

SECOND ORDER ARITHMETIC AND RELATED TOPICS *

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Received 1 November 1973

*... each problem they solve
creates ten problems more.*

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Introduction

Second order arithmetic A_2 is a theory which considers the properties of sets of natural numbers. Being an extension of Peano arithmetic, the second order arithmetic is a theory in which one can already state the induction axiom in the desired form. On the other hand, the restrictions of the language force us to construct the sets by using the comprehension scheme.

Although A_2 seems to be a fairly weak system, it is already possible to formalize within it quite a big part of classical mathematics. It is even possible to do that in some subsystems of A_2 with the scheme of comprehension suitably restricted. In this paper we do not consider the above subject and we always assume the full comprehension scheme.

A number of results concern the second order arithmetic with the scheme of choice. The intermediate systems – almost unknown so far¹ are not considered here.

* This paper is an extended version of a lecture which was read by the second author at the Logical Congress in Orleans, France. While preparing this lecture, the second author was partially supported by the S.U.N.Y. Research Foundation.

¹ Recently, important results in this direction have been obtained by W. Guzicki. Among others he has shown that the choice scheme without parameters does not imply in A_2^- , the choice scheme for Π_2^1 formulas with parameters. On the other hand, it is known that the choice scheme for Σ_2^1 formulas (with parameters) is provable in A_2^- (by the Kondo–Addison theorem which is, as a scheme, provable in A_2^-).

By the theorem of Gödel, second order arithmetic is incomplete, and thus it has models elementarily non-equivalent to the standard one $\mathcal{P}(\omega)$. A theorem of Rosser [59] states that such models can be found even among those which have standard natural numbers, i.e., the so-called ω -models. Thus ω -models form a large class of models of A_2 .

The aims for which second order arithmetic was created – a study of the properties of sets of natural numbers – suggest the restriction to ω -models in semantical considerations. In these models, sets are sets of natural numbers. Almost all the models studied in this paper are ω -models. These models preserve first order, i.e., predicative statements.

A_2 is an intermediate system between Peano arithmetic and set theory. Although it is much stronger than the former, yet it lacks some of the means available in the latter. For instance, we cannot yet define the notion of an ordinal. Fortunately, we may use the notion of well-ordering instead, which suffices in many considerations. Their properties, being similar to those of ordinals (for instance comparability), allow us to constructions like inductive definability, etc.

The importance of the notion of well-ordering suggests that we should consider those models of second order arithmetic for which this notion is absolute. These models, introduced by Mostowski, are called β -models.

They form a subclass of the class of ω -models and may be compared to transitive models of set theory (some more precise explanation of this analogy is contained in §6).

These two classes of models play the most important role in the study of models of A_2 , and considerations about them cover the main part of the paper.

The reader may sometimes need to consult Rogers' book [58] when we use recursion-theoretical results and Spector's paper [65] when we use results on inductive definitions.

We have divided the paper into 10 sections. Sections 1–10 start with a short summary together with an indication of the most important methods and results, thus enabling the reader to get a summary of the paper by reading them separately.

§0. Preliminaries and basic notions

The second order arithmetic is a theory formulated in a two sorted language $L(A_2)$ with equality and with the basic symbols $S, +, <, \cdot, \in$.

Small Latin letters denote variables ranging over natural numbers and capital Latin letters those ranging over sets of natural numbers. The axioms of second order arithmetic are the following:

(A) Peano's axioms for natural numbers (cf. Shoenfield [62, p. 22 and p. 204]).

(B) Axiom of extensionality.

(C) Induction axiom, i.e., $(0 \in X \ \& \ (x)(x \in X \rightarrow Sx \in X) \rightarrow (x)(x \in X))$.

(D) Comprehension scheme. Universal closure of the following:

$$(\exists X)(x)(x \in X \leftrightarrow \Phi(x)) \quad (X \text{ is not free in } \Phi).$$

(E) Scheme of choice. Universal closure of the following:

$$(x)(\exists Y)(\Phi(x, Y) \rightarrow (\exists Y)(x)\Phi(x, (Y)_x)),$$

where $(Y)_x = \{y: 2^x(2y+1) \div 1 \in Y\}$.

The theory based on axioms (A)–(E) is called full second order arithmetic and abbreviated throughout the paper as A_2 . The subtheory based on groups (A)–(D) is denoted by A_2^- . Strictly speaking, all our work is done in a certain recursive extension $A_2' (A_2'^-)$ of $A_2 (A_2^-)$ satisfying the following three conditions (cf. [62, p. 206] for the definition of a recursive extension):

(1) All the functions and the predicates defined in Paragraphs 6.4 and 6.6 of Shoenfield's book are included into the language. For the terms and the formulas denoting these – newly introduced – functions and predicates, we use the same denotations e.g. $\text{Seq}(x)$ corresponds to the predicate $\lambda x \text{Seq}(x)$.

(2) There is a binary function symbol $\bar{\cdot}(\cdot)$ with the following axiom:

$$\bar{X}(y) = a \leftrightarrow \text{Seq}(a) \ \& \ (\text{lh}(a) = y)$$

$$\ \& \ (i)(i < y \rightarrow [(i \notin X \rightarrow (a)_i = 1) \ \& \ (i \in X \rightarrow (a)_i = 0)]).$$

Informally speaking, $\bar{X}(y)$ is nothing else but $\bar{\chi}_X(y)$, where χ_X is the characteristic function of the set X .

(3) The functions $(X)_i, \langle X_1, \dots, X_n \rangle$ are introduced by appropriate

axioms e.g.

$$\langle X_1 \dots X_n \rangle = Y \leftrightarrow (\forall y)(y \in Y \leftrightarrow \text{Seq}(y) \ \& \ (\text{lh}(y) = n) \\ \& \ (i)(i < n \rightarrow (y)_i \in X_{i+1})) .$$

The extension obtained in this way is a conservative one. Therefore we denote it by A_2 (A_2^-) since it does not lead to confusion. If T is a set of formulas of $L(A_2)$, then $\ulcorner T \urcorner$ denotes the set of Gödel numbers of the formulas from T .

0.1. Definition. (a) A formula of $L(A_2)$ is called *arithmetical* if it is in prenex form and does not contain any quantifiers binding sets.

(b) A formula Φ of $L(A_2)$ is Σ_n^1 (Π_n^1) if there is an arithmetical formula Ψ such that $\Phi = (\exists X_1)(X_2) \dots (\exists X_n)\Psi$ ($(\exists X_1)(\exists X_2) \dots (\exists X_n)\Psi$),

where \exists is the appropriate quantifier.

A good reason for assuming the scheme of choice is the following:

0.2. Lemma. (a) For every formula Φ of $L(A_2)$, there exists a formula Ψ in $\mathbf{U}_n(\Sigma_n^1 \cup \Pi_n^1)$ such that $A_2 \vdash (\Phi \leftrightarrow \Psi)$.

(b) If Φ is Σ_n^1 , then there is a Σ_n^1 formula Ψ such that $A_2 \vdash ((\exists x)\Phi) \leftrightarrow \Psi$.

In the class of models of the language of A_2 , we distinguish those whose number-theoretic part is isomorphic to the set ω (the set of natural numbers); in other words, they are exactly those models whose natural numbers are well-ordered by the relation $<$ of the model in type ω . These models are called ω -models. Every ω -model determines (and is determined by) some family of subsets of ω . Thus we identify ω -models with families of subsets of ω . The cardinality of an ω -model is the power of this family of subsets of ω . By the standard model of A_2^- (and A_2) we mean $\mathcal{P}(\omega)$, the power set of ω . Throughout this paper we assume the theory ZFC as the meta-theory. This enables us to claim that $\mathcal{P}(\omega)$ is a model of A_2 . Let us note that second order arithmetic is often formulated in the language with functions and not sets as the second order objects. This is, however, a minor obstacle and the results can be translated from one language into the other in a simple way. (Cf. Rogers [58, p. 382] where

it is shown that the set of natural numbers is Σ_n^1 with respect to the function quantifier hierarchy iff it is Σ_n^1 with respect to the set quantifier hierarchy.) Let us finally mention that quite often we shall identify a definable relation with the formula defining it.

In the language of A_2 , we can express the fact that the relation X is a well-ordering.

$$\begin{aligned} \text{Bord}(X) \leftrightarrow & (x)(y)(x \in \text{Fld } X \ \& \ y \in \text{Fld } X \ \& \ (x, y) \in X \ \& \ (y, x) \in X \\ & \rightarrow x = y) \\ & \& \ (Y)(Y \neq \emptyset \ \& \ Y \subseteq \text{Fld } X \\ & \rightarrow (\text{E}z)(z \in Y \ \& \ (t)(t \in Y \rightarrow (z, t) \in X))), \end{aligned}$$

where $\text{Fld } X$ is $(X)_0 \cup (X)_1$.

The formula $\text{Bord}(X)$ is clearly equivalent to a Π_1^1 formula (even in A_2^-).

0.3. Definition (Mostowski [46]). A model M (of the language of A_2) is called a β -model if the formula Bord is absolute for it, i.e.,

$$M \models \text{Bord}[X] \rightarrow \{(x, y): \langle x, y \rangle \in X\}$$

is a well-ordering.

In particular, every β -model of A_2^- is an ω -model (since in A_2^- , we can prove that $<$ is a well-ordering).

0.4. Definition. (a) An ordinal α is *representable* in an ω -model M iff there is a set $X \in M$ such that $\{(x, y): \langle x, y \rangle \in X\} = \alpha$.

(b) $\text{Osp}(M)$ is the supremum of the ordinals representable in M .

(c) In the case where M is a β -model, we use $h(M)$ instead of $\text{Osp}(M)$.

Note that $\text{Osp}(M)$ is a limit ordinal. In fact, it is an admissible ordinal, and in the case of a β -model it is even a regular quasi cardinal (cf. Kripke [35]). (It need not to be even Σ_2 -admissible if M is not a β -model of A_2 .)

Let us define

$$\begin{aligned} A_\omega = & \{\varphi: \varphi \text{ is a sentence of } L(A_2) \\ & \& \ (M)(M \text{ is an } \omega\text{-model of } A_2^- \rightarrow M \models \varphi)\}, \end{aligned}$$

$$A_\beta = \{ \varphi : \varphi \text{ is a sentence of } L(A_2) \\ \& (M) (M \text{ is a } \beta\text{-model of } A_2^- \rightarrow M \models \varphi) \} .$$

Clearly, these theories satisfy the inclusions $A_2^- \subset A_\omega \subset A_\beta$. As we will see, both inclusions are proper.

§1. Representability

1.0. The question which sets are weakly or strongly representable in a system extending Peano arithmetic is a natural question about the strength of the system. In the case of A_ω , this problem has been solved by Grzegorzczuk, Mostowski and Ryll-Nardzewski [23]; and in the case of A_β by Mostowski [47] and Gandy and Putnam [57]. The results of Grzegorzczuk, Mostowski and Ryll-Nardzewski revealed for the first time the great analogy between the notions of recursiveness and hyperarithmeticality.

Namely recursive (recursively enumerable) sets are exactly the sets which are strongly (weakly) representable in Peano arithmetic, and hyperarithmetical (Π_1^1) sets are exactly those sets which are strongly (weakly) representable in A_ω .

1.1. **Definition.** Let M be a countable family of subsets of ω . Any set X such that $M = \{(X)_i; i < \omega\}$ is called a *code* for M . In this case we denote M by \sqcap_X .

Let us note that the downward Löwenheim–Skolem theorem enables us to restrict ourselves – in the definition of A_ω and A_β – to countable models (since an elementary submodel of a β -model is also a β -model). Once we consider only countable families of sets, we may pass to their codes and consider subsets of ω as ω -models.

1.2. **Theorem.** *The relation $Q(X, t, x, y) \leftrightarrow t$ is the Gödel number of a formula Φ of $L(A_2)$ with the free variables $V_1, \dots, V_m, v_1, \dots, v_n$ and $\sqcap_X \models \Phi[(X)_{(x)_0}, \dots, (X)_{(x)_{m-1}}, (y)_0, \dots, (y)_{n-1}]$ is Δ_1^1 .*

Proof. This follows from the fact that the expression “ Y is a satisfaction

sequence for Φ'' is an arithmetical one (cf. Mostowski [47]).

1.3. Definition. (a) $\text{Mod}_\omega(X) \leftrightarrow (x) (x \text{ is an axiom of } A_2^- \rightarrow Q(X, x, 0, 0))$.
 (b) $\text{Mod}_\beta(X) \leftrightarrow \text{Mod}_\omega(X) \ \& \ (Y)(x_0)((X)_{(x_0)} = Y \ \& \ Q(X, \ulcorner \text{Bord} \urcorner, x_0, 0)) \rightarrow \text{Bord}(Y))$.

Thus Mod_ω is the class of all codes for ω -models of A_2^- , and Mod_β is the class of all codes for β -models of A_2^- .

1.4. Lemma. (a) Mod_ω is a Δ_1^1 predicate.
 (b) Mod_β is a Π_1^1 predicate.

1.5. Proposition. $\ulcorner A_\omega \urcorner$ is a Π_1^1 set, $\ulcorner A_\beta \urcorner$ is a Π_2^1 set.

Proof. $x \in \ulcorner A_\omega \urcorner \leftrightarrow (X) (\text{Mod}_\omega(X) \rightarrow Q(X, x, 0, 0))$.
 $x \in \ulcorner A_\beta \urcorner \leftrightarrow (X) (\text{Mod}_\beta(X) \rightarrow Q(X, x, 0, 0))$.

1.6. Definition. Let T be a theory in the language of A_2 . A set $S \subseteq \omega$ is

- (a) *weakly representable* in T if there is a formula Φ of $L(A_2)$ such that for all natural numbers n , $n \in S \leftrightarrow T \vdash \Phi(n)$;
- (b) *strongly representable* in T if there is a formula Φ such that $(n)[(n \in S \leftrightarrow T \vdash \Phi(n)) \ \& \ (n \notin S \leftrightarrow T \vdash \neg \Phi(n))]$.

1.7. Theorem. (a) *If S is weakly representable in A_ω , then S is a Π_1^1 set.*
 (b) *If S is weakly representable in A_β , then S is a Π_2^1 set.*

Proof. It follows immediately from the fact that $\ulcorner A_\omega \urcorner$ is a Π_1^1 and $\ulcorner A_\beta \urcorner$ a Π_2^1 set, respectively.

1.8. Corollary. (a) *If S is strongly representable in A_ω , then S is a Δ_1^1 set.*
 (b) *If S is strongly representable in A_β , then S is a Δ_2^1 set.*

1.9. Lemma. (a) *If Φ is an arithmetical formula, \sqcap_X an ω -model and all A_i belong to \sqcap_X , then $\mathcal{P}(\omega) \models \Phi[A_1, \dots, A_n] \leftrightarrow \sqcap_X \models \Phi[A_1, \dots, A_n]$.*
 (b) *If Φ is a Π_1^1 formula and \sqcap_X an ω -model and all A_i belong to \sqcap_X ,*

then $\mathcal{P}(\omega) \models \Phi[A_1, \dots, A_n] \rightarrow \prod_X \models \Phi[A_1, \dots, A_n]$.

We proceed now to the proof that β -models preserve Σ_1^1 formulas.

1.10. Lemma. *Let Φ be a Σ_1^1 formula, then there is an arithmetical formula Ψ such that*

$$A_2^- \vdash \Phi(X_1, \dots, X_n) \leftrightarrow (\mathbf{E}Z)(t) \Psi(\bar{Z}(t), \bar{X}_1(t), \dots, \bar{X}_n(t)).$$

Proof. Note that the following equivalences are theorems of A_2^- :

$$x \in X \leftrightarrow (\forall y) (y = x \rightarrow (\bar{X}(y+1))_y = 0);$$

$$x \notin X \leftrightarrow (\forall y) (y = x \rightarrow (\bar{X}(y+1))_y = 1);$$

$$X = Y \leftrightarrow (\forall y) (\bar{X}(y) = \bar{Y}(y));$$

$$X \neq Y \leftrightarrow (\mathbf{E}z) (\bar{X}(z) \neq \bar{Y}(z)).$$

Now replace each of the above atomic formulas by its equivalent form. All the rest is just a transformation.

1.11. Theorem. *Let Φ be a Σ_1^1 formula. Then there is an arithmetical formula Ψ such that*

$$A_2^- \vdash \Phi(X) \leftrightarrow (\mathbf{E}Y) (\neg \text{Bord}(Y) \ \& \ (X) (x \in X \leftrightarrow \Psi(X, x, Y))).$$

The proof follows from Lemma 1.10 using the Brouwer–Kleene ordering of the appropriate sequences of natural numbers (cf. Rogers [58]).

1.12. Definition. Let T be a theory in $L(A_2)$. A formula Φ of $L(A_2)$ is T -provably Δ_n^1 if there exist Σ_n^1 formulas Φ_1 and Φ_2 such that $T \vdash (\Phi \leftrightarrow \Phi_1) \ \& \ (\Phi \leftrightarrow \neg \Phi_2)$.

1.13. Theorem. (a) *If Φ is a Π_1^1 formula, \prod_X a β -model of A_2^- and $A_1, \dots, A_n \in \prod_X$, then $\prod_X \models \Phi[A_1, \dots, A_n] \leftrightarrow \mathcal{P}(\omega) \models \Phi[A_1, \dots, A_n]$.*

(b) *If Φ is a Π_2^1 formula, \prod_X a β -model of A_2^- and $A_1, \dots, A_n \in \prod_X$, then $\mathcal{P}(\omega) \models \Phi[A_1, \dots, A_n] \rightarrow \prod_X \models \Phi[A_1, \dots, A_n]$.*

(c) *If Φ is a T -provably Δ_2^1 formula, \prod_X a β -model of T , $T \supseteq A_2^-$, $\mathcal{P}(\omega) \models T$ and $A_1, \dots, A_n \in \prod_X$, then*

$$\mathcal{P}(\omega) \models \Phi[A_1, \dots, A_n] \leftrightarrow \prod_X \vdash \Phi[A_1, \dots, A_n].$$

Proof. Clearly, (b) and (c) follow from (a). Let Φ be a Σ_1^1 formula. We use the reduction obtained in Theorem 1.11 and obtain (a).

Now we immediately obtain the following:

1.14. Corollary. (a) *Every Π_1^1 set is weakly representable in A_ω .*
 (b) *Every Π_2^1 set is weakly representable in A_β .*

Δ_1^1 sets are exactly the hyperarithmetical sets and it turns out that one can formalize the construction of the hyperarithmetical hierarchy in A_ω . Using this we get:

1.15. Theorem (Grzegorzcyk, Mostowski and Ryll-Nardzewski [23]).
Every Δ_1^1 set is strongly representable in A_ω .

By the comprehension scheme, every set which is strongly representable in A_ω (A_β) belongs to every ω -model (β -model) of A_2^- . Thus every Δ_1^1 set is in every ω -model of A_2^- . By relativation, we get that if X is Δ_1^1 in Y , then X is in every ω -model of A_2^- in which Y is (see §3 for strong representability in A_β).

§2. Development of the semantics of A_2

2.0. In this section we prove some results on ω -models which, either in formulation or in proof, have some connections with the recursion theory.

An important role in these considerations is played by the ω -consistency theorem, which links the ω -rule and ω -models. The importance of this theorem lies in the fact that it gives a method for building ω -models, which is used almost in every theorem about ω -models.

Kreisel's ω -compactness theorem shows that in certain situations the notion of hyperarithmeticity corresponds to the notion of finiteness rather than to that of recursiveness. This way of looking at the analogies arising here has led to the important development of the theory of infinitary languages. Barwise's compactness theorem (see e.g. Keisler [34]) is a generalization of the above theorem.

The Gandy–Kreisel–Tait theorem gives the characterization of hyperarithmetical sets in semantical terms, namely they form the intersection of all ω -models of A_2^- (A_2). Together with Kleene's result, which says that hyperarithmetical sets are not closed under the comprehension scheme, this shows that there is no smallest ω -model of A_2^- (A_2).

Friedman's theorem says much more: there is no minimal model of A_2 . This theorem shows that the class of ω -models of A_2 is very rich and complicated.

2.1. Definition. (i) The following infinitary rule of proof is called the ω -rule: from $\vdash \Phi(n)$ for each n infer $\vdash (x)\Phi(x)$.

(ii) $Cn_\omega(T)$ is the set of all ω -consequences of T , i.e., formulas provable in T using additionally the ω -rule.

(iii) A set of sentences T of $L(A_2)$ is ω -consistent if $Cn(T) = T$ and if for all formulas Φ with one free variable the fact that $\Phi(n) \in T$ for all n implies $(\exists x) \neg \Phi(x) \notin T$.

(iv) A set of sentences T of $L(A_2)$ is ω -complete if $Cn(T) = T$ and if for all Φ with one free variable the fact that $\Phi(n) \in T$ for all n implies $(x)\Phi(x) \in T$.

2.2. Remark. A consistent, complete set of sentences is ω -complete iff it is ω -consistent.

2.3. Theorem (ω -consistency theorem, Orey [56]). *Every consistent, ω -complete set T of sentences of $L(A_2)$ has an ω -model.*

Proof. It suffices to observe that the type $\Gamma = \{x \neq n : n < \omega\}$ is non-principal in $Cn_\omega(T)$; and use the omitting types theorem (see e.g. Shoenfield [62]).

2.4. Theorem (ω -completeness theorem, Orey [56]). *If T is a theory in $L(A_2)$, then $Cn_\omega(T)$ is exactly the set of all sentences true in all ω -models of T .*

Proof. In one direction the inclusion is immediate. In the other it follows from Theorem 2.3.

2.5. Theorem (ω -compactness theorem, Kreisel [33]). *Let T be a theory*

in $L(A_2)$ such that $\ulcorner T \urcorner \in \Pi_1^1$. If every $S \subseteq T$ such that $\ulcorner S \urcorner \in \Delta_1^1$ has an ω -model, then T has an ω -model.

2.6. Corollary. *Let T be as above. If $\Phi \in \text{Cn}_\omega(T)$, then $\Phi \in \text{Cn}_\omega(S)$ for some $S \subseteq T$ such that $\ulcorner S \urcorner \in \Delta_1^1$.*

The proof uses some facts concerning inductive definitions, which we review briefly.

2.7. Definition. Let Γ be an operator on $\mathcal{P}(\omega)$ which is monotone, i.e., $X \subseteq Y \rightarrow \Gamma(X) \subseteq \Gamma(Y)$.

(i) A subset S of ω is inductively defined with respect to the relation Γ if $\Gamma^0(X) = X$, $\Gamma^\alpha(X) = \bigcup_{\lambda < \alpha} \Gamma(\Gamma^\lambda(X))$ for $\alpha > 0$, and $S = \bigcup_\alpha \Gamma^\alpha(\emptyset)$.

(ii) The least α such that $\Gamma(\Gamma^\alpha(\emptyset)) = \Gamma^\alpha(\emptyset)$ is called the closure ordinal of Γ and denoted by $|\Gamma|$ (then $S = \Gamma^{|\Gamma|}(\emptyset)$ and $\Gamma(S) = S$).

(iii) Γ is Σ_8^1 if the relation $n \in \Gamma(X)$ is a Σ_8^1 relation.

2.8. Theorem (Spector [65]). (i) *If $S \in \Pi_1^1$, then there exists an arithmetical Γ such that S is inductively defined with respect to Γ .*

(ii) *If Γ is Π_1^1 , then $|\Gamma| \leq \omega_1$.*

(iii) *If Γ is Δ_1^1 , then $\Gamma^\alpha(\emptyset) \in \Delta_1^1$ for $\alpha < \omega_1$.*

(ω_1 denotes the first non-recursive ordinal; see Rogers [58].)

Proof of Theorem 2.5. Let Γ be an arithmetical monotone operator such that $\ulcorner T \urcorner = \bigcup_\alpha \Gamma^\alpha(\emptyset)$. Let Ω be a monotone operator such that for every set X of sentences of $L(A_2)$, $\Omega(\ulcorner X \urcorner)$ is the set of Gödel numbers of all logical consequences of X and the consequences obtained by a single application of the ω -rule. Then Ω is arithmetical.

Let Φ be an operator such that

$$\Phi^\alpha(\emptyset) = \Gamma^\alpha(\emptyset) \cup \bigcup_{\beta < \alpha} \Omega(\Phi^\beta(\emptyset)).$$

Then such a Φ can be chosen among Π_1^1 ones. By induction on α , we obtain that

$$\Phi^\alpha(\emptyset) \subseteq \bigcup_\beta \Omega^\beta(\Gamma^\alpha(\emptyset)) = \ulcorner \text{Cn}_\omega(\Gamma^\alpha(\emptyset)) \urcorner.$$

Thus, by Spector's results,

$$\begin{aligned} \lceil \text{Cn}_\omega(T) \rceil &= \bigcup_\alpha \Omega^\alpha(\lceil T \rceil) \subseteq \bigcup_\alpha \Phi^\alpha(\emptyset) \\ &= \bigcup_{\alpha < \omega_1} \Phi^\alpha(\emptyset) = \bigcup_{\alpha < \omega_1} \lceil \text{Cn}_\omega(\Gamma^\alpha(\emptyset)) \rceil. \end{aligned}$$

But for $\alpha < \omega_1$, $\Gamma^\alpha(\emptyset) \in \Delta_1^1$, i.e., by assumption, $\Gamma^\alpha(\emptyset)$ has an ω -model. Hence $\text{Cn}_\omega(T)$ is consistent, i.e., T has an ω -model.

2.9. Theorem (Hard core theorem, Gandy, Kreisel and Tait [21]). *Let T be a theory in $L(A_2)$ such that $T \supseteq A_2^-$, and $\lceil T \rceil$ is a Π_1^1 set. Assume that there exists an ω -model of T . Then a set of natural numbers is Δ_1^1 iff it belongs to every ω -model of T .*

Proof. One direction follows from the Grzegorzczuk–Mostowski–Ryll-Nardzewski Theorem 1.15. Suppose now that Z belongs to every ω -model of T . If the type $\Delta = \{n \in X : n \in Z\} \cup \{n \notin X : n \notin Z\}$ was non-principal in $\text{Cn}_\omega(T)$, then, by omitting types Γ and Δ , we would get an ω -model of T to which Z does not belong. Thus Δ is principal in $\text{Cn}_\omega(T)$. This leads to the conclusion that Z is a Π_1^1 set because $\lceil \text{Cn}_\omega(T) \rceil$ is a Π_1^1 set. But $\omega - Z$ also belongs to every ω -model of T , therefore Z is Δ_1^1 .

Observe that the same result was proved in [24]. Notice that Mostowski [52] proved that if $\{A_n\}_{n < \omega}$ is a family of non-hyperarithmetic sets, then there exists an ω -model of A_2^- disjoint with this family.

The “hard core” Theorem 2.9 implies that there is no smallest ω -model of A_2 since Δ_1^1 sets do not form a model of A_2 . (Indeed, Kleene [31] proved that the comprehension scheme is false in Δ_1^1 .) This would not exclude the existence of a \subseteq -minimal ω -model of A_2 , but we soon prove that this is not the case.

2.10. Definition. (a) $\text{Def } M = \{a \in M : \text{there is a formula } \Phi \text{ such that } M \models ((\exists! x) \Phi) \ \& \ \Phi[a]\}$.

(b) M is called *pointwise definable* iff $\text{Def } M = M$.

2.11. Theorem (Friedman [17]). *If M is an ω -model of A_2 , then there is an $N \subsetneq M$ which is also an ω -model of A_2 .*

Proof. Let M be an ω -model of A_2 . We may as well assume that M is not

a β -model since the statement $(EX) (\text{Mod}_\omega(X))$ is preserved by β -models. We may also assume that M satisfies $(X) \text{Constr}(X)$ (i.e., analytical form of the axiom of constructibility ²) and therefore we may additionally require that M is pointwise definable because if $M \models (X) \text{Constr}(X)$, then $\text{Def}(M) \prec M$ (by the definability of Skolem functions).

2.12. Definition. A theory T in $L(A_2)$ is ω - n -complete iff

- (a) $A_2 \cup \{(X) \text{Constr}(X)\} \subseteq T$,
- (b) $T = (\{\varphi: \varphi \in T \ \& \ \varphi \text{ is a } \Sigma_n^1 \cup \Pi_n^1 \text{ sentence}\} \cup A_2 \cup \{(X) \text{Constr}\})_\omega$,
- (c) T decides all formulas with at most n quantifiers,
- (d) T is consistent.

Let U be the family of all ω - n -complete theories belonging to the model M (for all $n < \omega$). We define a certain tree-like structure between some finite sequences of elements of M . Let $\tilde{T} = \langle V, \succ \rangle$, where $V = \{\langle \ulcorner T_0 \urcorner, \dots, \ulcorner T_n \urcorner \rangle: \ulcorner T_j \urcorner \in M, T_j \text{ is } \omega\text{-}j\text{-complete and } i < j \rightarrow T_i \subseteq T_j, X \succ Y \leftrightarrow (\text{lh}(X) < \text{lh}(Y) \text{ and } X = Y \upharpoonright \text{lh}(X))\}$. Now consider $B = \{\langle \ulcorner T_0 \urcorner, \dots, \ulcorner T_n \urcorner \rangle: T_j \text{ is the set of all sentences of } L(A_2) \text{ with at most } j \text{ quantifiers, true in } M\}$. Since the partial truth is definable in $A_{2,2}$, each $\ulcorner T_n \urcorner$ belongs to M , and so B is an infinite descending branch in \tilde{T} , so finally \tilde{T} is not well founded. Yet \tilde{T} is definable over M .

Now Friedman's proof breaks into two facts.

- (I) Every infinite descending path in \tilde{T} determines a submodel of M .
Different branches determine different models.

Proof. Let G be an infinite branch in the tree \tilde{T} . Consider UG . UG determines the set

$$M(UG) = \{y \subseteq \omega: (E\Phi) (y = \{n: \ulcorner \Phi(n) \urcorner \in UG\})\}.$$

Since G is a complete, ω -complete consistent theory, it has an ω -model M_1 . Now since $(X) \text{Constr}(X) \in UG$, $\text{Def}(M_1) \prec M_1$. Now by comprehension, $\text{Def}(M_1) = M(UG)$ and so $M(UG) \models A_2$. Thus we only need to show that $M(UG) \subseteq M$. Let Φ be a formula. Then there is a k , so that for

² An exact exposition of the facts connected with the notion of constructibility in A_2 will be developed later.

all n , $\ulcorner \Phi(n) \urcorner \in \text{UG}$ iff $\ulcorner \Phi(n) \urcorner \in T_k$. Since $T_k \in M$, $\{n: \ulcorner \Phi(n) \urcorner \in T_k\} \in M$. Thus for all $y \in M(\text{UG})$, $y \in M$. Clearly, our demonstration shows that different branches determine non-elementarily equivalent models.

(II) There are at least two different branches in T .

Proof. We consider all subtrees of T which are (coded as) elements of M .

Case (a). There is a tree which is not well founded inside of the model. Then this tree has an infinite descending branch inside of the model. The branch gives an ω -model of A_2 which cannot be the original model since, if it was, we could code M inside itself, which by diagonal reasoning leads to a contradiction. Thus we may pass to Case (b).

Case (b). All trees under consideration are well founded inside the model. Then to every well-founded tree, we may uniquely correlate the rank of this tree in the model (here we use once more the axiom of constructibility). The rank is a well-ordering.

There are two possible subcases.

(1) There is a nonstandard well-ordering which is a rank for such a tree. Then – outside the model – one can find a descending, infinite branch. It does not belong to the model and hence it can not be unique (since then, having a code of this branch implicitly defined, it would be hyperarithmetical in this tree and so would be inside the model). Thus there are at least two branches and we are done.

(2) All the ranks of the trees under consideration are real well-orderings. Friedman shows that this is impossible. Namely let α_0 be $\text{Osp}(M)$. By the definition, α_0 is not representable in M . For $X \in \tilde{T}$, we define $F(X)$ to be the supremum of the ranks of all the subtrees of \tilde{T} in which X appears.

The proof splits once more:

(i) There is a bound for $F(X)$ for all $X \in \tilde{T}$, and this bound is less than α_0 . Then \tilde{T} is well founded, which is false since we exhibited an infinite descending branch in \tilde{T} .

(ii) The bound for $F(X)$ is α_0 . Then α_0 has to be represented in M . Indeed, it is so since all the trees under consideration have restricted height (in M). Thus there must be a well-ordering (in the sense of M) which is a bound for all of them. Since they all are real well-orderings, the bound has to be real as well, and thus α_0 is represented in M , which

is a contradiction. Thus subcase (2) is excluded and so we are done.

Note that the same proof applies for any extension T of A_2 such that $T \in \Pi_1^1$ and $T \vdash \text{Constr}(T)$. Moreover, we can keep the set of natural numbers fixed inside the smaller model.

§3. The ramified analytical hierarchy

3.0. In the previous section we have found that there is no smallest ω -model of A_2 . The situation in the case of β -models is different: there exists the smallest β -model of A_2^- (it is even a model of A_2). The model is constructed as a result of a certain hierarchy, called the ramified analytical hierarchy.

The construction of the initial segment of this hierarchy up to recursive ω_1 has been introduced by Kleene, who has shown that the sets that are obtained up to this point are exactly the hyperarithmetical sets. From the semantical point of view the construction strongly resembles that of the smallest transitive model of ZF.

An important paper of Boyd, Hensel and Putnam [8] has revealed that from the recursion-theoretical point of view the ramified analytical hierarchy is an extension of the hyperarithmetical jump hierarchy.

This deep fact shows that from the recursion-theoretical point of view there is an analogy between the intersection of all β -models of A_2^- and the intersection of all ω -models of A_2^- which is hard to observe in semantical considerations. Also, similarly as in the case of ω -models, the sets strongly representable in A_β form the intersection of all β -models of A_2^- .

3.1. **Definition.** The hyperjump of X is Kleene's universal $\Pi_1^{1,X}$ set.

The following theorem is part of the "mathematical folklore" and can be found explicitly in [8].

3.2. **Theorem.** *An ω -model of A_2^- is a β -model iff it is closed under the operation of the hyperjump.*

Proof. By Kleene's basis theorem (every nonempty $\Sigma_1^{1,X}$ family of sets contains an element which is recursive in the hyperjump of X), the implication from right to left follows. The other direction follows from the absoluteness of Π_1^1 formulas with respect to β -models.

3.3. Definition. (a) Let $\mathfrak{A} \subseteq \mathcal{P}(\omega)$. We define $D(\mathfrak{A})$ to be the family of all subsets of ω definable in the relational system

$$\langle \mathfrak{A}, \omega, +, \cdot, 0, \in, X \rangle_{X \in \mathfrak{A}}.$$

(b) We define inductively the ramified analytical hierarchy as follows:

$$\begin{aligned} \text{R.A.}_0 &= \emptyset, \\ \text{R.A.}_{\alpha+1} &= D(\text{R.A.}_\alpha), \\ \text{R.A.}_\lambda &= \bigcup_{\xi < \lambda} \text{R.A.}_\xi \quad \text{for } \lambda \text{ limit,} \\ \text{R.A.} &= \bigcup_\xi \text{R.A.}_\xi. \end{aligned}$$

By a cardinality argument, there is a ξ such that $\text{R.A.}_\xi = \text{R.A.}_{\xi+1}$ (the existence of such ξ may be proved already in $\text{ZF}^- + \text{"}\mathcal{P}(\omega) \text{ exists"}$). The smallest ξ such that $\text{R.A.}_\xi = \text{R.A.}_{\xi+1}$ is called β_0 . The ramified analytical hierarchy was introduced by Kleene [31]. Cohen [9] observed that one can prove in ZF that β_0 is countable. Namely one can carry out the whole construction of the hierarchy faithfully within a countable transitive model of $\text{ZF}^- + \text{"}\mathcal{P}(\omega) \text{ exists"}$, and the existence of such a model is provable in ZF. (ZF^- denotes ZF set theory without the power set axiom.)

The following theorem of Gandy and Putnam shows the importance of R.A. in studying β -models.

3.4. Theorem (Gandy and Putnam). (a) R.A._{β_0} is the smallest β -model of A_2^- . It is also a model of A_2 .

$$\begin{aligned} \text{(b)} \quad h(\text{R.A.}_{\beta_0}) &= \beta_0. \\ \text{(c)} \quad \text{R.A.}_{\beta_0} &= \{X : X \text{ is strongly representable in } \text{A}_\beta\} \\ &= \{X : X \text{ is } \text{A}_\beta\text{-provably } \Delta_2^1\}. \end{aligned}$$

The proof of this theorem was published in [8]. In fact, Boyd, Hensen and Putnam prove much more. To present the content of their paper we need some definitions.

3.5. Definition. Let A denote the degree of unsolvability of the set A and let $j(a)$ denote the jump of the degree a . If Γ is a collection of degrees, then by $RU(\Gamma)$ we mean the collection of all sets recursive in sets whose degrees are in Γ .

3.6. Definition. If Γ is a set of degrees, we say that a degree b is a *uniform upper bound* (u.u.b.) on $RU(\Gamma)$ if the class $RU(\Gamma)$ is uniformly recursive in a member of b . A degree b is a *n-least* u.u.b. on $RU(\Gamma)$ if for every c which is an u.u.b. on $RU(\Gamma)$, $c \leq_T j^{(n)}(b)$.

3.7. Definition. Let d be a function from an initial countable segment of ordinals into the degrees, d is called an *admissible degree hierarchy* if d satisfies the following three conditions:

- (1) $d(0) = \Phi$,
- (2) $d(\alpha + 1) = j(d(\alpha))$,
- (3) $d(\lambda)$ is an n -least u.u.b. on $RU(\{d(\alpha) : \alpha < \lambda\})$ for some n (λ -limit).

Condition (3) says that every u.u.b. on $RU(\{d(\alpha) : \alpha < \lambda\})$ is arithmetical in $d(\lambda)$. Such a hierarchy is a natural extension of the arithmetical hierarchy into the transfiniteness.

The main result of Boyd, Hensel and Putnam is the following:

3.8. Theorem. (1) *There exists an admissible degree hierarchy on β_0 , and for any such a.d.h.,*

$$RU(\{d(\alpha) : \alpha < \beta_0\}) = \text{R.A.}_{\beta_0}.$$

(2) *If d is an a.d.h. on α , then d can be extended to an a.d.h. on some $\beta > \alpha$ iff $\alpha < \beta_0$.*

We give here a rough sketch of the proof. By induction on λ , we easily find that if d_1, d_2 are two a.d.h. on a limit λ , then $RU(\{d_1(\alpha) : \alpha < \lambda\}) = RU(\{d_2(\alpha) : \alpha < \lambda\})$. Also if $d_1 \subseteq d_2 \subseteq \dots$ is a sequence of a.d.h.'s, then $\bigcup_{n < \omega} d_n$ is an a.d.h. too. From the first fact we infer that if d is an a.d.h. on the recursive ω_1 , then $RU(d(\alpha) : \alpha < \omega_1) = \Delta_1^1$, because the hyperarithmetical hierarchy is (by Spector's "uniqueness" theorem) an a.d.h. Using the fact that "the truth in an ω -model" is a Δ_1^1 relation, we get that there exists a Δ_1^1 formula $\text{def}(X, Y)$ such that if $\text{Bord}(Y)$, then

$(E!X) \text{ def}(X, Y)$ and for this X , $\Pi_X = \text{R.A.}_{\bar{Y}}$. This fact together with some information concerning the hyperarithmetical hierarchy implies Kleene's theorem: $\text{R.A.}_{\omega_1} = \Delta_1^1$.

Now relativizing the proof of Kleene's theorem and using the Spector–Gandy hyperarithmetical quantifier theorem (every Π_1^1 set is Σ_1^1 definable over Δ_1^1 ; see [64]), we get that $\text{R.A.}_{\omega_1^{0(n)}} = \Delta_1^{1,0(n)}$ ($0^{(n)}$ denotes the n th iteration of the hyperjump of \emptyset). By the same reasoning, there exists an a.d.h. d_n on $\omega_1^{0(n)}$ and for any such d ,

$$RU(\{d(\alpha): \alpha < \omega_1^{0(n)}\}) = \Delta_1^{1,0(n)}.$$

These hierarchies d_n may be set in a nested sequence, so we get an a.d.h. on $\omega_1^\infty = \sup\{\omega_1^{0(n)}: n < \omega\}$.

Define α to be a HYP-ordinal if R.A._α is closed under the hyperjump operation. By the above, ω_1^∞ is the least HYP-ordinal. Suppose now that α is a HYP-ordinal such that there exists an a.d.h. d on α such that $RU(\{d(\lambda): \lambda < \alpha\}) = \text{R.A.}_\alpha$. It turns out that the further extension of d depends on the following claim: there exists a set $X \in \text{R.A.}_{\alpha+1}$ such that $\text{Bord}(X)$ and $\bar{X} = \alpha$. If the claim holds, then relativizing the proof concerning ω_1^∞ to the set X we can extend the hierarchy d to $\omega_1^{\infty, X}$ still keeping the appropriate equality between this hierarchy and the ramified analytical hierarchy until $\omega_1^{\infty, X}$. In such a situation, $\omega_1^{\infty, X}$ will be the next HYP-ordinal after α .

The claim is guaranteed by the following *main step* in the proof: if $\alpha < \beta_0$, then over R.A._α we may define a well-ordering of type α . This technically non-trivial result follows mainly from the fact that there is a definable well-ordering of R.A._α for all $\alpha \leq \beta_0$ (it is interesting that for all α such that $\omega_1^\infty \leq \alpha \leq \beta_0$, the same formula defines this well-ordering). Thus the main step implies that β_0 is a HYP-ordinal (because the limit of HYP-ordinals is a HYP-ordinal) and we get the proof of part (1).

The proof of part (2) depends on the following result (based on a “generic” argument): if M is a countable ω -model of A_2^- , then for any n , no u.u.b. on M is an n -least one.

On the other hand, just by the definition of β_0 , $\text{R.A.}_{\beta_0} \models A_2^-$. These facts together with part (1) imply that no a.d.h. can be extended further than β_0 .

Proof of Theorem 3.4. The proof now easily follows from the above results. Since $R.A._{\beta_0}$ is closed under the hyperjump operation and has a definable well-ordering, we obtain that $R.A._{\beta_0}$ is a β -model of A_2 . Since $R.A._{\beta_0}$ is countable, we may write

$$x \in R.A._{\beta_0} \leftrightarrow (\exists Y) (\exists S) (\text{Bord}(Y) \ \& \ \text{def}(S, Y) \ \& \ (\exists n) (X = (S)_n)).$$

Denote this Σ_2^1 formula by $R.A.(X)$.

Let M be a β -model of A_2^- . Then $R.A._{h(M)} = R.A.^M \subseteq M$. By the main step, if $h(M) < \beta_0$, then over $R.A.^M$ we can define a well-ordering of the type $h(M)$ which would then belong to M . Thus $R.A._{\beta_0}$ is the smallest β -model and our main lemma assures that $h(R.A._{\beta_0}) = \beta_0$. Since there is a definable well-ordering of $R.A.$, $\text{Def}(R.A._{\beta_0}) < R.A._{\beta_0}$, and by the minimality of $R.A.$, $\text{Def}(R.A._{\beta_0}) = R.A._{\beta_0}$, i.e., $R.A._{\beta_0}$ is pointwise definable, which, by definability of $R.A._{\beta_0}$ in any other β -model, settles (c).

The fact that $R.A._{\beta_0}$ is a β -model can be proved in a much more direct way (the reasoning presented here comes from the original proof of Gandy).

Observe that already in A_2^- we can prove that any two well-orderings are comparable. This implies that if M is an ω -model of A_2^- , then the non-standard well-orderings in M are all “longer” than all the standard ones.

Suppose that $R.A._{\beta_0}$ is not a β -model. Let $X_0 \in R.A._{\beta_0}$ be such that $R.A._{\beta_0} \models \text{Bord}(X_0)$, but $\neg \text{Bord}(X_0)$. Let $S \in R.A._{\beta_0}$.

Case (I): β_0 is limit. Then $S \in R.A._\gamma$ for some $\gamma < \beta_0$. By the main step and the fact mentioned above, for some n , we have $\text{Bord}(X_0 \upharpoonright n)$ and $\overline{X_0 \upharpoonright n} = \gamma$. Thus by the properties of the formula $\text{def}(X, Y)$, we get that S is Δ_1^1 in X_0 .

Case (II): β_0 is not limit. The first ordinal which cannot be coded by a set from an ω -model of A_2^- must be limit, thus by the same reasons as above, for some n , we have $\text{Bord}(X_0 \upharpoonright n)$ and $\overline{X_0 \upharpoonright n} = \beta_0$ and S is Δ_1^1 in X_0 as well.

Hence $R.A._{\beta_0} = \Delta_1^{1, X_0}$ which is impossible by the relativization of Kleene’s result [31].

Let us observe that there is a close analogy between strongly constructible sets of Cohen [9] and the ramified analysis. In both cases we

successively close the previous steps of construction under certain operations and the operations are closed in such a way that we obtain at the critical point just a model for the appropriate theory.

We already noticed that $R.A._{\beta_0}$ turns out to be defined by means of a Σ_2^1 formula of $L(A_2)$ and defines in every β -model an ω -submodel. By the Kondo–Addison basis theorem (every nonempty Σ_2^1 family of sets contains a Δ_2^1 element), we find that $R.A.$ has a Δ_2^1 code. Indeed, the predicate $P(X) \leftrightarrow \text{Mod}_\beta(X) \ \& \ \sqcap_X \models (Y) R.A.(Y)$ is a Π_1^1 one and is nonempty (by the Gandy–Putnam Theorem 3.4). Clearly, P consists of all codes for $R.A._{\beta_0}$. Thus $R.A.$ has a Δ_2^1 code.

Therefore we get the following:

- 3.9. Theorem.** (a) $R.A._{\beta_0} \subsetneq \Delta_2^1$.
 (b) β_0 is an Δ_2^1 ordinal.

Let us note the following:

3.10. Theorem (Leeds and Putnam [38]). *If in the definition of $R.A.$ we do not allow parameters (in the construction of the next step), we get the same hierarchy (starting from the step ω).*

The connection between the ramified analysis and the constructible sets (i.e., the ramified set theory) was shown by Boolos [7].

3.11. Theorem (Boolos). *For all $0 < \alpha < \beta_0$, $R.A._\alpha = L_{\omega+\alpha} \cap \mathcal{P}(\omega)$.*

(Further strengthening of this result was proved by R. Jensen in his “Habilitationsschrift” – as pointed out to us by R.O. Gandy.)

Let us note that if a set X is definable by an A_β -provably Δ_2^1 formula, then it belongs to every β -model of A_2^- . Thus β_0 is not a A_β -provably Δ_2^1 ordinal and by point (c) of the Gandy–Putnam Theorem 3.4, it is the first non A_β -provably Δ_2^1 ordinal. (This fact was pointed out to us by R.O. Gandy.) Thus β_0 is not an A_2 -provably Δ_2^1 ordinal. It is even not the first non A_2 -provably Δ_2^1 ordinal. However, we are not able to evaluate exactly the latter.

§4. Definable quantifiers

4.0. While trying to extend ω -models one wants to retain the set of natural numbers while enlarging the family of sets of natural numbers. Often we want to preserve also some other properties of the model. A definable quantifier, a natural notion of size turned out to be a useful tool for this purpose.

In order to behave like a notion of size, a definable quantifier has to satisfy some natural conditions. The properties that are “small” with respect to the quantifier may be preserved whereas the “big” ones get new elements.

If one has a countable ω -model of A_2^- in which there exists a definable quantifier, then, by a model-theoretic reasoning, one can elementarily extend the model to a bigger ω -model. The choice of an appropriate quantifier allows us to produce ω -models with the desired properties.

This subject – in the general situation – is studied by Krivine and McAloon [36] and we refer the reader to this paper.

We noticed that an elementary submodel of a β -model is also a β -model. This, however, does not hold for elementary extensions. Namely we have the following:

4.1. Theorem (Mostowski and Suzuki [55]). *Every countable β -model M of A_2 has a countable elementary extension N which is an ω -model but not a β -model.*

The corresponding result for the ZF set theory case was proved, with the use of similar methods, by Keisler and Morley [30]. Note that the fact that there are ω - but not β -models followed already from the different nature of the sets A_ω and A_β .

Recently, Miss M. Dubiel, using a theorem of Krivine and McAloon [36] (see Theorem 4.9), has proved that additionally we may require that $\text{Osp}(N)$ is equal to $h(M)$.

On the other hand, the continuum of Lévy’s model of ZF in which $\aleph_1 = \aleph_\omega^L$ is a counterexample to Theorem 4.1 in the case of A_2^- . One easily checks that every ω -model which extends this model and is ele-

mentary equivalent to it must be a β -model with the same height. On the other hand, Guzicki proved that this β -model of A_2^- has a proper elementary extension which is an ω -model (cf. Guzicki [27]).

We also have the following:

4.2. Theorem (Keisler [29], Mostowski [53]). *Every countable ω -model of A_2 has a countable proper elementary extension which is an ω -model.*

In the proofs of the above theorems, we shall use the suggestive notion of a definable quantifier, introduced by Mostowski.

4.3. Definition. Let F be a symbol not occurring in $L(A_2)$. We add to the formulas of $L(A_2)$ new atomic formulas of the form $F(X)$, where X is a variable of $L(A_2)$. Each sentence of this new language containing the symbol F is called a *definable quantifier*.

4.4. Definition. Let Q be a definable quantifier, Φ a formula of $L(A_2)$ and X a variable. By $(QX)\Phi$ we denote the formula of $L(A_2)$ obtained by means of the following operations:

(i) all bound variables of Q are replaced by variables occurring neither in Φ nor in Q ;

(ii) for every variable Y , each occurrence of an atomic formula $F(Y)$ in Q is replaced by $\Phi_Y[X]$.

4.5. Definition. Let Q be a definable quantifier, Φ, Φ_1, Φ_2 formulas of $L(A_2)$ with exactly one free set variable and Ψ a formula of $L(A_2)$ with exactly two free set variables. Then:

(i) Q is *non-trivial* if for every formula Φ ,

$$A_2 \vdash (QX)\Phi(X) \rightarrow (EX)(EY)(X \neq Y \ \& \ \Phi(X) \ \& \ \Phi(Y)) ;$$

(ii) Q is *monotone* if for every Φ_1, Φ_2 ,

$$A_2 \vdash (X)(\Phi_1 \rightarrow \Phi_2) \rightarrow ((QX)\Phi_1(X) \rightarrow (QX)\Phi_2(X)) ;$$

(iii) Q is *additive* if for every Φ_1, Φ_2 ,

$$A_2 \vdash (QX)(\Phi_1(X) \vee \Phi_2(X)) \rightarrow ((QX)\Phi_1(X) \vee (QX)\Phi_2(X));$$

(iv) Q is σ -additive if for every Φ ,

$$A_2 \vdash (QX)(Ey)\Phi(X, y) \rightarrow (Ey)(QX)\Phi(X, y);$$

(v) Q is normal with respect to formulas Φ, Ψ if

$$A_2 \vdash (QX)\Phi(X)$$

and

$$A_2 \vdash (X)(\Phi(X) \rightarrow \neg(QY)\neg\Psi(X, Y)).$$

4.6. Theorem (Mostowski [53]). *Let Q be a definable quantifier which is non-trivial, monotone, additive, σ -additive and normal with respect to formulas $\tilde{\Phi}, \tilde{\Psi}$. If M is a countable ω -model of A_2 , then there exists an ω -model N such that $M < N$ and a $C \notin M$ such that $N \models \tilde{\Phi}[C]$ and $N \models \Psi[X, C]$ for each $X \in M$ such that $M \models \tilde{\Phi}[X]$.*

Proof. (The proof presented here is in fact Keisler's proof of the Keisler–Mostowski Theorem 4.2.) Let $L(M)$ denote the language obtained from $L(A_2)$ by adding for each $X \in M$ a new constant. Let M_M denote the expansion of M to the model of $L(M)$, and let C be a new constant symbol. Let $T = \text{Th}(M_M) \cup \{\Phi(C): M \models \neg(QX)\neg\Phi(X)\}$.

Fact 1. $\Phi(C)$ is consistent with T iff $M \models (QX)\Phi(X)$.

The implication from left to right is immediate. Using additivity and monotonicity of Q (and Theorem 2.5), we easily obtain the converse implication.

By the assumptions on Q , we have $M \models (QX)\tilde{\Phi}(X)$, thus T is consistent.

Fact 2. T is ω -complete.

It follows immediately from the σ -additivity of Q and Fact 1.

By the ω -consistency theorem, theory T has an ω -model. Let N be the reduct of it to the language $L(A_2)$. Then $M < N$ and in virtue of the normality of Q and Fact 1, N is the desired model.

Proof of Theorem 4.2. Let Q be the following definable quantifier:

$$(X)(EY)[F(Y) \& \neg(Ex)(Y = (X)_x)] .$$

$(QX)\Phi(X)$ has the following intuitive meaning: the family of sets satisfying Φ is uncountable. Let $\tilde{\Phi}$ be $X = X$ and $\tilde{\Psi}$ be $X \neq Y$. It is easy to

check that \mathbb{Q} satisfies all conditions required in the above theorem.

Proof of Theorem 4.1. (This proof, due to Mostowski, is not the original one.) Let Θ be the following definable quantifier:

$$(X) (\text{Bord}(X) \rightarrow (\text{E}Y) (F(Y) \ \& \ \text{Bord}(Y) \ \& \ X \prec Y)),$$

where $X \prec Y$ means that X is similar to an initial segment of Y . $\Theta X \Phi(X)$ means that there are arbitrary large well-orderings satisfying Φ . By $\tilde{\Phi}$, we now take $\text{Bord}(X)$ and by $\tilde{\Psi}$, $\text{Bord}(Y) \rightarrow X \prec Y$. (We stress the fact that the proofs of σ -additivity of \mathbb{Q} and Θ use the axiom of choice.)

The properties of Θ imply that every countable β -model of A_2 has the proper elementary extension which is either an ω - but not a β -model, or it is a β -model of bigger height. Let M be a countable β -model of A_2 . Suppose that every elementary extension of M which is an ω -model is also a β -model. By the above, we get an increasing chain

$$M = M_0 \prec M_1 \prec \dots \prec M_\xi \prec \dots, \quad \xi < \aleph_1,$$

of countable β -models with increasing heights.

Thus the following equivalence holds: $\text{Bord}(X) \leftrightarrow (\text{E}f) (\text{E}Y) (\text{E}N) (M \prec N \ \& \ (N \text{ is a countable } \omega\text{-model}) \ \& \ Y \in N \ \& \ N \models \text{Bord}(Y) \ \& \ (f \text{ is an isomorphism between the orderings coded by } X \text{ and } Y, \text{ respectively}))$. By applying the facts from §2, we obtain that $\text{Bord}(X)$ is a $\Sigma_1^{1,A}$ relation, where A is a code of M , this is, by a classical result of Kuratowski, impossible.

Recently, Mostowski obtained the following result:

4.7. Theorem (Mostowski). *Let M be a countable β -model of A_2 such that for some ω -model N of $\text{ZFC} + 2^{\aleph_0} > \aleph_1$, $M = N \cap \mathcal{P}(\omega)$. Then there exists a proper elementary extension of M which is a β -model with the same height. (Note that in this case, N is automatically \aleph_1 -standard.)*

A different proof of Theorem 4.7 than the one we give later, based on a theorem of Krivine and McAloon (Theorem 4.9), was given by M. Srebrny.

4.8. Definition. (a) Formula $A(\cdot)$ is countable-like ($c - \iota$) in a model M

$$\dots$$

iff

$$M \models \text{Q}x(\text{E}x)(\varphi \ \& \ A) \rightarrow (\text{E}x)\text{Q}x(\varphi \ \& \ A)$$

for all formulas $\varphi(y, x)$ such that y is not free in M .

(b) N is a complete end extension (c.e.e.) of M if $M \prec N$ and

(i) for every formula $A(\cdot)$ which is $c - \iota$ in M , if $N \models A[a]$, then $a \in M$;

(ii) for every formula $A(\cdot)$ which is not $c - \iota$ in M , there is $a \in N - M$ such that $N \models A[a]$.

4.9. Theorem (Krivine and McAloon). *M has a complete end extension.*

(Proof of this theorem may be obtained using similar reasoning as was used in Mostowski's proof of Theorem 4.7.) The theorem of Krivine and McAloon, when applied to the model of set theory, in which $2^{\aleph_0} > \aleph_1$ and with the quantifier "There is more than $\aleph_1 x$ such that ...", gives a model whose continuum N is the desired model. Indeed, N is a β -model since in ZFC every well-ordering is isomorphic with some ordinal and since we are interested in countable ordinals which are in N the same as in M .

Proof of Theorem 4.7. We give a brief sketch of the proof. The idea of the proof of this theorem somewhat resembles the original proof of Mostowski and Suzuki of Theorem 4.1 in the following sense:

In both cases one constructs step by step a sequence of finite sets of sentences. Yet there is an important difference. In the Mostowski–Suzuki Theorem 4.1, one has to produce a "false well-ordering". In Theorem 4.7, we assure that there is no new type of well-ordering. In the language of set theory we describe this situation as increasing the width but not the length of the model. In the proof of the Mostowski–Suzuki theorem, apart from the new ordering which has been constructed, many other things could have been added to the model. Here we want to make our construction as economic as possible. So we go one step further and construct directly a Henkin–Orey model as follows:

Firstly, we construct a Skolemization (as usual in a Henkin-type proof). (Let Φ^* denote an open formula of the extended language equivalent to Φ .) Now we add to the extended language L an additional constant C (a

name for the new set of natural numbers). L contains also names for all elements of $N \cap \mathcal{P}(\omega)$.

Secondly, we enumerate several sets.

(a) $\{F_j\}_{j < \omega}$ is an enumeration of all sentences of the extended language.

(b) $\{A_j\}_{j < \omega}$ is an enumeration of $\text{Th}(N \cap \mathcal{P}(\omega))$ and $\{A_j^*\}_{j < \omega}$ is the corresponding enumeration of the equivalent open formulas of the extended language.

(c) $\{t_j\}_{j < \omega}$ is an enumeration of all terms (note that here we have two kinds of terms, first order terms and second order terms).

(d) $\{D_j\}_{j < \omega}$ is an enumeration of $N \cap \mathcal{P}(\omega)$.

Finally, we are able to construct a sequence of finite sets of formulas $\{\mathfrak{B}_j\}_{j < \omega}$ subject to two groups of conditions. The first group contains auxiliary conditions needed in the Henkin proof. The second one contains some specific conditions which ensure that the construction succeeds.

Let t, t_1, t_2 denote terms in the extended language.

Group 1:

$$(1.1) \quad t_1 = t_2 \in \mathfrak{B}_n \ \& \ A(t_1) \in \mathfrak{B}_n \rightarrow A(t_2) \in \mathfrak{B}_{n+1}.$$

$$(1.2) \quad t \text{ occurs in a formula from } \mathfrak{B}_n \rightarrow t = t \in \mathfrak{B}_{n+1}.$$

$$(1.3) \quad t_1 = t_2 \in \mathfrak{B}_n \rightarrow t_2 = t_1 \in \mathfrak{B}_{n+1}.$$

$$(1.4) \quad t_1 = t_2 \in \mathfrak{B}_n \ \& \ t_2 = t_3 \in \mathfrak{B}_n \rightarrow t_1 = t_3 \in \mathfrak{B}_{n+1}.$$

(1.5) If t_1, t_2 are second order terms & $\neg(t_1 = t_2) \in \mathfrak{B}_n$, then there is a first order term s such that

$$\neg(s \in t_1 \equiv s \in t_2) \in \mathfrak{B}_{n+1}.$$

Group 2:

(2.1) \mathfrak{B}_n decides F_n .

(2.2) $(j) (j < n \rightarrow A_j^* \in \mathfrak{B}_n)$.

(2.3) If t_n is a first order term, then for some $k \in \omega, t = k \in \mathfrak{B}_{n+1}$.

(2.4) If t is a second order term and $(\text{Bord}(t))^* \in \mathfrak{B}_n$, then there is an $A \in N \cap \mathcal{P}(\omega)$, such that $N \cap \mathcal{P}(\omega) \models \text{Bord}[A]$ and such that

$\text{Sim}(t, A) \in \mathfrak{B}_{n+1}$ (where $\text{Sim}(\cdot, \cdot)$ is a formula expressing “there is a similarity map”).

(2.5) $(j) (j < n \rightarrow C \neq D_j \in \mathfrak{B}_n)$.

(2.6) $\Phi(t_{i_1}, \dots, t_{i_k}) \in \mathfrak{B}_n \rightarrow \Phi(t_{i_1}, \dots, t_{i_{j-1}}, f(\vec{t}), t_{i_{j+1}}, \dots, t_{i_k}) \in \mathfrak{B}_{n+1}$,
where f is an appropriate Skolem function for Φ .

(2.7) $N \models \overline{\{c \subseteq \omega : \mathfrak{A} \mathfrak{B}_n(c)\}} > \aleph_1$.

The reduct to $L(A_2)$ of the Henkin model constructed from terms (with the help of the equivalence relation $r \approx s \stackrel{\text{df}}{\leftrightarrow} r = s \in \bigcup_{n < \omega} \mathfrak{B}_n$) has the following properties:

- (a) It is an ω -model (by (2.3)).
- (b) It extends elementarily $N \cap \mathcal{P}(\omega)$ (by (2.2)).
- (c) It has the same types of well-orderings and so it is a β -model (by (2.4)).
- (d) It is bigger than $N \cap \mathcal{P}(\omega)$ (by (2.5)).

((2.6) and (2.1) are used to make the sequence closed under Skolem functions and to make it complete, and then (2.7) is the main thing used to make the construction of \mathfrak{B}_{n+1} possible.)

§5. The method of trees

5.0. In second order arithmetic we have only objects of type 0 (natural numbers) and of type 1 (sets of natural numbers). Thus second order arithmetic is at a great disadvantage with respect to set theory. We cannot directly perform any operations on sets of natural numbers and we have to use codes for this purpose. (We encountered this phenomenon while studying countable models of second order arithmetic.) As a natural antidote against this disadvantage we use trees.³ They provide a natural way of coding (some) higher type objects within sets of natural numbers. Thus with the help of trees we are able to model hereditarily countable sets, a natural model for ZFC^- within the continuum, the standard model for A_2 . Trees turn out to be an interpretation of

³ Let us notice that the method of trees (in various contexts) has been used by several people, e.g., Addison, Kreisel, Scott and Specker.

ZFC^- within A_2 and owing to this fact we find that under the natural embedding of A_2 into ZFC^- the latter theory is a conservative extension of the former. Every model of A_2 determines a model of ZFC^- whose continuum is isomorphic to the original model. In the case of β -models, the model obtained is well founded. This useful fact, due to Kreisel and Zbierski, allows us to use interchangingly the results concerning transitive models of ZFC^- and those concerning β -models of A_2 .

The models of ZFC^- which arise in this way may be viewed as closures of models of A_2 under certain set-theoretical operations (a similar phenomenon exists in the case of admissible sets).

5.1. Definition. (a) The theory ZFC^- is the theory formulated in the language L_{ZF} of set theory and arises from ZF as follows: We omit the power-set axiom and add the following scheme of substitution-choice:

$$(x)_z (E y) \Phi(x, y) \rightarrow (E f) (\text{Func}(f) \ \& \ \text{Dom}(f) = z \\ \& \ (x)_z \Phi(x, f(x))) .$$

(b) $V = HC$ denotes the statement “everything is countable”.

5.2. Definition. A subset X of ω is called a *tree* if

- (i) the relation $a <_X b \leftrightarrow \langle a, b \rangle \in X$ is a partial ordering with no loops;
- (ii) there is a unique maximal element MAX_X in the partial ordering $<_X$;
- (iii) every linear subordering of $<_X$ is finite.

By $AMAX_X$ we denote the set of direct predecessors of MAX_X in the ordering $<_X$.

5.3. Definition. A tree X is called a *reduced tree* if $<_X$ has no automorphisms.

We can express the fact that a set X is a reduced tree as a Π_1^1 formula, thus absolute with respect to β -models.

5.4. Definition. Let X, Y be reduced trees. Then

- (a) $X \text{Eq} Y \leftrightarrow <_X$ is isomorphic to $<_Y$,

(b) $X \text{Eps } Y \leftrightarrow X \text{Eq } Y_a$ for some $a \in \text{AMAX}_Y$ (where $Y_a = Y \cap \{\langle x, y \rangle : y <_X a\}$).

We can express both Eq and Eps relations as Σ_1^1 conditions. Take the following interpretation of L_{ZF} in $L(A_2)$: as the universe of interpretation let us take the reduced trees. Equality is interpreted as Eq and membership as Eps, respectively.

5.5. Theorem (Kreisel [34], Zbierski [67]). *Under the interpretation described above, the theory $ZFC^- + V = HC$ is interpretable in A_2 .*

The proof is straightforward but laborious and we omit all the details. The interpretation allows us to produce from models of A_2 , models of $ZFC^- + V = HC$. Let M be a model of A_2 . Let M' be the model consisting of the trees from M . Clearly, M' has no absolute notion of equality. Let M'' be the model M'/Eq . Being a model of $ZFC^- + V = HC$, M'' determines its set of natural numbers and continuum, $\omega^{M''}$ and $(\mathcal{P}(\omega))^{M''}$, respectively.

5.6. Lemma. $\langle \omega^{M''}, (\mathcal{P}(\omega))^{M''}, \in^{M''} \rangle \simeq \langle \omega^M, (\mathcal{P}(\omega))^M, \in \rangle$.

Thus we get the following:

5.7. Theorem (Kreisel and Zbierski). *The theory $ZFC^- + V = HC$ is a conservative extension of A_2 (under the usual embedding of A_2 into ZF).*

A specially interesting case is when M is a β -model of A_2 . Then the relation Eps is well founded and the model M'' is thus also well founded (the formulas $X \text{Eq } Y$ and $X \text{Eps } Y$ are absolute with respect to β -models and therefore the well-foundedness is preserved). Using Mostowski's collapsing theorem, we get \bar{M} , the transitive collapse of M'' .

Using Lemma 5.6, we get the following:

5.8. Theorem (Zbierski). *If M is a β -model of A_2 , then $M = \bar{M} \cap \mathcal{P}(\omega)$.*

5.9. Corollary. *M is a β -model of A_2 iff there is transitive N , $N \models ZFC^- + V = HC$, such that $M = N \cap \mathcal{P}(\omega)$.*

It is easy to see that $h(M) = h(\bar{M})$ and since $M \models \text{ZFC}^-$, $h(M)$ is Σ_n admissible for every n , i.e., $h(M)$ is a regular quasicardinal (cf. Kripke [35]).

Trees serve as codes for higher type sets.

5.10. Definition. Let X be a tree. By $\|X\|$ we denote the realization of the tree, i.e., the operation defined as follows:

$$\|a\|_X = \{\|b\|_X : a \text{ is a } <_X\text{-successor of } b\},$$

$$\|X\| = \|\text{MAX}_X\|_X.$$

By induction on the rank, we get that for every $a \in \text{HC}$ (the collection of all hereditarily countable sets), there exists a tree X such that $\|X\| = a$. Conversely, if $\|X\| = a$ for X a tree, then $a \in \text{HC}$.

Also it is easy to see that the realizations of trees from a β -model M (which are thus real trees) form a transitive model of $\text{ZFC}^- + V = \text{HC}$. From the definition of the collapsing isomorphism we find that this model is just \bar{M} .

The above interpretation is the formalization of the above facts.

A natural question arises if the analogue of the above theorem on the conservative extension may be proved when the scheme of choice is deleted from both theories. We have two results here.

Let Z be the set theory of Zermelo.

5.11. Theorem (Kreisel [34]). Z^- (i.e. the theory of Zermelo without the power set axiom) is a conservative extension of A_2^- .

In fact, the same interpretation works.

5.12. Theorem (Gandy [19]). ZF^- is not a conservative extension of A_2^- .

Thus we see that the interpretation of the replacement scheme cannot be proved without using the scheme of choice.

The model theoretic counterpart of Zbierski's Theorem 5.8 on β -models of A_2 is in the case of A_2^- somewhat bizarre.

We have the following:

5.13. Theorem. *Let M be an ω -model of A_2^- . Then there exists a transitive model N of Z^- such that $M = N \cap \mathcal{P}(\omega)$.*

Note that M does not need to be a β -model.

Proof. In the standard way, we define the height of a tree. To get the model N we consider all the trees of height less than $\omega + \omega$ which are elements of M . N is now the set of their realizations.

This striking difference in relationship between A_2^- and A_2 and their conservative extensions Z^- and ZFC^- , respectively, is connected with the different status of the formula “ x is a well-ordering” in Z^- and ZFC^- . In the first it is Π_1 whereas in the second Δ_1 .

Addison [2] showed that there exists a Σ_2^1 formula $\text{Constr}(X)$ which is satisfied in $\mathcal{P}(\omega)$ exactly by the constructible subsets of ω . This formula, roughly speaking, says that there exists a well-ordering T , such that X is $F'_{|T|}$, where F is the function from Gödel’s monograph [22].

By the absoluteness results concerning β -models, we have the following:

5.14. Lemma. *If M is a β -model of A_2^- , then $(M \models \text{Constr}[X]) \rightarrow X \in L$, i.e., $\mathcal{L}^M = \{X: M \models \text{Constr}[X]\} \subseteq M \cap L$.*

Let us notice that a simple Cohen argument allows us to state the following lemma:

5.15. Lemma. *There is a β -model M of A_2^- such that $M \in L$ and $\mathcal{L}^M \neq M \cap L$.*

5.16. Theorem (Enderton [13]). *If M is a β -model of A_2^- , then so is \mathcal{L}^M .*

5.17. Corollary. $\text{R.A.}_{\beta_0} \models (X) \text{Constr}(X)$.

The proof is based on a careful inspection of the nature of Addison’s formula.

In the case when M is model of A_2 , the proof due to Zbierski easily follows from our previous considerations. Namely,

$$\begin{aligned} \mathcal{L}^M &= \{X: M \models \text{Constr}[X]\} = \{X: \bar{M} \models \text{Constr}[X]\} \\ &= \{x: \bar{M} \models x \subseteq \omega \ \& \ L(x)\} = \{x: L^{\bar{M}} \models x \subseteq \omega\} = L^{\bar{M}} \cap \mathcal{P}(\omega). \end{aligned}$$

$L^{\bar{M}}$ is a transitive model of ZFC^- , thus \mathcal{L}^M is a β -model.

We also have the following:

5.18. Theorem (Zbierski). *If M is an ω -model of A_2 , then so is \mathcal{L}^M .*

Proof. Let φ be an axiom of A_2 . Then $\text{ZFC}^- \vdash \varphi$. An analysis of Gödel's proof shows that L is an inner model of ZFC^- . Thus $\text{ZFC}^- \vdash (\varphi)^L$. On the other hand, an analysis of Addison's proof shows that we can dispense with the power set axiom, i.e., $\text{ZFC}^- \vdash (x) (x \subseteq \omega \rightarrow (\text{Constr}(x) \leftrightarrow L(x)))$. Hence $\text{ZFC}^- \vdash \varphi \text{ Constr}$ ($\varphi \text{ Constr}$ is the formula which results from φ by the relativization of its set quantifiers to the formula Constr). But ZFC^- is a conservative extension of A_2 , thus $A_2 \vdash \varphi \text{ Constr}$, hence $M \models \varphi \text{ Constr}$, i.e., $\mathcal{L}^M \models \varphi$. (In fact, the proof shows that $\text{Constr}(\cdot)$ is an inner interpretation of A_2 in A_2 , which preserves natural numbers.)

Yet the fact that \mathcal{L}^M is a β -model does not ensure that M is also a β -model.

To see this, consider a transitive model of $\text{ZFC}^- + V = \text{HC} + \text{“}\omega_1^L \text{ exists”}$. A model of this sort is easily obtained by the Cohen forcing. Applying the Mostowski–Suzuki Theorem 4.1 to the continuum of this model, we get an ω -model with the same constructible sets, which is not a β -model (this proof is based on an observation of W. Powell).

As an additional profit from this remark, we note that the height of \mathcal{L}^M may be smaller than the height of the β -model M (take M to be the continuum of a transitive model of $\text{ZFC}^- + V = \text{HC} + \text{“}\omega_1^L \text{ exists”}$).

The following theorem easily follows from an analysis of the operation $\bar{\cdot}$.

5.19. Theorem (Zbierski). *If N is a transitive model of ZFC^- , then $\overline{N \cap \mathcal{P}(\omega)} = \text{HC}^N$. In particular, if $N \models V = \text{HC}$, then $\overline{N \cap \mathcal{P}(\omega)} = N$.*

The method of trees gives us some additional information about β -models of $A_2 + (X) \text{ Constr}(X)$.

5.20. Theorem (Zbierski). *Let M be a β -model of $A_2 + (X) \text{Constr}(X)$. Then $M = L_{h(M)} \cap \mathcal{P}(\omega)$ and $L_{h(M)} = \bar{M}$.*

5.21. Corollary. *L_{β_0} is the smallest transitive model of ZFC^- .*

Proof. We have already proved $L^{\bar{M}} \cap \mathcal{P}(\omega) = M$. But $h(L^{\bar{M}}) \leq h(M)$, therefore $h(L^{\bar{M}}) = h(M)$. \bar{M} is transitive, thus $L^{\bar{M}}$ is equal to some L_α so $L^{\bar{M}} = L_{h(M)}$. By Theorem 5.20, $\bar{M} = \overline{L_{h(M)} \cap \mathcal{P}(\omega)} = \text{HC}^{L_{h(M)}} \subseteq L_{h(M)} = L^{\bar{M}}$, i.e., $\bar{M} = L_{h(M)}$. Now the corollary follows from the facts that $\text{R.A.}_{\beta_0} \models A_2 + (X) \text{Constr}(X)$ and $h(\text{R.A.}_{\beta_0}) = \beta_0$.

‘We now establish some further connections between β -models of $A_2 + (X) \text{Constr}(X)$ and the constructible hierarchy.

5.22. Definition. (a) α is called a *gap* iff $(L_{\alpha+1} - L_\alpha) \cap \mathcal{P}(\omega) = \emptyset$.

(b) α is called the *beginning of a gap* if α is a gap, but

$$(\beta)_\alpha (L_\beta \cap \mathcal{P}(\omega) \neq L_\alpha \cap \mathcal{P}(\omega)).$$

5.23. Theorem (Leeds and Putnam [37], Marek and Srebrny [44]).

(a) *If α is a gap, then $L_\alpha \cap \mathcal{P}(\omega)$ is a β -model of $A_2 + (X) \text{Constr}(X)$.*

(b) *α is the beginning of a gap iff L_α is a model of $\text{ZFC}^- + V=L=\text{HC}$.*

Leeds and Putnam prove (a) using the fact that in this case L_α is closed under the hyperjump operation. Another proof is obtained using the method of trees.

In fact, (a) follows from (b). We prove (b) as follows: just because α is a gap, $L_\alpha \cap \mathcal{P}(\omega)$ is an ω -model of A_2^- . Now using the method of arithmetical copies of Boolos and Putnam [7]⁴ (a subset X of ω is called an arithmetical copy of L_α if $\langle L_\alpha, \in \rangle$ is isomorphic to $\langle \text{Fld}(X), <_X \rangle$, where $a <_X b \leftrightarrow (a, b) \in X$), we find that $V=\text{HC}$ holds in L_α . Then we use the fact that if α is a gap, then the following form of the scheme of choice holds in L_α :

$$(n)_\omega (\text{EX})_{\mathcal{P}(\omega)} \Phi(n, X) \rightarrow (\text{EY})_{\mathcal{P}(\omega)} (n)_\omega \Phi(n, Y^{(n)})$$

(for all set-theoretical Φ).

⁴ This method was first used by Boolos in proving his theorem on the connections between the ramified analytical hierarchy and the constructible hierarchy.

These facts together imply the reflection property for L_α which combined with the fact that α is limit (and so Δ_0 -comprehension holds) settle replacement. There is a definable well-ordering of L_α , so the scheme of substitution–choice holds as well.

5.24. Proposition. *If $L_\alpha \models Z^- + V = HC$, then $L_\alpha \models ZFC^- + V = HC = L$.*

We may interpret it as a weak semantical counterpart of the following result:

5.25. Theorem. *Each of the theories $A_2^-, A_2, ZFC^-, ZFC^- + V = HC, Z^-, Z^- + V = HC$ is interpretable in the other.*

A number of results on ω -models and β -models, which have analogues in set theory, have been established. Let us note among them the following:

5.26. Theorem (Mostowski [50]). *Every countable ω -model of $A_2 + DC$ may be expanded by a binary relation, which is a linear ordering such that in the expanded model the following conditions exist:*

- (1) *Every definable class (in the extended language) has a first element.*
- (2) *Comprehension and the scheme of choice (in the extended language) hold.*

DC is the following scheme:

$$(X) (\exists Y) \Phi(X, Y) \rightarrow (\exists X) (n) \Phi((X)_n, (X)_{n+1}).$$

The similar fact for ZFC set theory has been proved by Felgner [16].

5.27. Lemma.⁵ *The smallest transitive model of ZFC^- is pointwise definable and has no proper transitive elementary extension.*

As a simple application of Zbierski's theorem and Lemma 5.27, we get the following:

⁵ Mostowski [48] proved that the smallest transitive model of ZF has no proper transitive elementary extensions.

5.28. Theorem. *The smallest β -model of A_2 has no proper elementary extension being a β -model.*

In connection with this fact, Marek [41] studied the so-called “Dam Hypothesis”.

5.29. Dam Hypothesis. Let M be a countable β -model of A_2 . Then there is an elementary extension of M , say M_1 , being a β -model such that if M_2 is an elementary extension of M_1 , then either M_2 is not a β -model or $h(M_2) = h(M_1)$.

It may be shown that the hypothesis holds if $\mathcal{P}(\omega) \subseteq L$ or $M \models (X) \text{Constr}(X)$.

Zarach [66] developed the theory of forcing with classes for the case of ZFC^- . Using this, Guzicki [25] obtained the following result on uncountable β -models of A_2 .

5.30. Theorem. (a) *If M is a countable β -model of A_2 , then there exists an uncountable β -model of A_2 , $M_1 \supseteq M$ such that $h(M_1) = h(M)$.*

(b) *Assuming Martin’s axiom (cf. Martin and Solovay [45]), M_1 can be found of the power continuum.*

The proof uses Zbierski’s theorem and the striking fact that in the case of ZF^- some generic extensions of some models may happen to be elementary ones (which is impossible for the full ZF case).

Part (b) provides (under the assumption of Martin’s axiom) a positive answer to the following question of Mostowski: Do there exist two different ω -models of A_2 of the power of the continuum?

Recently, G. Sacks (private communication) has shown a positive solution of this question.

§6. ω -models of the theory of $\mathcal{P}(\omega)$

6.0. Among extensions of A_2 specially interesting is the theory of the continuum $\text{Th}(\mathcal{P}(\omega))$. Unfortunately, not too much is known about models of it. They have been studied by Ellentuck [11] and Mostowski

[52]. Let us note that the theory of $\mathcal{P}(\omega)$ depends on the set theory in which the considerations are carried out and thus the structure of the family of ω -models of $\text{Th}(\mathcal{P}(\omega))$ may vary.

6.1. Theorem (Ellentuck [11]). *The following facts are equivalent:*

- (i) $\text{Def}(\mathcal{P}(\omega)) \models \text{Th}(\mathcal{P}(\omega))$,
- (ii) $\text{Def}(\mathcal{P}(\omega))$ is the smallest ω -model of $\text{Th}(\mathcal{P}(\omega))$,
- (iii) $\text{Def}(\mathcal{P}(\omega)) \prec \mathcal{P}(\omega)$,
- (iv) the analytical basis theorem (every non-empty analytical family of sets contains an analytical element) holds.

(Recall that $\text{Def}(\mathcal{P}(\omega))$ is the family of all subsets of ω definable in $\mathcal{P}(\omega)$ by a formula of $L(A_2)$, i.e., the family of all analytical sets).

Note that (iv) is true if there is an analytical (without parameters) well-ordering of the continuum. The last is true when the axiom of constructibility (or $\mathcal{P}(\omega) \subseteq \check{L}$) holds. In that case, there is a Δ_2^1 well-ordering of the continuum (see Addison [2]).

Let us note that one can show the following (see Marek [39, 42]):

6.2. Theorem. (i) *If M is an ω -model of A_2 , then $M \models \text{Th}(\mathcal{P}(\omega))$ iff $\bar{M} \models \text{Th}(\text{HC})$.*

(ii) *There is an analytical well-ordering of the continuum iff there is a definable well-ordering of HC.*

Mostowski [51] considers the following relation between countable ω -models: $M \in N$ iff there is a code of M which is an element of N .

6.3. Theorem (Mostowski). *The type $\eta \cdot \aleph_1$ can be embedded into the partial ordering ϵ of the ω -models of $\text{Th}(\mathcal{P}(\omega))$.*

Some other results of a similar type are proved in [46, 51].

§7. Models of higher order arithmetics and Kelley–Morse set theory

7.0. Consecutively adjoining higher types, we get so-called higher order arithmetics. Some methods introduced in the case of second order arith-

work here as well. In particular, the method of trees gives fairly strong results. Another theory which – roughly speaking – plays the same role with respect to ZF set theory as does A_2 with respect to Kelley–Morse arithmetic (this has been pointed out to us by G. Kreisel) is the Kelley–Morse set theory.

A common feature here is the “highest” level and the full comprehension scheme (or the full choice scheme). In the case of Kelley–Morse theory, we get similar results which seem to suggest deeper analogies behind. The analogy between the results obtained should not obscure some very important differences between A_2 on the one hand and the higher order arithmetics and the Kelley–Morse theory on the other. In particular, the position of the notion of well-ordering is different in the two cases. This, in turn, leads to a different status of constructibility in these theories.

Theorem (Zbierski [67]). *Let $n \geq 2$.*

-) *The full n th order arithmetic A_n (i.e., the theory with the full axiom of choice) is mutually interpretable with the theory $ZFC^- + \mathcal{P}^{(n-2)}(\omega)$ exists”.*
-) *$ZFC^- + \mathcal{P}^{(n-2)}(\omega)$ exists” is a conservative extension of A_n .*
-) *The operation $\bar{\cdot}$ works as in the case of $n = 2$ (see page 206), and $\mathcal{P}^{n-1}(\omega) = M$ in the case when M is a β -model of A_n .*
-) *$\mathcal{P}^{(n-1)}(\omega) = H(I_{n-2})$ (the sets of the power hereditarily $\leq I_{n-2}$).*

The connections between $A_2 + (X) \text{Constr}(X)$ and the gaps in the constructible universe found by Leeds and Putnam [37] in the case of A_2 were extended by Marek and Srebrny [44] for the case of the full second order arithmetic.

Definition. α is an n th order gap iff

$$(L_{\alpha+1} - L_\alpha) \cap \mathcal{P}(\omega) = \dots = (L_{\alpha+1} - L_\alpha) \cap \mathcal{P}^{(n-1)}(\omega) = \emptyset.$$

Theorem. *If α is an n th order gap, then $L_\alpha \cap \mathcal{P}^{(n-1)}(\omega)$ is a β -model of the full n th order arithmetic + the appropriate form of the axiom of constructibility.*

The proof, using the method of trees, is similar to that of the A_2^- case with the following change: One considers a copy of L built not from integers but from elements of $\mathcal{P}^{(n-2)}(\omega)$.

The Kelley–Morse theory of classes (called below \mathfrak{M}) (cf. Mostowski [49]) resembles the second order arithmetic in many points, yet there are striking differences. Roughly speaking it is an impredicative extension of ZF set theory and thus arises in a similar way as second order arithmetic arises from the Peano arithmetic. (One can imagine second order arithmetic simply as the Kelley–Morse theory without the axiom of infinity and third order arithmetic as the Kelley–Morse theory without the axiom of the powerset in the sense of mutual interpretability – as shown in Marek and Srebrny [43].)

Several similarities between A_2 and \mathfrak{M} were noted by Marek [40] and Guzicki [26]. The most striking difference is the status of the formula Bord in both theories. In A_2 , Bord is Π_1^1 (i.e., not predicative) whereas in \mathfrak{M} it is predicative. This difference leads to a “one–down” phenomenon occurring in \mathfrak{M} with respect to A_2 . For instance, the constructibility formula is Σ_1 in \mathfrak{M} (being Σ_2^1 in A_2). This in turn allows us (under the assumption that the class form of the axiom of constructibility is true) to generalize Addison’s results on the separation and reduction principles (Σ_n definable superclasses possess the reduction property). Also under the same assumptions one gets a Δ_1 (over \mathfrak{M}) definable well-ordering of all classes. (These results are due to Guzicki [26].)

Another analogy was noted by Marek [40]. Let T be a set-theoretical statement asserting the existence of a strongly inaccessible family (i.e., a transitive family X containing ω , closed under the power set and the image of elements under functions included in X). Let \mathfrak{M}' be the Kelley–Morse theory together with the full impredicative scheme of choice; then we have the following:

- 7.4. Theorem.** (i) \mathfrak{M}' is mutually interpretable with $ZFC^- + T$.
(ii) $ZFC^- + T$ is a conservative extension of \mathfrak{M}' (under the interpretation $\text{Cls}(x) \leftrightarrow x \subseteq X$, where X is an inaccessible family).
(iii) The operation $\bar{\cdot}$ works as in the case of A_2 .
(iv) If M is a β -model of \mathfrak{M}' , then $\bar{M} \cap \mathcal{P}(X) = M$, where X is the maximal inaccessible family in \bar{M} .

The proof of this theorem strongly resembles Zbierski's proof. The analogue of the Mostowski–Suzuki Theorem 4.1 for the case of \mathfrak{M}' holds. Analogous results for higher order set theories were obtained by Marek and Zbierski.

Let us note finally that Guzicki (in his thesis [26]) finds an interesting extension of \mathfrak{M}' mutually interpretable with $ZFC + T$.

§8. β_n -models

8.0. The hierarchy of formulas of $L(A_2)$ naturally introduces a hierarchy of ω -models of A_2^- . As we know, ω -models preserve arithmetical statements and β -models Σ_1^1 statements. β_n -models are thus introduced as ω -models preserving Σ_n^1 statements.

As in the case of the smallest β -model, there exists a smallest β_2 -model. It may also be characterized as a result of union of certain hierarchy. In the case of $n > 2$, the situation becomes more complicated. Under suitable conditions (the n th basis assumption) there is a smallest β_n -model and an analogous hierarchy may be used to characterize it. The situation is analogous to that in higher levels of the analytical hierarchy where – in order to get certain results – we have to assume additional hypotheses.

8.1. Definition. (a) If $M \subseteq N \subseteq \mathcal{P}(\omega)$, then $M <_n^1 N$ iff for every Σ_n^1 formula Φ and sequence of parameters \vec{X} from M , we have

$$N \models \Phi[\vec{X}] \leftrightarrow M \models \Phi[\vec{X}] .$$

(b) An ω -model M is called a β_n -model iff $M <_n^1 \mathcal{P}(\omega)$.

Obviously, in this case, M preserves Π_{n+1}^1 formulas downward and Σ_{n+1}^1 formulas upward.

8.2. Theorem. *If M is an ω -model, then M is a β_n -model iff M is a basis for the Π_{n-1}^1 collections of sets with the parameters from M .*

Using Kleene's universal Σ_n^1 relation, we get that the statement “ X is a code of a β_n -model” is a Π_n^1 formula. Therefore we get the following corollary:

8.3. Corollary. *Every β_{n+1} -model contains a code of a β_n -model.*

Using Shoenfield's lemma (which states that $L \cap \mathcal{P}(\omega)$ is a β_2 -model) and the fact that $\text{Constr}(X)$ is a Σ_2^1 formula, we get the following:

8.4. Theorem (Enderton and Friedman [14]). *If M is a β_2 -model of A_2^- , then*

- (i) $\mathcal{L}^M = L \cap M$,
- (ii) $\mathcal{L}^M < \frac{1}{2} M$, i.e., \mathcal{L}^M is a β_2 -model of A_2^- .

As a corollary, we get that there exists a smallest β_2 -model of A_2^- . In this model the scheme of choice and $(X)\text{Constr}(X)$ hold also.

8.5. Definition. (a) Let $M \subseteq N$ be structures for the language of set theory. M possesses the property of Σ_n reflection with respect to N ($M <_n N$) if for every Σ_n formula Φ (in the sense of Lévy) of L_{ST} and sequence parameters \vec{x} from M ,

$$M \models \Phi[\vec{x}] \leftrightarrow N \models \Phi[\vec{x}] .$$

M possesses the property of Σ_n reflection if $M <_n V$.

- (b) An ordinal α is stable if L_α possesses the property of Σ_1 reflection.

A characterization of β_2 -models of A_2 , analogous to the one obtained by Zbierski in the case of β -models of A_2 is the following:

8.6. Theorem (Marek [42]). ⁶ *M is a β_2 -model iff \bar{M} is a model of $\text{ZFC}^- + V = \text{HC}$ with the property of Σ_1 reflection.*

Using similar methods one gets:

8.7. Theorem. *Let α be the beginning of a gap. Then $L_\alpha \cap \mathcal{P}(\omega)$ is a β_2 -model of A_2 iff α is stable.*

Now using the Enderton–Friedman Theorem 8.4 and Zbierski's theorems, we obtain the following corollary:

⁶ Using this theorem we may prove that the smallest β_2 -model of A_2 cannot be elementarily extended to a β -model.

8.8. Corollary. *The smallest β_2 -model of A_2^- (A_2) is equal $L_\alpha \cap \mathcal{P}(\omega)$, where α is the least stable ordinal such that $L_\alpha \models \text{ZFC}^-$ (i.e., α is the least stable gap).*

The proofs of both above theorems may be obtained using the following translation lemma:

8.9. Lemma. *Let M be a β -model of A_2 . Then:*

(a) *for every formula φ of L_{ST} , there exists a formula φ^T of $L(A_2)$ such that if $\|A_1\| = a_1, \dots, \|A_n\| = a_n$, then*

$$M \models \varphi^T[A_1, \dots, A_n] \leftrightarrow \bar{M} \models \varphi[a_1, \dots, a_n];$$

(b) *in the case when φ is a Δ_0 formula, φ^T may be chosen as Δ_1^1 , if φ is a Σ_n formula ($n \geq 1$), then φ^T may be chosen as a Σ_{n+1}^1 formula.*

Moreover the choice of a formula does not depend on a model M and the lemma holds even in the case of arbitrary models after suitable precautions.

8.10. Theorem (Enderton and Friedman [14]). *The smallest β_2 -model of A_2^- has height less than the constructible δ_3 and has a (constructible) Δ_3^1 code.*

Using Theorems 8.7 and 8.10, we get the following:

8.11. Corollary. *There are stable gaps less than the constructible δ_3 (which is less than or equal to the “real” δ_3).*

(We recall that δ_n denotes the least non Δ_n^1 ordinal.) Let us notice that the assumption that α is the beginning of a gap cannot be omitted in Theorem 8.10. δ_2 serves as a counterexample. Namely Shoenfield [62] proved that $L_{\delta_2} \cap \mathcal{P}(\omega) = \Delta_2^1$ and Δ_2^1 is not an ω -model of A_2 (see Remark 8.16). On the other hand, δ_2 is stable ⁷ (Kripke [35]).

Another construction of the smallest β_2 -model of A_2^- is due to Shilleto [60]. His point of departure is the following corollary of the Kondo–Addison theorem.

⁷ A proof that δ_2 is stable, which uses the method of trees, can be found in Marek [42].

8.12. Corollary. *An ω -model M is a β_2 -model iff it is closed under relative Δ_2^1 -ness.*

This “ramified style” construction of the smallest β_n -model is the following (we give here a simpler construction of Enderton and Friedman [14]):

$$\begin{aligned} F_0^2 &= \emptyset, \\ F_{\alpha+1}^2 &= \bigcup_{X \in \text{Def}(F_\alpha^2)} \Delta_2^{1,X}, \\ F_\lambda^2 &= \bigcup_{\alpha < \lambda} F_\alpha^2 \quad \text{for } \lambda \text{ limit.} \end{aligned}$$

8.13. Theorem (Enderton and Friedman [14]). *The smallest β_2 -model of A_2^- is equal to F_α^2 for α the smallest such that $F_{\alpha+1}^2 = F_\alpha^2$.*

This construction is generalized for the case $n > 2$ by Shilleto [60] under the assumption $\mathcal{P}(\omega) \subseteq L$ and by Enderton and Friedman [14] under a weaker n th basis theorem assumption: the $\Delta_n^{1,X}$ sets form a basis for $\Sigma_n^{1,X}$ families of sets.

8.14. Remark. Using basis theorems, we can obtain a very easy proof of the fact that for no n , A_2^- (A_2) is axiomatizable using only Π_n^1 formulas. Assume $V=L$. Then the n th basis theorem holds (see Addison [2]). Hence by the Tarski–Vaught test, $\Delta_n^1 <_n^1 \mathcal{P}(\omega)$ which implies that $D(\Delta_n^1) \supseteq \Pi_n^1 \cup \Sigma_n^1$ and so Δ_n^1 is not an ω -model of A_2^- . (For $n=2$, we may omit the assumption $V=L$ by using the Kondo–Addison theorem.) Suppose that for some n , A_2^- (A_2) is axiomatizable by using Π_n^1 formulas. Then, by the above, Δ_n^1 is an ω -model of A_2^- , which gives a contradiction. Now we can eliminate the assumption $V=L$, because the statement “For any n , A_2^- (A_2) is not axiomatizable by using Π_n^1 formulas” is a Π_1^1 sentence, thus absolute with respect to L .

8.15. Theorem (Enderton and Friedman). *Assume the n th basis theorem ($n \geq 2$). Then the sets strongly representable in A_{β_0} (the set of all sentences true in all β_n -models of A_2^-) are exactly the elements of the smallest β_n -model of A_2^- .*

8.16. Remark. It is easy to see that under the same assumptions, this smallest β_n -model cannot be elementarily extended to a β_n -model. Under the yet stronger assumption $\mathcal{P}(\omega) \subseteq L$ one may prove that this smallest β_n -model cannot be elementarily extended to a β -model.

The proof of Theorem 8.15 uses a ramified style construction analogous to the above. The basis assumption implies that the ω -model of A_2^- obtained in such a way is a β_n -model. The heart of the proof is the lemma analogous to the main lemma from the proof that the ramified analytical hierarchy is the smallest β -model of A_2^- . However, the proof of this lemma is much more difficult, because the construction of the model is more complicated.

In the general case of the β_n -models, the following characterization holds:

8.17. Theorem. *A β -model M of A_2 is a β_n -model iff $\bar{M} \models \text{ZFC}^-$ and $M \prec_{n-1} \text{HC}$.*

Observe that the arithmetical axiom of constructibility $(X) \text{Constr}(X)$ is a Π_3^1 sentence, thus if it is true, then it is true in every β_2 -model. On the other hand, Shoenfield's lemma says that $L \cap \mathcal{P}(\omega)$ is a β_2 -model of $A_2 + (X) \text{Constr}(X)$.

In opposition to β -models and β_2 -models of A_2 , β_3 -models have the following feature:

8.18. Lemma. (a) *There exists a β_3 -model of $A_2 + (X) \text{Constr}(X)$ iff $\mathcal{P}(\omega) \subseteq L$.*

(b) *If M is a β_3 -model, then $h(L^M) = h(M)$ iff $\omega_1 = \omega_1^L$.*

The existence of a smallest β_3 -model of A_2^- (A_2) is relatively consistent with the existence of a Ramsey cardinal. This follows from the fact that the required third basis assumption holds in $L[0^\#]$ (cf. Solovay [63]).

§9. Infinitistic rules of proof

9.0. The various applications of the ω -consistency theorem suggest that we should look for some other infinitary rules of proof which would satisfy the consistency theorem and the completeness theorem. The existence of such rules would lead to some new methods of construction of models of A_2^- .

Enderton has introduced the \mathcal{A} -rule which comes from the notion of Souslin's quantifier. However, it turns out that this rule does not satisfy either the consistency or the completeness theorem. On the other hand, the syntactical properties of the \mathcal{A} -rule give interesting characterizations of the recursion in $E_1^\#$, an object of type 2.

The question: "does there exist a syntactical β -rule?" seems to be especially interesting. We shall see that the existence of such a rule is a special case of a general fact. On the other hand, the theorem which states that no β -rule can be found among those which result from the notion of a generalized quantifier shows that it seems hopeless to find such a rule among natural rules. This theorem, imposing severe restrictions on the methods of constructing β -models, indicates a radical difference between the notion of an ω -model and a β -model.

We begin our considerations in the general situation.

9.1. **Definition.** Let L be the classical language.

(i) f is a rule of proof if f is a partial mapping from the power set of the formulas into formulas. If T is a set of sentences of L , then $(T)_f$ denotes the closure of T under logical consequences and the rule f .

(ii) A rule f is sound in a structure \mathcal{U} if $(\text{Th}(\mathcal{U}))_f = \text{Th}(\mathcal{U})$.

(iii) A set of sentences T is f -consistent if $\text{Cn}(T) = T$ and $X \subset T$ & $X \in \text{dom } f \rightarrow \neg f(X) \notin T$.

(iv) A set of sentences T is f -complete if $\text{Cn}(T) = T$ and $X \subset T$ & $X \in \text{dom } f \rightarrow f(X) \in T$.

9.2. **Definition.** (i) A rule f satisfies the completeness theorem if for every set T of sentences $(T)_f = \{\Phi: \Phi \text{ is a sentence and } \mathcal{U} \models \Phi \text{ for every } \mathcal{U} \text{ such that } \mathcal{U} \models T \text{ and } f \text{ is sound in } \mathcal{U}\}$.

(ii) A rule f satisfies the consistency theorem if every consistent, f -complete set of sentences has a model in which f is sound.

(iii) A class \mathbf{K} of structures is a semantics of the rule f if for every set

T of sentences, $(T)_f = (T)_\mathbf{K} = \{\Phi: \Phi \text{ is a sentence and } \mathfrak{A} \models \Phi \text{ for every } \mathfrak{A} \text{ such that } \mathfrak{A} \in \mathbf{K} \text{ and } \mathfrak{A} \models T\}$.

9.3. Remark. (i) If a rule f has a semantics, then f satisfies the completeness theorem.

(ii) If a rule has a semantics and satisfies the consistency theorem, then every consistent f -complete set of sentences has a model belonging to the semantics.

(iii) If a rule satisfies the completeness theorem, then it satisfies the consistency theorem.

The following general result holds.

9.4. Theorem. *Let L be a language and \mathbf{K} a class of structures for L . Then there exists a rule f such that \mathbf{K} is a semantics of f and every f -consistent set of sentences has a model belonging to \mathbf{K} .*

Let $f_T: \text{Cn}(\emptyset) \xrightarrow{\text{onto}} (T)_\mathbf{K}$. Then the following rule is the required one:

$$\text{dom } f = \bigcup_{\substack{T=\text{Cn}(T) \\ \Phi \in \text{Cn}(\emptyset)}} \{(T - \text{Cn}(\emptyset)) \cup \{\Phi\}\}$$

and

$$f((T - \text{Cn}(\emptyset)) \cup \{\Phi\}) = f_T(\Phi).$$

In the above terminology, the ω -rule has a semantics, namely ω -models, and it satisfies the consistency theorem and the completeness theorem.

An interesting rule of proof has been introduced by Enderton [12]. Enderton worked in the second order arithmetic with function variables and therefore we have to change his rule in order to express it in $L(A_2)$.

\mathcal{A} -rule. For any set X from $\vdash_{\mathcal{A}} \Phi(\bar{X}(n))$ for each n , infer $\vdash_{\mathcal{A}} (\text{EX})(x) \Phi(\bar{X}(x))$.

It turns out that the ω -rule is derivable in A_2^- from the \mathcal{A} -rule, but not conversely.

9.5. Definition (Enderton [12]). A model M of $L(A_2)$ is called a $d\beta$ -model if $M \models \text{Bord}[X]$ implies that X is a well-ordering provided that $X \in \text{Def}(M)$.

Enderton noted that every $d\beta$ -model of A_2^- is an ω -model but not conversely. Also, by the Mostowski–Suzuki Theorem 4.1, every countable β -model of A_2 has an elementary extension which is a $d\beta$ -model but not a β -model.

Also the following theorem, based on an observation of Friedman, holds.

9.6. Theorem. *Every $d\beta$ -model of A_2 contains as a subset a β -model of A_2 .*

Proof. If M is a $d\beta$ -model of A_2 , then \mathcal{L}^M is also a $d\beta$ -model of A_2 . But then, since \mathcal{L}^M has definable Skolem functions, $\text{Def } \mathcal{L}^M < \mathcal{L}^M$. Finally, $\text{Def } \mathcal{L}^M$ is a pointwise definable $d\beta$ -model and so it is a β -model.

The following simple theorem gives the reason for introducing the notion of a $d\beta$ -model here.

9.7. Theorem (Enderton [12]). (i) *An ω -model M of A_2^- is a $d\beta$ -model iff the \mathcal{A} -rule is sound in M .*

(ii) *If $\Phi \in (A_2^-)\mathcal{A}$, then Φ is true in every $d\beta$ -model of A_2^- .*

On the other hand, we have the following:

9.8. Theorem (Apt [5]). *The \mathcal{A} -rule does not satisfy either the completeness theorem or the consistency theorem.*

Proof. It is easy to see that if the \mathcal{A} -rule satisfies the consistency theorem (Theorem 2.3), then it satisfies the completeness theorem (Theorem 2.4). By the above results, it is now sufficient to prove that $d\beta$ -models do not form the semantics of the \mathcal{A} -rule.

One easily checks that $\ulcorner (A_2 \cup \{(X) \text{ Constr}(X)\})_{\mathcal{A}} \urcorner \in \Sigma_2^1$ whereas by the above facts $\ulcorner \{\Phi: \mathfrak{A} \models \Phi \text{ for every } \mathfrak{A} \text{ such that } \mathfrak{A} \text{ is a } d\beta\text{-model of } A_2 + (X) \text{ Constr}(X)\} \urcorner = \ulcorner (A_2 + (X) \text{ Constr}(X))_{\beta} \urcorner \in \Pi_2^1 - \Sigma_2^1$. It even turns out that one can replace here $A_2 + (X) \text{ Constr}(X)$ by A_2 .

The problem of the weak and strong representability in $(A_2^-)_{\mathcal{A}}$ (i.e., the closure of A_2^- under the \mathcal{A} -rule) was solved by Aczel [1].

Let $E_1^{\#}$ be the functional such that for partial functions f ,

$$E_1^\#(f) \simeq \begin{cases} 0 & \text{if } \forall \alpha \exists n f(\bar{\alpha}(n)) = 0, \\ 1 & \text{if } \exists \alpha \forall n f(\bar{\alpha}(n)) > 0. \end{cases}$$

Aczel slightly changes Kleene's definition (see Kleene [32]) of the recursion in the functional defined on total functions in order to define the recursion in functionals defined on partial functions.

9.9. Theorem (Aczel). (a) *The following are equivalent for $X \subseteq \omega$:*

- (i) *A set X is weakly representable in $(A_2^-)_\mathfrak{A}$,*
- (ii) *X is semirecursive in $E_1^\#$,*
- (iii) *X is inductively definable with respect to a Σ_1^1 monotone relation*

(b) *The following are equivalent for $X \subseteq \omega$:*

- (i) *X is strongly representable in $(A_2^-)_\mathfrak{A}$,*
- (ii) *X is recursive in $E_1^\#$,*
- (iii) *X and $\omega - X$ are weakly representable in $(A_2^-)_\mathfrak{A}$.*

9.10. Definition. We call a rule f a *syntactical β -rule* if β -models form the semantics of f and every consistent f -complete set of sentences has a β -model.

9.11. Theorem (Apt). (i) *There exists no syntactical β -rule with a Σ_2^1 graph (after Gödlization).*

(ii) *There exists a syntactical β -rule with a Π_2^1 graph.*

Proof. (i) If the graph of f is Σ_2^1 , then the set $(A_2^-)_f$ is inductively defined with respect to a Σ_2^1 monotone relation, thus it is a Σ_2^1 set (an unpublished result of Gandy). On the other hand, $\ulcorner A_\beta^- \urcorner \in \Pi_2^1 - \Sigma_2^1$.

(ii) The required rule is obtained by taking the appropriate rule from the general result.

Unfortunately, such an artificially constructed rule cannot be used at all for building β -models. This suggests to look for a β -rule among rules which are in a certain sense natural.

Such a class of rules has been introduced by Aczel [1].

9.12. Definition. Let F be an analytical subset of $\mathcal{P}(\omega)$ which is monotone, i.e., $X \subset Y$ & $X \in F \rightarrow Y \in F$. Then F determines the following rule of proof f_F -rule: from the fact that $\{n: \vdash_{f_F} \Phi(n)\} \in F$ infer $\vdash_{f_F} (EX)(F(X) \& (x)(x \in X \leftrightarrow \Phi(x)))$.

Of course, the ω -rule and the \mathcal{A} -rule belong to this class of rules.

In the considerations about this class of rules, the following criterion found by Aczel is very much useful.

9.13. Theorem (Aczel). *Let f be a rule of proof formulated in $\mathbf{L}(A_2)$ such that the graph of f is analytical.*

Suppose that for every sentence Φ and every β -model M of A_2^- , we have

$$M \models \ulcorner \Phi \urcorner \in \ulcorner (A_2^-)_f \urcorner \rightarrow \Phi .$$

Then $(E\Phi) (\Phi \in A_\beta - (A_2^-)_f)$.

Proof. It is easy to check that the Gödelian sentence Φ such that $A_2^- \vdash \Phi \leftrightarrow \neg(\ulcorner \Phi \urcorner \in \ulcorner (A_2^-)_f \urcorner)$ is the required one.

Using this criterion we obtain the following:

9.14. Theorem. *Let f be a rule of proof from the above class of rules. Then*

- (i) $(E\Phi) (\Phi \in A_\beta - (A_2^-)_f)$,
- (ii) *there exists a consistent f -complete set of sentences which does not have a β -model.*

Proof. For every n , there exists an f_F -rule stronger than those which come from $F \in \Pi_n^1$. This rule comes from the following F :

$$F_{n+1}(X) \leftrightarrow (X_1)(EX_2) \dots (QX_{n+1})(\tilde{Q}x)(\langle \bar{X}_1(x), \dots, \bar{X}_{n+1}(x) \rangle \in X) ,$$

where Q is the appropriate quantifier, and \tilde{Q} the dual one. But the f_{F_n} -rules satisfy Aczel's criterion which can be checked (it is Aczel's idea) by induction on the proof (outside the β -model).

(ii) Let Φ be the sentence found in (i). Then $\text{Cn}((A_2^-)_f \cup \{\neg\Phi\})$ is the required set of sentences.

Thus there is no syntactical β -rule in this class of rules.

Besides of the ω -rule we know only one natural infinitary rule of proof with a semantics and which satisfies the consistency theorem. It is the following:

9.15. Def-rule. Infer $(X) \Phi X$ from $\{\Phi(Y) : Y \text{ is analytical set}\}$.

The proof of both properties is analogous as in the case of the ω -rule. Namely pointwise definable models of A_2^- form a semantics for the Def-rule and every consistent Def-complete set of sentences has a pointwise definable model. The Def-rule is sound in $\mathcal{P}(\omega)$ iff the analytical basis theorem is true.

§10. Non-standard models and expandability

10.0. Apart from general model theoretical results, not too much is known about non-standard models of the second order arithmetic. Since there are sentences of the language of Peano arithmetic unprovable in Peano arithmetic, but provable in A_2^- (for instance Gödel's sentence stating consistency of Peano arithmetic), there are models of Peano arithmetic non-expandable to the model of A_2^- (i.e. with the preservation of type 0 objects).

Much deeper results have been obtained by Ehrenfeucht and Kreisel [10].

10.1. Theorem. *There is a model M being an elementary extension of the standard model of Peano arithmetic such that for no $\mathfrak{A} \subseteq \mathcal{P}(M)$, $\langle M, \mathfrak{A}, +_M, \cdot_M, <_M, S_M, \in \rangle$ is a model of A_2^- .*

A similar phenomenon in the case of the higher order arithmetic has been obtained by Zbierski.

10.2. Theorem (Zbierski [67]). *The smallest model of A_n is not expandable to the model of A_{n+1} .*

The proof follows from the fact that in A_{n+1} we are able to prove strong reflection property for the n th order objects.

Similar results hold in the case of ZF set theory; here we have the following:

10.3. Theorem. *The smallest transitive model of ZF is not expandable to the model of Kelley–Morse set theory.*

In fact, stronger results hold: Let ξ_α be a consecutive enumeration of the heights of transitive models of ZFC. Let ρ_α be a consecutive enumeration of continuity points of the function ξ , then we have the following:

10.4. Theorem. (a) *If $\alpha < \xi_\alpha$, then a transitive model of ZFC of the height ξ_α is expandable to the model of Kelley–Morse theory.*

(b) *The same result holds for ρ_α .*

One can obtain several further improvements of the theorem along this line.

Since all nonstandard models of Peano arithmetic contain as an initial segment a copy of ω , it seems reasonable to consider a trace on ω of a nonstandard model of A_2 . Let M be a nonstandard model of A_2 . We may assume that $M = \langle X, \mathcal{Y}, \dots \rangle$ such that $\mathcal{Y} \subseteq \mathcal{P}(X)$.

(Notice that in that case $\mathcal{Y} \neq \mathcal{P}(X)$, moreover we may assume that ω is an initial segment of X).

Form M' as follows: $M' = \langle \omega, \mathcal{U}, +, \cdot, \dots \rangle$, where $S \in \mathcal{U} \leftrightarrow (\exists Y)_{\mathcal{Y}} (S = \omega \cap Y)$.

10.5. Theorem (Mostowski [54]). *There is an M elementarily equivalent to $\mathcal{P}(\omega)$ such that M' is not a model for A_2 .*

§ 11. Problems ⁸

(1) Let $\mathcal{U} \subseteq \mathcal{P}(\omega)$, $\overline{\mathcal{U}} = \aleph_0$. Does there exist the smallest β -model of A_2^- (A_2) which contains \mathcal{U} ?

(2) Can the smallest β -model of A_n ($n > 2$) be characterized “from below” similarly to the “ramified analytical” characterization in the case of A_2 ?

(3) Is the theorem of Keisler–Mostowski true for ω -models of A_2^- ?

(4) Characterize set theoretical assumptions under which there is the smallest β_n -model ($n > 2$) of A_2 .

⁸ Several problems in this list have already been considered by other authors and no claim to originality is made for any of them.

(5) Is it consistent to assume that Δ_n^1 sets for some $n > 2$ form an ω -model of A_2^- ?

(6) Is the Dam Hypothesis that is formulated in the paper true?

(7) Does there exist an extension T of A_2^- such that there is no smallest ω -model of T but there is a minimal ω -model of T ?

(8) Characterize $\{X: X \leq \omega \ \& \ (\exists \mathfrak{U}) (\mathfrak{U} \text{ is an } \omega\text{-model of } A_2^- \ \& \ \mathfrak{U} \models \text{Bord}[X])\}$. Clearly, each such object is a pseudo well-ordering in the sense of Feferman and Spector [15], yet one can easily show that these classes do not coincide.

(9) Can the axiom of choice be finitely axiomatizable over the comprehension scheme, i.e., does there exist a sentence φ such that $A_2^- \cup \{\varphi\}$ is equivalent to A_2 ?

(10) Characterize the conditions under which a countable β -model of A_2 has an elementary proper extension which is a β -model.

(11) Define $X \leq_\beta Y \leftrightarrow (\mathfrak{U}) (\mathfrak{U} \text{ is a } \beta\text{-model of } A_2 \ \& \ Y \in \mathfrak{U} \rightarrow X \in \mathfrak{U})$ and $X \sim_\beta Y \leftrightarrow X \leq_\beta Y \ \& \ Y \leq_\beta X$. $\mathcal{P}(\omega)/\sim_\beta$ gives a structure of β -degrees. (Note that analogous \leq_ω relation leads simply by the Gandy–Kreisel–Tait result (relativised version) to hyperdegrees.)

It is easy to see that X is Δ_1^1 in $Y \rightarrow X \leq_\beta Y \rightarrow X$ and is Δ_2^1 in Y .

Both implications cannot be reversed. A forcing argument leads to the existence of incomparable β -degrees. What else can be proved about this structure?

Acknowledgments

We are indebted to Professor R.O. Gandy, Wojtek Guzicki and Professor A. Mostowski and the referee who read the manuscript and pointed out several improvements.

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