICS
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ω-Models in Analytical Hierarchy

by

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Summary. We consider ω-models of the second order arithmetic A_2, which are the subsets of the class of analytical sets. It is proved that there exist ω-models of A_2 $\mathfrak{A}$, such that $\mathfrak{A}\cap(\Pi_1^1\cup\Sigma_1^1)=\Delta_1^1$ and $\mathfrak{A}\cup(\Pi_1^1\cup\Sigma_1^1)\nsubseteq\Delta_2^1$, and that there are no ω-models which lie lower in the analytical hierarchy. It is shown that there exist ω-models of A_2 in which every set is analytical but no code is analytical. Finally, we prove some results concerning hyperdegrees, e.g. the existence of "many" uncountable antichains of hyperdegrees.

1. Introduction. We consider the second order arithmetic A_2 (with the set variables) Every ω-model is identified with its class of sets, and so is a subset of the power set of ω.

It seems sensible to look for such ω-models which are least complicated in their structure. We must look for such ω-models among those that are the subsets of the class of all analytical sets, because only in his case we can examine the complication of the structure of the ω-model.

The complication of the structure of such ω-models we may understand in two ways: as a level of the analytical hierarchy, of which the ω-model is already a subset, or as the least level of the analytical hierarchy to which a certain code of the ω-model belongs. We show that these possibilities are quite different.

By N_X we denote the set $\{(X)_k : k<\omega\}$, where $(X)_k=\{y : 2^k\cdot 3^y\in X\}$.

If $\mathfrak{A}$ is a countable ω-model, then every set X such that $N_X=\mathfrak{A}$ is called the code of $\mathfrak{A}$.

Terminology and notations are well-known in the recursion theory. In particular $X\leqslant_h Y$ denotes that X is hyperarithmetical in Y, X denotes the hyperdegree of the set X.

By 0 we denote the hyperdegree of the hyperarithmetical sets, and by $0'$ its hyperjump.

2. ω-models of A_2. Every ω-model of A_2 contains all Δ_1^1 sets. By a theorem of Kleene (see [1]) Δ_1^1 does not satisfy the comprehension schema, i.e. it is not the ω-model of A_2.

The next level of the analytical hierarchy, which could contain an ω-model of A_2 is $\Pi_1^1\cup\Sigma_1^1$. However we have the following simple

THEOREM 1. *If* $\mathfrak{A} \subseteq \Pi_1^1 \cup \Sigma_1^1$, *then* $\mathfrak{A}$ *is not an* ω*-model of* A_2.

Proof. Suppose, on the contrary, that $\mathfrak{A} \models A_2$. By the above remarks there exists the set $X \in \Pi_1^1 - \Sigma_1^1$, such that $X \in \mathfrak{A}$. By the comprehension schema the set $Y = \{2k : k \in X\} \cup \{2k+1 : k \notin X\}$ belongs to $\mathfrak{A}$. Hence $Y \in \Pi_1^1 \cup \Sigma_1^1$, which implies that $X \in \Delta_1^1$, which is a contradiction.

Spector proved (see e.g. theorem XXX II p. 411 in [2]) that all $\Pi_1^1 - \Sigma_1^1$ sets have the same hyperdegree, namely $0'$. Every ω-model of A_2 is hyperarithmetically closed (i.e. $\mathfrak{A} \models A_2$, $X \in \mathfrak{A}$, $Y \leqslant_h X$ implies that $Y \in \mathfrak{A}$) and therefore we have the following.

Remark. If an ω-model of A_2 contains a set from $\Pi_1^1 - \Sigma_1^1$ (or from $\Sigma_1^1 - \Pi_1^1$), then it contains every set from $\Pi_1^1 \cup \Sigma_1^1$.

The next level of the analytical hierarchy is Δ_2^1. In this case we have already got the positive result. Namely we prove

THEOREM 2. *There exists an* ω*-model of* A_2 $\mathfrak{A}$ *such that* $\mathfrak{A}$ *has* Δ_2^1 *code and*

$$(*)\qquad \mathfrak{A} \cap (\Pi_1^1 \cup \Sigma_1^1) = \Delta_1^1 \quad \text{and} \quad \mathfrak{A} \cup (\Pi_1^1 \cup \Sigma_1^1) \nsubseteq \Delta_2^1.$$

Proof. It follows from Mostowski [3], that the relation $\mathrm{Mod}\,\omega\,(X) \leftrightarrow N_X \models A_2$ is a Δ_1^1 one. By the Skolem-Löwenheim theorem $\exists X\, \mathrm{Mod}\,\omega\,(X)$, hence by Gandy's basis theorem (see [4]) there exists a set X, such that $X <_h 0'$ and $\mathrm{Mod}\,\omega\,(X)$. X is Δ_1^1 in Π_1^1 sets, and so is Δ_2^1 set (see e.g. the exercise 18 b, p. 198 in [5]). If $Y \in N_X$, then $Y <_h 0'$, i.e. N_X does not contain the sets from $0'$. This means, by the above remark, that $N_X \cap (\Pi_1^1 \cup \Sigma_1^1) = \Delta_1^1$.

It suffices now to show that $\mathfrak{A} \models A_2$, $\mathfrak{A} \subseteq \Delta_2^1$ and

$$\mathfrak{A} \cap (\Pi_1^1 \cup \Sigma_1^1) = \Delta_1^1 \quad \text{implies that} \quad \mathfrak{A} \cup (\Pi_1^1 \cup \Sigma_1^1) \nsubseteq \Delta_2^1.$$

Let $Y \in 0''$ Then.

$$Y \in \Delta_2^1 - (\Pi_1^1 \cup \Sigma_1^1) \quad \text{and (because } 0' <_h Y) \quad Y \notin \mathfrak{A}.$$

The ω-model obtained in the above Theorem has the least complicated structure in every sense. Its code lies as low as possible in the analytical hierarchy and in the structure of the hyperdegrees (in the sense of the levels determined by $0, 0', 0'', \ldots$).

The following two theorems show that both notions of the complication of the structure of the ω-model are quite different.

THEOREM 3. *There exists an* ω*-model of* A_2 $\mathfrak{A}$ *which satisfies the conditions* $(*)$ *from the Theorem* 2, *but has no* Δ_2^1 *code.*

Proof. Let AC denote the choice schema for A_2 (i.e. the following one: $\forall x \exists Y \varphi(x, Y) \rightarrow \exists Y \forall x \varphi(x, (Y)_x)$).

By [3] the relation $\mathrm{Mod}'\,\omega\,(X) \leftrightarrow N_X \models A_2 + \mathrm{AC}$ is Δ_1^1 one. From the proof of Theorem 2 there exists $X_0 \in \Delta_2^1$, such that $N_{X_0} \models A_2 + \mathrm{AC}$ and $N_{X_0} \cap (\Pi_1^1 \cup \Sigma_1^1) = \Delta_1^1$.

The relation $\mathrm{EL}\,(X, Y) \leftrightarrow N_X \prec N_Y$ is Δ_1^1 one (see Mostowski [6]), hence the relation $\mathrm{ELM}\,(X, Y) \leftrightarrow N_X \models A_2 + \mathrm{AC}$ and $\mathrm{EL}\,(X, Y)$ and $N_Y \cap (\Pi_1^1 \cup \Sigma_1^1) = \Delta_1^1$ is Δ_2^1 one. Indeed, provided $N_Y \models A_2$, we have on the base of Remark:

$$N_Y \cap (\Pi_1^1 \cup \Sigma_1^1) = \Delta_1^1 \leftrightarrow \forall k\, ((Y)_k \neq Z),$$

where $Z \in \Pi_1^1 - \Sigma_1^1$, hence it is easily seen that it is Δ_2^1 relation with respect to Y.

Keisler proved in [7] (p. 155) that (we give a weaker version of the theorem) if $\mathfrak{A}$ is a countable ω-model of A_2+AC and $\mathfrak{C} \subseteq 2^\omega$ is a countable set, such that $\mathfrak{A} \cap \mathfrak{C} = \emptyset$, then there exists a ω-model $\mathfrak{B}$, such that $\mathfrak{A} \prec \mathfrak{B}$ and $\mathfrak{B} \cap \mathfrak{C} = \emptyset$. This implies that $\exists Y\, \mathrm{ELM}(X_0, Y)$. Using the relativized form of the Kondo—Addison basis theorem (see [2]) we obtain the set X_1 which is Δ_2^1 in X_0 (i.e. really Δ_2^1 one) and such that ELM (X_0, X_1). Iterating this procedure we get the sequence $X_0, X_1, \ldots$ of Δ_2^1 sets such that ELM (X_n, X_{n+1}) for $n \geq 0$. By Tarski's Lemma, $\bigcup_{n<\omega} N_{X_n} \models A_2+AC$.

Suppose that the thesis is false. Because $\bigcup_{n<\omega} N_{X_n}$ satisfies the conditions (*) (see the proof of Theorem 2), there exists the set $X_\omega \in \Delta_2^1$ such that $N_{X_\omega} = \bigcup_{n<\omega} N_{X_n}$. Continuing this procedure we obtain a strictly increasing elementary chain of ω-models of the length ω_1, whose sum is the subset of Δ_2^1, which gives a contradiction.

THEOREM 4. *There exists an ω-model of A_2 which has no analytical code, but is a subset of the class of analytical sets.*

Proof. Suppose that the thesis is false. We show, that then exists a strictly increasing elementary chain of ω-models A_2+AC of the length ω_1, such that the code of each of these ω-models is analytical, which of course is impossible. Let $N_{X_\tau} \models A_2$+AC for some $X_\tau \in \Delta_k^1$, where $k \geq 2$ $(\tau < \omega_1)$. On the ground of the theorems of Keisler and Kondo-Addison there exists the set $X_{\tau+1}$ which is Δ_2^1 in Δ_k^1 sets, (i.e. is Δ_k^1 set) such that EL $(X_\tau, X_{\tau+1})$.

On the limit steps the sum of ω-models constructed till now has (by our assumption) the analytical code, because on the base of Tarski's Lemma it is a model of A_2+AC.

It seems that the following conjecture is true.

CONJECTURE. *There exists an ω-model of A_2 $\mathfrak{A}$ such that for some k $\mathfrak{A} \subseteq \Delta_k^1$, but $\mathfrak{A}$ has no analytical code.* However the author was unable to prove it.

No level of the analytical hierarchy is ω-model of A_2. Namely, we have the following

THEOREM 5. a). *$\Pi_n^1 \cup \Sigma_n^1$ is not the ω-model of A_2 $(n \geq 1)$.*

b). *Δ_2^1 is not the ω-model of A_2.*

c). *If every non-empty Σ_{n+1}^1 relational has Δ_{n+1}^1 set as its element, then Δ_{n+1}^1 is not the ω-model of A_2.*

d). Boolean closure of $\Pi_n^1 \cup \Sigma_n^1$ $((\Pi_n^1 \cup \Sigma_n^1)_B)$ is not the ω-model of A_2 $(n \geq 1)$.

The condition in c) follows from the axiom of constructibility, (see [8]).

Proof. a) The proof is analogous to the proof of Theorem 1, b) and c) see [9]. d) Suppose that $(\Pi_n^1 \cup \Sigma_n^1)$ is the ω-model of A_2. Then it is hyperarithmetically closed. If $X \in (\Pi_n^1 \cup \Sigma_n^1)_B$ then X is recursive in a Π_n^1 complete set, say P_n. Hence $(\Pi_n^1 \cup \Sigma_n^1)_B$ would be just the collection of all Δ_1^1 sets in P_n, what would contradict the relativized form of Kleene's theorem, that Δ_1^1 is not the ω-model of A_2.

3. Hyperdegrees. Up to now we have used the connection between models and hyperdegrees only in one direction. Now we show how by using the theorems about ω-models we can examine hyperdegrees.

THEOREM 6. *Let $\mathfrak{C}$ be the countable set of nonhyperarithmetic sets. Then there exists a set A which is hyperarithmetically incomparable with every set from $\mathfrak{C}$.*

Proof. By a theorem of Mostowski (see [10]) there exists the countable ω-model of A_2 $\mathfrak{A}$, such that $\mathfrak{A} \cap \mathfrak{C} = \emptyset$. From the already cited theorem of Keisler there exists ω-model $\mathfrak{B}$ of power ω_1, such that $\mathfrak{A} \prec \mathfrak{B}$ and $\mathfrak{B} \cap \mathfrak{C} = \emptyset$. The set $D = \{Z: Z \leqslant_h C$ for some $C \in \mathfrak{C}\}$ is the countable one. Hence there exists $A \in \mathfrak{B} - D$. If $Z <_h A$ then $Z \in \mathfrak{B}$, what shows that A is the required set.

THEOREM 7. *If $X_h > 0$, then there exists the antichain of hyperdegrees of power ω_1 which has X as its element.*

Proof. By Zorn's lemma there exists a maximal antichain of hyperdegrees which has X as its element. This antichain cannot be countable on the ground of Theorem 6.

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К. Р. Апт, **ω-модели в аналитической иерархии**

Содержание. В работе рассматриваются ω-модели арифметики второго порядка A_2, которые являются подмножествами класса всех аналитических множеств. Доказано, что существуют ω-модели A_2 $\mathfrak{a}$ такиеж чето $\mathfrak{a} \cap (\pi_1^1 \cup \Sigma_1^1) = \Delta_1^1$ и $\mathfrak{a} \cup (\pi_1^1 \cup \Sigma_1^1) \nsubseteq \Delta_2^1$ и что нет ω-моделей, которые лежат ниже в аналитичесюкой иерархии. Показано, что существуют ω-модели A_2, не имеющие аналитического кода, но в которых каждое множество аналитическое. В заключение работы, доказаны некоторые теоремы о гиперступенях, как например существование неперечислимых актицепей гиперступеней.