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CONTROL AND FILTERING OF A CLASS OF NONLINEAR BUT "HOMOGENEOUS" SYSTEMS

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CONTROL AND FILTERING OF A CLASS OF NONLINEAR BUT "HOMOGENEOUS" SYSTEMS

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ABSTRACT. One striking aspect of the class of linear systems is that the controls enter in a way which is independent of the state; that is they are homogeneous, w.r.t. the underlying vectorspace (additive Lie group) structure as far as the controls are concerned, and the autonomous term enjoys reminiscent but not identical "homogeneity properties". Another class of systems which enjoys such properties is the class of systems on Lie groups and coset spaces (E.g. $\dot{g} = (A+\Sigma u_i B_i)g$, $g \in \underline{\underline{GL}}_n$, A, $B_i \in g\ell_n$) studied by Brockett, Jurdjevic-Sussmann, Hirschhorn and others. However, in the case the Lie group G is the additive group this class does not specify to the familiar class of linear systems (but to $\dot{x} = a + \Sigma u_i b_i$, a, $b_i \in \underline{\underline{R}}^n$). Yet the analysis of these two classes of control systems suggests certain "family" characteristics.

In this paper I discuss several aspects of classes of systems, which in one-way or another - there are several different choices one can make - generalize both the familiar linear systems and the class on Lie groups mentioned above.

1. INTRODUCTION.

This paper, or more precisely the research program which this paper tries to describe, resulted from the following two considerations: (i) nonlinear systems theory in general is, at the moment, too difficult and - as a research area - not well enough structured: we have relatively little feeling for the right problems and questions to ask and perhaps little intuition for the phenomena (pathologies) which can occur, and (ii) if in LQG one changes either L, Q or G things get unstuck immediately and rather severely; the three interact rather closely and it seems to follow that to find interesting generalizations all three at once must be adjusted (changed) simultaneously and in a compatible manner.

The lines above are of course the personal opinion of the present author; they may not, as far as I know, reflect the consensus, if such an unlikely thing exists, of the systems theory community.

A situation as described in (i) above is not unusual in mathematics. It has occurred before, e.g. in the theory of Riemannian manifolds. In this particular instance the theory of symmetric spaces came to the rescue. To quote from [Helgason, 1962] (or the revised 1978 edition):

"By their definition, symmetric spaces form a special topic in Riemannian geometry; their theory, however, has merged with the theory of semi-simple Lie groups. This is the source of very detailed and exhaustive information about these spaces. They can therefore often serve as examples on the basis of which general conjectures in differential geometry can be *made* and *tested*".

At the same time symmetric spaces are general enough to serve as a real testing ground.

It seems to me that nonlinear systems and control theory could do with a class of examples like that. And the classes of "homogeneous", but nonlinear systems described below are mainly intended (by me) as a possible testing ground for ideas, conjectures and concepts in general nonlinear system theory. Special cases, though, do occur naturally in science and engineering, cf. e.g. [Brockett,1972] in connection with theorem 3.14 below.

Consideration (ii) above also points naturally to Lie groups and homogeneous spaces (and some kind of "homogeneous" system on them) as a natural possible class of candidates for generalized

LQG. Especially in view of the theory of "Gaussian processes" on general Lie groups based on Bochner's theorem and a definition of positive definite function which makes sense on any Lie group.

The main philosophy behind what is described below is to study linear systems on $\underline{\underline{R}}^n$ and to formulate their characteristic properties either in terms of the additive Lie group $\underline{\underline{R}}^n$ or in terms of the natural connection on $\underline{\underline{R}}^n$. Not surprisingly these two possible characterization give rise to different possible generalizations when these characteristic properties are formulated for general Lie groups (and homogeneous spaces), even when we restrict attention to (left-) invariant connections on Lie groups.

Two classes of systems arise this way: "Group linear systems" and "connection linear systems". In addition there is a small section on a third class of systems: "fibre linear systems". The "connection linear systems" discussed below are in the torsion-free, zero-curvature case precisely the systems discussed by Brockett in this volume.

What follows below is an outline of a research program rather than a full grown paper. In particular, also to avoid excessive length, I concentrate on ideas and concepts, and proofs are only sketched. A more complete (and longer) account will, hopefully, appear in the future.

All manifolds in the following will be C^∞ and so will all functions and vectorfields defined on them. If M is a C^∞-manifold F(M) denotes the ring of R-valued C^∞-functions (i.e. infinitely often differentiable functions) on M and V(M) denotes the Lie-algebra of all C^∞-vectorfields on M.

2. WHAT MAKES A LINEAR SYSTEM LINEAR

The reason we are asking this question is that we are interested in formulating the conditions for linearity of a system in such a way that natural generalizations on (noncommutative) Lie groups suggest themselves. Let us consider the familiar class of linear systems on $\underline{\underline{R}}^n$

(2.1) $$\dot{x} = Ax + Bu, \quad y = Cx$$

and see whether we can capture its characteristic properties in some "coordinate free way". If $\phi : \mathbb{R}^n \to \mathbb{R}^n$ is any diffeomorphism, then the nonlinear state space transformation $z = \phi(x)$ transforms (2.1) into a set of highly nonlinear looking equations, viz.

(2.2) $$\dot{z} = (J\phi)(\phi^{-1}(z))(A\phi^{-1}(z) + Bu), \quad y = C\phi^{-1}(z)$$

where $(J\phi)(x')$ is the Jacobian matrix of ϕ at z'. These equations still have the form

(2.3) $$\dot{x} = \alpha(x) + \sum_{i=1}^{m} \beta_i(x)u_i, \quad y = \gamma(x)$$

where α, β_i, $i = 1,\ldots,m$, are vectorfields on $\mathbb{R}^n$ and γ is a nonlinear function $\mathbb{R}^n \to \mathbb{R}^m$ but beyond that there is little at first sight which might tip one of that we are really dealing with a linear system written down in the wrong coordinates. Up to nonlinear state space equivalence and nonlinear feedback the question of when a system like (2.3) is linear has been considered and solved by [Brockett 1978], and an answer to the question whether a system (2.3) is locally like (2.1) is given by [Krener 1973] in terms of the Lie-algebras generated by the vectorfields $\alpha(x), \beta_i(x)$ (locally around 0).

As a very small simple example consider the example with $A = \begin{pmatrix} 1 & 2 \\ 3 & 0 \end{pmatrix}$, $B = \begin{pmatrix} 1 \\ 1 \end{pmatrix}$, $C = (2,0)$ in (2.1) and $z = \phi(x)$ given by the diffeomorphism

$$\begin{pmatrix} x_1 \\ x_2 \end{pmatrix} \to z = \begin{pmatrix} 1+x_2^2+2x_2x_1^2+x_1+x_1^4 \\ x_2+x_1^2 \end{pmatrix}$$

which gives us the system

(2.5) $$\begin{aligned} \dot{z}_1 &= 2z_2+(4+6z_2+8z_2^2)(z_1-1-z_2^2)+(4z_2-2)(z_1-1-z_2^2)^2 \\ &\quad -8z_2(z_1-1-z_2^2)^3 + \{(2+2z_2) + 4z_2(z_1-1-z_2^2)\}u \\ \dot{z}_2 &= (3+4z_2)(z_1-1-z_2^2) + 2(z_1-1-z_2^2)^2 \\ &\quad -4(z_1-1-z_2^2)^3 + (2z_1-1-2z_2^2)u \end{aligned}$$

Returning to our original system (2.1), viewing it as a special case of systems of the form (2.3), and concentrating for the moment on the input part the following "homogeneity properties" could be noticed

(2.6) The input vectorfields $\beta_i(x)$ are invariant with respect to the group structure.

This means the following. Let M be a C^∞-manifold, F(M) the ring of C^∞-functions on M. Then a vectorfield on M is a derivation X: F(M) → F(M), i.e. an $\underline{\underline{R}}$-linear map with the property X(fg) = X(f)g + fX(g). Let Φ be a diffeomorphism M → M, then the translated vectorfield X^Φ is defined by $(X^\Phi)(f) = (Xf^{\Phi^{-1}})^\Phi$ where $f^{\Phi^{-1}} = f \circ \Phi^{-1}$. If G is a Lie group then X is said to be left invariant if $X^{L_\sigma} = X$ for all $\sigma \in G$ where L_σ stands for the diffeomorphism $g \to \sigma g$, $g \in G$.

Indeed a vectorfield on $\underline{\underline{R}}^n$ can be written as

$$X = \sum f_i(x)\frac{\partial}{\partial x_i} \tag{2.7}$$

Then the requirement that $X^{L_\sigma} = X$ for all $\sigma \in \underline{\underline{R}}^n$ becomes

$$\sum f_i(x-\sigma) \frac{\partial f}{\partial x_i}(x) = \sum f_i(x) \frac{\partial f}{\partial x_i}(x) \tag{2.8}$$

for all functions f (and for all $\sigma \in \underline{\underline{R}}^n$). This means that the $f_i(x)$ in (2.7) must be constants so that the left invariant vectorfields in $\underline{\underline{R}}^n$ are precisely the vectorfields $\sum b_i \frac{\partial}{\partial x_i}$, $b_i \in \underline{\underline{R}}$ which are the vectorfields multiplying the controls in (2.1).

The "vectorfield Ax", or more precisely the vectorfield

$$\alpha(x) = \sum_i (\sum_j a_{ij}x_j) \frac{\partial}{\partial x_i} \tag{2.9}$$

does not have an equally obvious invariance property. But it does have the property

(2.10) Let $\mathfrak{g}$ be the Lie algebra of left invariant vectorfields on $\underline{\underline{R}}^n$, then $[\alpha, X] \in \mathfrak{g}$ for all $X \in \mathfrak{g}$.

The obvious generalization of properties (2.6) and (2.10) will define the class of what I like to call "group linear systems". They will be discussed in some more detail below in section 3. At the moment they are my favourite class of "nonlinear but homogeneous systems".

A totally different way of saying that the vectorfields $\beta_i(x)$ in (2.1) are as they are is to remark that the coefficients b_i in

(2.11) $$\sum b_{ij} \frac{\partial}{\partial x_i} = \beta_i(x)$$

do not vary with x, i.e. that" $\frac{\partial}{\partial x_j} b_{ik}$" = 0 all k,j. This concept, however, is not defined on general manifolds but requires a "manifold with connection" to be properly defined. This will lead to "connection linear systems" a second class of nonlinear but homogeneous systems which will probably repay detailed study. Connection linear systems and their relation with group linear systems are the topic of section 4 below.

3. GROUP LINEAR SYSTEMS.

3.1. <u>Definition of Group Linear Systems</u>. Let G be a Lie group, finite dimensional and X a homogeneous space for G, i.e. X = G/H where H is a closed subgroup of G. Let $\mathfrak{m}$ be the Lie algebra of G invariant vectorfields on X. (This is a Lie algebra because $[V_1^\phi, V_2^\phi] = [V_1, V_2]^\phi$ for any two vectorfields V_1, V_2 on a manifold M and any diffeomorphism $\phi: M_1 \to M_2$). A group equivariant system on X now looks like

(3.2) $$\dot{x} = \alpha(x) + \sum \beta_i(x) u_i, \quad y = \gamma(x)$$

where

(3.3) $$\beta_i(x) \in \mathfrak{m} \quad \text{for all } i,$$

(3.4) $[\alpha, \beta] \in \mathfrak{m}$ for all $\beta \in \mathfrak{m}$

(3.5) γ is a collection of quotient maps $X \to G/K_j$ where K_j is a closed subgroup of G containing H.

3.6. Example. Translation Invariant Systems. An example is afforded by the systems on Lie groups and spheres studied by [Brockett 1972,1973], [Jurdjevic-Sussmann, 1972], [Hirschhorn 1977]. Let G be a closed subgroup of $GL_n(\mathbb{R})$ and $\mathfrak{g}$ the Lie algebra of G, viewed as a subalgebra of $g\ell_n(\mathbb{R})$. Consider systems of the form

$$\dot{g} = g(A + \sum B_i u_i), \quad y = \gamma(g) = Kg$$

The invariant vectorfields on G are the vectorfields gC, $C \in \mathfrak{g}$, or more explicitly the vectorfields $\sum_{i,j,k} g_{ij} c_{jk} \frac{\partial}{\partial g_{ik}}$

(restricted to G) in the coordinates $g_{11}, \ldots, g_{nn}$ for $GL_n(\mathbb{R})$ More precisely translation invariant systems are of the form

$$\dot{g} = \alpha(g) + \sum \beta_i(g) u_i, \quad y = \gamma(g) = gK, \tag{3.7}$$

where α, β_i are left invariant vectorfields, and K is a closed subgroup of G.

3.8. Example. Bilinear systems. Let $X = \mathbb{R}^n \setminus \{0\}$ and view X as a coset space for $GL_n(\mathbb{R})$ by letting $GL_n(\mathbb{R})$ act on $\mathbb{R}^n$ in the usual manner, i.e. $X = GL_n(\mathbb{R})/H$ where H is e.g. the stabilizer of e_1;

that is H is the subgroup $H = \{\begin{pmatrix} 1 & 0 \\ x & y \end{pmatrix} : x \in \mathbb{R}^{n-1},\ y \in GL_{n-1}(\mathbb{R})\}$. Then the vectorfields $Ax, B_i x$ are right invariant under $GL_n(\mathbb{R})$, so that (modulo right invariance versus left invariance) the familiar bilinear systems

$$\dot{x} = Ax + \sum (B_i x) u_i, \quad y = Cx \tag{3.9}$$

are examples of group equivariant systems. This also makes it probable that the complete study of group equivariant systems will not be a totally trivial matter. Note that the equilibrium point $x = 0$ has been removed in the above set up. Results pertaining to this approach to bilinear systems can be found in [Hirschhorn 1977].

3.10. Remark. Consider $\mathbb{R}^n$ as a (vector) Lie group, and consider the systems of type (3.7) on it. E.g. embed $\mathbb{R}^n$ by

$x \to \begin{pmatrix} I_{n-1} & x \\ 0 & 1 \end{pmatrix} \in GL_{n+1}(\mathbb{R})$. This gives us systems of the form

$$\dot{x} = a + \sum b_i u_i, \; a, \; b_i \in \mathbb{R}^n, \; y = Cx \tag{3.11}$$

i.e. not the class of systems $\dot{x} = Ax + Bu$, $y = Cx$. This accounts to some extent for the lesser elegance of the results in the inhomogeneous case ($A \neq 0$) with respect to the homogeneous case ($A=0$) in the controllability/reachability results of [Brockett 1972, Jurdjevic-Sussmann 1972].

3.12. Proposition. Consider $\mathbb{R}^n$ as a Lie group. Then the group equivariant systems (according to definition 3.1) on $\mathbb{R}^n$ are the systems of the form

$$\dot{x} = a + Ax + Bu, \; y = Cx, \qquad a \in \mathbb{R}^n, \; A \in g\ell_n(\mathbb{R}), \; B \in \mathbb{R}^{n\times m}, \; C \in \mathbb{R}^{p\times n} \tag{3.13}$$

Proof. Easy exercise. Indeed let $\alpha(x) = \sum f_i(x) \frac{\partial}{\partial x_i}$. Then $[\alpha(x), \frac{\partial}{\partial x_j}]$ left invariant, i.e. constant, means

$(\frac{\partial}{\partial x_j} f_i)(x) = 0$ for all i,j and the result follows.

3.14. Theorem. Let G be a semi-simple or compact Lie group. Then every group equivariant system over G is of the form (3.7).

Proof. Let G be semisimple and let (Σ) be a system of type (3.2). Let $\mathfrak{g}$ be the Lie algebra of G viewed as a subalgebra of $V(G)$ the Lie algebra of all vectorfields on G. The vectorfield α has the property $[\alpha, \mathfrak{g}] \subset \mathfrak{g}$ and hence defines a derivation of $\mathfrak{g}$. Because $\mathfrak{g}$ is semi-simple every derivation of $\mathfrak{g}$ is inner so that there exists a vectorfield $V \in \mathfrak{g}$ such that $[\alpha,\beta] = [V,\beta]$ for all $\beta \in \mathfrak{g}$. Now the vectorfields β for every $g \in G$ span a basis for the tangent space

T_gG at g and it follows by the easy lemma below that $\alpha = V$ proving the theorem in this case.

If G is compact consider the translated vectorfields α^{L_σ} for all $\sigma \in G$. Let $d\mu$ be unit mass left invariant Haar measure on G, and define $V = \int \alpha^{L_\sigma} d\mu$. Then V is left invariant and the remaining bit of the proof is as before.

3.15. Lemma. Let $V_1, \ldots, V_n$ be a set of vectorfields on the connected manifold M such that $V_1(x), \ldots, V_n(x)$ is a basis for the tangent space T_xM for all $x \in M$. Let V,W be two more vectorfields on M and suppose that $[V_i,V] = [V_i,W]$, $i = 1, \ldots, n$ and $V(x_0) = W(x_0)$ for some $x_0 \in M$. Then V = W.

Proof. This is an immediate consequence of standard uniqueness results for solutions of differential equations.

Another pleasing consequence of lemma 3.15 is that the dimension of the space of all group linear systems on a Lie group G is finite, exactly as in the case of linear systems. This is a property of the space of all linear systems (of a given dimension, with a given number of outputs and inputs) which is important in identification problems.

3.16. Proposition. Let G be an n-dimensional Lie group. Then the space of all systems $\dot{x} = \alpha(x) + \sum_{i=1}^{m} u_i \beta_i(x)$ satisfying (3.3), (3.4) is of dimension $\leq n^2 + n + mn$.

Indeed, the control vectorfields β_i, $i = 1, \ldots, m$ account for mn dimensions. The vectorfield α induces an endomorphism of the n-dimensional vectorspace $\mathfrak{g}$, the Lie algebra of G and is uniquely determined by this endomorphism and its value $\alpha(e)$ (by lemma 3.15). Note that if $G = \mathbb{R}^n$ then the upper bound $n^2 + n + mn$ is reached. It is maybe also worth noticing that the control systems (3.2) satisfying (3.3) - (3.5) are automatically analytic.

3.17. Remarks. Thus the familiar linear systems $\dot{x} = Ax + Bu$ and the systems (3.7) are the extreme examples of the class of group equivariant systems, corresponding respectively to the abelian and semi-simple cases. Their theory though exhibits considerable similarly which gives reasonable grounds for optimism for the whole class.

The following example shows that there are nontrivial intermediate cases.

3.18. Example. The Heisenberg group. Let H be the following subgroup of $GL_3(\mathbb{R})$, the socalled Heisenberg group

$$H = \left\{\begin{pmatrix} 1 & x & z \\ 0 & 1 & y \\ 0 & 0 & 1 \end{pmatrix} : x,y,z \in \mathbb{R}\right\} \tag{3.19}$$

Using the global coordinates given by this embedding one finds that all the left invariant vectorfields are linear combinations of

$$b_1 = \frac{\partial}{\partial x}, \quad b_2 = \frac{\partial}{\partial y} + x\frac{\partial}{\partial z}, \quad b_3 = \frac{\partial}{\partial z} \tag{3.20}$$

and that the vectorfields a which have the property that for all $i = 1,2,3$, $[a,b_i] \in \mathfrak{g}$, the Lie algebra spanned by b_1, b_2, b_3 are linear combinations of b_1, b_2, b_3 and the six further vectorfields

$$\begin{array}{l} x\frac{\partial}{\partial x} - y\frac{\partial}{\partial y}, \quad x\frac{\partial}{\partial y} + \frac{1}{2}x^2\frac{\partial}{\partial z}, \quad x\frac{\partial}{\partial z} \\ y\frac{\partial}{\partial x} + \frac{1}{2}y^2\frac{\partial}{\partial z}, \quad y\frac{\partial}{\partial z}, \quad z\frac{\partial}{\partial z} + y\frac{\partial}{\partial y} \end{array} \tag{3.21}$$

3.22. A slight generalization. Complete vectorfields and a theorem of Palais. Let M be a differentiable manifold such that there is a finite dimensional Lie algebra of vectorfields $\mathfrak{m}$ such that the vectors $V(x)$, $V \in \mathfrak{m}$ span the tangent space T_xM for all $x \in M$. If $\dim \mathfrak{m} = \dim M$ this makes M parallellizable of course. Now consider systems of the type

$$\dot{x} = \alpha(x) + \sum u_i \beta_i(x) \tag{3.23}$$

with α such that $[\alpha, \mathfrak{m}] \subset \mathfrak{m}$, $\beta_i \in \mathfrak{m}$. Suppose that the vectorfields α, β_i are all complete. Then the Lie algebra generated by α and the β_i is finite dimensional (it is contained in $\mathfrak{m} + \mathbb{R}\alpha$) and it follows from a theorem of [Palais, 1957] (as was pointed out to me by Roger Brockett) that there will be no finite escape time phenomena for (3.23) (for bounded inputs $u_i(t)$).

3.24. Reachability Conditions. Both for group linear systems and the slight generalization mentioned just above one expects to find pleasing conditions for reachability/controllability, (and observability, invertability) guided and stimulated by the results of[Brockett 1972], [Jurdjevic-Sussmann 1972], [Hirschhorn 1977] and of course the results of the linear theory. The most natural, coordinate invariant object to consider with respect to controllability is probably the Lie-sub-algebra of generated by the $ad^i\alpha(\beta_j)$, $j = 1, \ldots, m; i=0,1,2,\ldots$.
Here $ad^o\alpha(\beta) = \beta$, $ad^i\alpha(\beta) = [\alpha, ad^{i-1}\alpha(\beta)]$, $i = 1,2,\ldots$. One has e.g.

3.25. Proposition. Let $\dot{x} = \alpha(x) + \Sigma\, u_i\beta_i(x)$ be a group linear control system on the Lie group G with Lie algebra, and suppose that $\alpha(e) = 0$. Then the system is weakly locally reachable around e iff the Lie algebra generated by the $ad^i\alpha(\beta_j)$, $j = 1, \ldots, m$; $i = 0,1,2,\ldots$ is equal to $\mathfrak{g}$.Here locally reachable around e means that for every open neighbourhood U of e the set of points reachable from e such that the trajectory does not leave U contains e in its interior. The sufficiency of the condition for weak local reachability at e is wellknown, cf. e.g. [Hermann-Krener 1977]. Here "weak" means that one is allowed to travel backwards along the vectorfield α (negative time). The example $\alpha = \frac{1}{2} x^2 \frac{\partial}{\partial z} + x \frac{\partial}{\partial y}$, $\beta = \frac{\partial}{\partial x}$ on the Heisenberg group (cf. 3.18 above) shows that "weakly" cannot be removed from the statement of the proposition. If all β's are in the centre of $\mathfrak{g}$ (cf. (4.27) below) then weakly can be removed by a result of Hirschhorn.

The proof of the necessity of the condition is most easily done via connections and a sketch is postponed till we have discussed these. That proof in fact yields the stronger result that all trajectories remain in the connected subgroup H of G corresponding to the Lie algebra generated by the $ad^i\alpha(\beta_j)$, so that being able to move far away does not improve the reachability, precisely as in the case of linear systems.

4. CONNECTION LINEAR SYSTEMS.

To be able to say how a vectorfield $\Sigma f_i(x) \frac{\partial}{\partial x_i}$ changes as x varies on a general manifold we need the idea of a connection (or covariant differentiation).

4.1. Connections. Let M be a C^∞-manifold; V(M) the Lie algebra of C^∞-vectorfields on M; F(M) the algebra of C^∞-functions on M. A linear connection on M by definition assigns to each $X \in V(M)$ a derivation $\nabla_X: V(M) \to V(M)$, of V(M) as a F(M) module; i.e. a map ∇_X which satisfies

(4.2) $$\nabla_X(fV) = X(f)V + f\nabla_X(V), \quad f \in F(M), \; V \in V(M)$$

Moreover the assignment $X \to \nabla_X$ must satisfy

(4.3) $$\nabla_{fX+gY} = f\nabla_X + g\nabla_Y, \quad f,g \in F(M); \; Y \in V(M)$$

4.4. Example. Canonical connection on $\underline{\underline{R}}^n$. Assign to $\frac{\partial}{\partial x_i} \in V(\underline{\underline{R}}^n)$ the derivation

(4.5) $$\Sigma f_j(x) \frac{\partial}{\partial x_j} \to \Sigma \frac{\partial f_j}{\partial x_i}(x) \frac{\partial}{\partial x_j}$$

4.6. Torsion and Curvature. Given a connection ∇ on M its torsion and curvature tensors are defined by

(4.7) $$T(X,Y) = \nabla_X(Y) - \nabla_Y(X) - [X,Y]$$

(4.8) $$R(X,Y) = \nabla_X\nabla_Y - \nabla_Y\nabla_X - \nabla_{[X,Y]}$$

The manifold with connection (M,∇) is said to be torsionfree if $T(X,Y) = 0$ and flat if $R(X,Y) = 0$ (in some texts the terminology "flat" is supposed to imply also torsion free). The canonical connection on $\underline{\underline{R}}^n$ is both flat and torsionfree.

4.9. Geodesics and Completeness. Let $\gamma: (a,b) \to M$ be a curve in M. It is called a geodesic if $\nabla_X(X) = 0$ along γ where X is the vectorfield $\dot{\gamma}(t)$, i.e. $d\gamma(\frac{\partial}{\partial t})$, along $\gamma(a,b) \subset M$.

Given $m \in M$, $v \in T_m M$ there is a unique (local) geodesic $\gamma: (a,b) \to M$, $0 \in (a,b)$ such that $\gamma(0) = m$, $\dot\gamma(0) = v$. The manifold with connection (M,∇) is called complete if every geodesic can be extended indefinitely.

4.10. Flat, torsion free manifolds. Let (M,∇) be a flat, torsion-free manifold with connection. The universal covering space $\tilde M$ of a manifold with connection carries a natural connection $\tilde\nabla$ (cf. e.g. [Wolf, 1976]) and if (M,∇) is flat torsion free then $(\tilde M,\tilde\nabla)$ is diffeomorphic to $(\mathbb{R}^n,\nabla_o)$ where ∇_o is the canonical connection on $\mathbb{R}^n$ described above in example 4.4.

More precisely let E_n be the Lie group of affine motions of $\mathbb{R}^n$, i.e. $E(n) = \mathbb{R}^n \times GL_n(\mathbb{R})$ as a space acting on $\mathbb{R}^n$ by $(x,g)(v) = x + g(v)$, which also defines the group action on E_n. Then every flat, torsion free, connected manifold M with connection is diffeomorphic to $\mathbb{R}^n/\Gamma$ where Γ is a discrete subgroup of E_n acting properly discontinuously, so that M is a product of a torus and an $\mathbb{R}^m$

In particular if (M,∇) is flat, torsion free, connected and simply connected then $M = \mathbb{R}^n$ with the canonical connection (up to connection preserving diffeomorphism) and this gives a not very practical answer to the question of what makes a system (2.3) linear up to diffeomorphism (neglecting outputs). This will be the case if and only if there is a flat, torsion free connection ∇ such that $\nabla\beta_i = 0$ for all i and all vectorfields V (such vectorfields are called constant) and $\nabla_X\alpha$ is constant for all constant vectorfields X and finally there is an equilibrium point for zero controls.

4.11. Connection Linear Systems. This brings us quite naturally to the definition of a connection linear system. A control system

$$\dot x = \alpha(x) + \sum \beta_i(x)u_i \tag{4.12}$$

on a manifold with connection (M,∇) will be called *connection linear* if

$$\nabla_V\beta_i = 0 \text{ all } V \in V(M) \tag{4.13}$$

so that the β_i are constant vectorfields, and

(4.14) $\nabla_X \alpha$ = constant for all constant vectorfields X.

It would I think perhaps be even more interesting to consider the class of control systems (4.12) which satisfy (4.13) and

(4.15) $[\alpha, V]$ = constant for all V.

Warning. On an arbitrary manifold with connection (M,∇) there may very well be no constant vectorfields other than the zero vectorfield.

A last interesting class of connection defined systems, more or less analogous to 3.19 above, consists of systems (4.12) such that the β_i belong to a finite dimensional Lie algebra $\mathfrak{m}$ such that the $\mathfrak{m}(x)$ form a basis (or span) T_xM for all $x \in M$ and which satisfy

(4.16) $\nabla_X \alpha \in \mathfrak{m}$ for all $X \in \mathfrak{m}$

In the case of a connected, simply connected, flat torsion free manifold both (4.13) + (4.14) and (4.13) + (4.15) lead to control systems $\dot{x} = a + Ax + Bu$. If the manifold with connection (M,∇) is connected, flat, torsion free (but not simply connected) then these conditions result in the class of systems described by Roger Brockett in these proceedings (and some of these naturally occur in engineering, loc. cit.).

4.17. Intermezzo on foliations and distributions and the distributions defined by a control system. A *foliation* of an n-dimensional manifold M by q-dimensional submanifolds is a collection of q-dimensional submanifolds (called the leaves) such that through every $x \in M$ there passes exactly one leaf and such that locally around every point the partitioning of M by the leaves looks like $\underline{\underline{R}}^n$ partioned by the
$a + \underline{\underline{R}}^q$, $a \in \{x \in \underline{\underline{R}}^n : x_1 = \ldots = x_q = 0\}$,

$\mathbb{R}^q = \{x \in \mathbb{R}^n: x_{q+1} = \ldots = x_n = 0\}$.

A *distribution* of dimension q on M assigns to every $x \in M$ a q-dimensional subspace $D(x) \subset T_xM$ of the tangent space of M at x such that D(x) varies differentiably with x.

Obviously a q-dimensional foliation defines a distribution, viz. $x \to T_xF_x$ where F_x is the unique leaf of the foliation passing through x. Such distributions are called *integrable*. They have the following property (obviously): if X,Y are two vectorfields on M such that $X(x), Y(x) \in D(x)$ for all x then also $[X,Y](x) \in D(x)$. Such distributions are called *involutive*. It is a theorem of Frobenius that such distributions are integrable, i.,e., come from foliations.

Now consider a control system (2.3). For each $x \in M$ define a nested series of subspaces of the tangent space T_xM

$$B_i(x) = \text{subspace spanned by } \mathrm{ad}^j\alpha(\beta_k)(x), \quad j = 0,\ldots,i;\ k = 1, \ldots, m \tag{4.18}$$

If the system (2.3) is linear the B_i form a nested system of integrable distributions. And inversely [Brockett 1979] for a control system (2.3) on $\mathbb{R}^n$, if dim $B_i(x)$ is constant as a function of x (so that the B_i are distributions) and these distributions are all integrable then the control system is linear up to nonlinear feedback (and nonlinear base change in input and state space).

There is a version of the results described in 4.10 above relative to a foliation [Blumenthal, 1980] (in which the conditions are stated in terms of a connection "adapted to" the foliation, a socalled basic connection) which – it seems to me – will be worth considering in this constant (e.g. to obtain similar results on more general spaces like the $\mathbb{R}^n/\Gamma$, Γ a discrete subgroup of $\mathbb{R}^n \times GL_n(\mathbb{R})$).

4.19. Parallel displacement. Let (M,∇) be a manifold with connection. Let $X \in V(M)$ and $\gamma: [a,b] \to M$ an integral curve of X, i.e. $d\gamma(\frac{\partial}{\partial t}) = X(\gamma(t))$ for all $t \in [a,b]$. Let Y be another vectorfield. The vectorfield Y is called *parallel along* γ *if*

$\nabla_X(Y)(\gamma(t)) = 0$ for all t. This definition does not depend of course on the vectorfield X but only on γ. This notion can be used to identify the tangent spaces T_xM for $x \in \gamma[a,b]$ (parallel displacement along γ) with $v \in T_xM$ corresponding to $v' \in T_{x'}M$ iff there is a parallel vectorfield Y along γ with $v = Y(x)$, $v' = Y(x')$.

4.20. Intermezzo on Riemannian manifolds and the Levi-Civita connection. A pseudo-Riemannian (resp. Riemannian) manifold is a manifold equipped with a nondegenerate (resp. positive definite) symmetric bilinear form on each tangent space T_xM which varies differentiably with x. Given a pseudo-Riemannian manifold there exists a unique torsion-free connection which preserves the bilinear form (inner product) under parallel displacements along geodesics. This connection is called the Levi-Civita connection. It will perhaps be advantageous to analyse connection linear systems first for connections of this type.

4.21. Group-linear versus connection linear systems. Now let G be a Lie group. More generally similar things can be discussed for homogeneous spaces. There are at least three rather special connections on G which stand out and seem to deserve special attention. All three are left-invariant where a connection ∇ on G is called left invariant if for all $X,Y \in V(M)$ we have

$$\nabla_X(Y) = \nabla_{X^\sigma}(Y^\sigma)^{\sigma^{-1}} \tag{4.22}$$

where I have simply written σ for the left translation $L_\sigma : G \to G$, $g \to \sigma g$.

Left-invariant connections on G correspond biuniquely to bilinear forms $\alpha: \mathfrak{g} \times \mathfrak{g} \to \mathfrak{g}$, where $\mathfrak{g}$ is the Lie algebra of G. Here α is simply equal to $\alpha(X,Y) = \nabla_{\tilde{X}}(\tilde{Y})(e)$, where $\tilde{X},\tilde{Y}$ are the left-invariant vectorfields whose tangent vectors at $e \in G$ are equal to $X,Y \in \mathfrak{g}$ respectively. Cf. e.g. [Helgason 1978] for this.

Let $\nabla^1,\nabla^2,\nabla^3$ be the three connections on G defined by the bilinear forms

(4.23) $\alpha^1(X,Y) = 0$ (the zero-connection)

(4.24) $\alpha^2(X,Y) = [X,Y]$ (the + connection)

(4.25) $\alpha^3(X,Y) = \frac{1}{2}[X,Y]$ (the - connection)

Under ∇^1 the constant vectorfields are precisely the left-invariant ones. So that using ∇^1 conditions (4.12) and (4.14) together precisely define what we called a group linear system in section 3 above.

∇^3 is the only torsion free connection among these 3 and seems to be by far the most natural torsion free connection on G. It is perhaps worth remarking here that there exist no left-invariant torsion free flat connections on reductive homogeneous spaces ([Doi 1979], cf. also [Matsushima-Okamoto 1979] for the case of real semisimple Lie groups. This very nicely distinguishes $\mathbb{R}^n$ from the reductive homogeneous spaces (such as $\mathbb{R}^n \setminus \{0\}$, the natural state space of bilinear systems).

Finally ∇^2 is such that $\nabla^2_X(V)$ is left-invariant for all left invariant X if and only if [X,V] is left-invariant for all left-invariant X so that under ∇^3 conditions (4.16) and (4.4) are equivalent, cf. also 4.15.

Indeed any vectorfield Y on G can be written as $\sum f_i(x)X_i$ where $X_1, \ldots, X_n$ is a basis for $\mathfrak{g}$. So that for $X \in \mathfrak{g}$

$$\nabla^2_X(Y) = \sum_i X(f_i)X_i + \sum_i f_i[X,X_i]$$

On the other hand $[X,Y](\phi) = \Sigma X((f_iX_i)(\phi)) - \Sigma f_iX_i(X(\phi)) =$ $= \Sigma X(f_i)X_i(\phi) + \Sigma f_i X(X_i(\phi)) - \Sigma f_iX_i(X(\phi))$ So that for the + connection,

(4.26) $$\nabla^2_X(Y) = [X,Y], \quad X \in \mathfrak{g}, \quad Y \in V(M).$$

However, under ∇^2 the left-invariant vectorfields are no longer the constant ones, so that if G is noncommutative "connection linear" systems and "group linear" systems are different objects.

But of course the vectorfields in the centre of $\mathfrak{g}$ are constant. This defines a special class of systems

$$\dot{x} = \alpha(x) + \Sigma\, u_i \beta_i(x) \tag{4.27}$$

with $\beta_i \in Z(\mathfrak{g})$, the centre of $\mathfrak{g}$ and $[\alpha, \mathfrak{g}] \subset \mathfrak{g}$. This class is intermediate between linear systems and group linear (and bilinear) systems and certainly will repay detailed further investigation. I would also not be surprised if this class yielded further examples of finite dimensional estimation algebras (cf. section 6 below for this notion).

4.28. On the necessity of the controllability condition of proposition 3.25.
Consider a group linear control system on the Lie group G. Let H be the connected Lie subgroup of G corresponding to the sub Lie algebra $\mathfrak{h}$ of $\mathfrak{g}$ generated by the $ad^i\alpha(\beta_j) \in \mathfrak{g}$. We show that any trajectory starting in $e \in G$ remains in H. To see this consider the + connection on G. First notice that this connection restricts to a connection on H so that parallel displacements of vectors tangent to H at e along a curve γ in H results in vectors in $T_{\gamma(t)}G$ which are tangent to H. Now let $h \in H$ and γ a curve from e to $h = \gamma(1)$ in H. Then identifying tangent vectors in the various tangent spaces to G along γ by means of parallel displacement along γ we have

$$\alpha(h) = \alpha(e) + \int_0^1 (\nabla_{\gamma'(t)}\alpha)(\gamma(t))dt$$

(cf. [Helgason 1978, thm 7.1, page 41).
Now $\gamma'(t) \in T_{\gamma(t)}H$, $\alpha(e) = 0$ and $\nabla_X\alpha = [X,\alpha]$ by (4.26) and $[\alpha,\mathfrak{h}] \subset \mathfrak{h}$ and it follows by the remark made above that $\alpha(h)$ is tangent to H (at $\gamma(1)$), so that $\alpha(h) + \Sigma\, u_i\beta_i(h)$ is in $T_hH \subset T_hG$ for all $h \in H$.

4.29. Another example. Consider the linear Lie group G consisting of all 2 x 2 matrices of the form $\begin{pmatrix} x & z \\ 0 & 1 \end{pmatrix}$, $x,z \in \mathbb{R}$, $x > 0$. The Lie algebra $\mathfrak{g}$ of G consists of all real 2 x 2 matrices of the form $\begin{pmatrix} a & b \\ 0 & 0 \end{pmatrix}$. In the coordinates x,z the invariant vectorfields are linear combinations of

$$x\frac{\partial}{\partial x},\ x\frac{\partial}{\partial z}$$

and the vectorfields α such that $[\alpha,\mathfrak{g}]\subset\mathfrak{g}$ and $\alpha(e) = 0$ are linear combinations of the three vectorfields

$$x\ell nx\frac{\partial}{\partial x},\ z\frac{\partial}{\partial z},\ \frac{\partial}{\partial z} - x\frac{\partial}{\partial z}$$

5. FIBRE LINEAR SYSTEMS.

A rather different class of nonlinear systems with enough special structure to make one optimistic is what I like to call fibre linear systems. As an example consider a system whose total state x can be partioned into two parts $x = \binom{x_1}{x_2}$ evolving according to

$$(5.1)\qquad \dot{x}_1 = A_1x_1 + B_1u_1, \qquad x_1 \in \mathbb{R}^{n_1}$$

$$(5.2)\qquad \dot{x}_2 = A_2(x_1,u_1) + B_2(x_1u_1)u_2, \qquad x_2 \in \mathbb{R}^{n_2}$$

where A_1 and B_1 are constant matrices and A_2 and B_2 depend only on x_1 and u_1. Thus the total system consists of an ordinary linear system on the base and the state and controls of this influence the systems in the fibre which are also linear given x_1,u_1. One can of course even write down the input-output map of such a system explicitly (more or less).

More generally the first system in the base can itself be nonlinear, perhaps itself a fibre linear system with linear base giving rise so to speak to a three stage tower of linear systems. Generalizations on arbitrary rather than trivial vectorbundles now are easy to define.

5.3. The Heisenberg group again. Consider the Heisenberg group H example of section 4 above again.
Write $x_1 = (x,y)$, $x_2 = z$. Then for all the group linear systems on H, x_1 evolves as a linear system and given x_1 then $z = x_2$ evolves as a slightly generalized linear system

$$\dot{z} = a(x_1,y_1) + A(x_1)z + B(x_1)u_2$$

So that these systems are also fibre linear with linear base. This is a general phenomenon: every group linear system on a unipotent Lie group can be considered as a tower of linear systems in the sense suggested above.

6. REMARKS ON FILTERING FOR GROUP-LINEAR SYSTEMS.

Consider the general nonlinear filtering problem (Ito equations)

$$(6.1) \qquad dx_t = f(x_t)dt + G(x_t)dw_t, \quad dy_t = h(x_t)dt + dv_t,$$

where w_t, v_t are independent Wiener noise processes also independent of the initional random variable x_0. Here h,f,G are vector and matrix valued functions of the appropriate dimensions. Given enough regularity so that the density of the p(x,t) of $\hat{x}_t = E[x|y_s, 0 \leq s \leq t]$, the conditional state at time t given the observations $y^t = \{y_s : 0 \leq s \leq t\}$ exists, a certain unnormalized version $\rho(x,t)$ of p(x,t) satisfies the socalled Duncan-Mortenson-Zakai equation (which is driven by the observations)

$$(6.2) \qquad d\rho = \frac{1}{2} \sum_{i,j} \frac{\partial^2}{\partial x_i \partial x_j}((GG^T)_{ij}\rho) - \sum_i \frac{\partial}{\partial x_i}(f_i\rho) - \\ - \frac{1}{2} \sum_i h_i^2 \rho - \sum_i h_i dy_i$$

(cf. e.g. [Davis-Marcus 1981] for a derivation of this equation). This equation is in Fisk-Stratonovic form. The Lie algebra generated by the differential operator

$$\mathcal{L} = \frac{1}{2} \sum_{i,j} \frac{\partial^2}{\partial x_i \partial x_j}(GG^T)_{ij} - \sum_i \frac{\partial}{\partial x_i} f_i - \frac{1}{2} \sum_i h_i^2$$

(where $(GG^T)_{ij}$ is the (i,j)-th entry of the matrix GG^T, f_i, h_i the i-th component of the vector f,h) and the operators (multiplication with) $h_1, \ldots h_p$ is called the estimation algebra. It is likely to be of considerable importance in the analysis of the filtering problem (= building finite dimensional systems driven by the observations which produce $\hat{x}_t$ as outputs), cf. [Brockett 1981], [Hazewinkel-Marcus, 1980] and several more papers in [Hazewinkel-Willems, 1981].

The most general group linear stochastic Ito equation on the Heisenberg group is

$$(6.4)\qquad \begin{pmatrix} dx_1 \\ dx_2 \\ dx_3 \end{pmatrix} = \begin{pmatrix} a_1x_1 + a_4x_2 \\ -a_1x_2 + a_2x_1 + a_6x_2 \\ \tfrac{1}{2}a_2x_1^2 + a_3x_1 + \tfrac{1}{2}a_4x_2^2 + a_5x_2 + a_6x_3 \end{pmatrix} dt + \sum_{i=1}^{m} \begin{pmatrix} b_{1i} \\ b_{2i} \\ x_1b_{2i}+b_{3i} \end{pmatrix} dw_i$$

$a_1,\dots,a_6$; $b_{ji} \in \mathbb{R}$, and the most general observation equations coming from a group homomorphism $H \to \mathbb{R}$ are of the form

$$(6.5)\qquad dy_i = (c_{1i}x_1+c_{2i}x_2)dt + dv_i$$

6.6. Proposition. Consider a system on the Heisenberg group given by a signal equation of type (6.4) with observation equations of type (6.5). Then the observation Lie algebra is always pro-finite dimensional.

A Lie algebra L is pro-finite dimensional if there exists a sequence of ideals $L_1 \supset L_2 \supset \dots$ such that L/L_i is finite dimensional for all i and $\bigcap L_i = 0$. Cf.e.g.[Hazewinkel-Marcus, 1980] for a number of remarks on the relevance of this property for filtering problems.

Indeed writing out the various operators explicitly one observes that they are sums of operators of the type

$$x^\alpha \frac{\partial^\beta}{\partial x^\beta} \frac{\partial^i}{\partial z^i}, \quad i = 0,1,2\dots; \ |\alpha|, |\beta| \le 2,$$
$$x = \binom{x_1}{x_2}, \quad z = x_3$$

where α and β are multiindices $|\alpha| = \alpha_1 + \alpha_2$. The operators $x^\alpha \frac{\partial^\beta}{\partial x^\beta}$, $|\alpha|, |\beta| \leq 2$ span a finite dimensional Lie algebra LS_2 (of dimension 15) so that the estimation algebra is a subalgebra of the "current-algebra"

$$LS_2 \otimes \mathbb{R}\left[\frac{\partial}{\partial z}\right]$$

which is of course profinite dimensional. As a finite dimensional Lie algebra LS_2 can of course be embedded in a Lie algebra of vectorfields on $\mathbb{R}^N$, some large N) and this then easily gives rise to an inbedding of the current algebra $LS_2 \otimes \mathbb{R}[\frac{\partial}{\partial z}]$. In this case, however, there exists an inbedding of LS_2 modulo its centre in the vectorfields on $\mathbb{R}^5$ which comes from all Kalman-Bucy filters put together (and is closely related to the Segal-Shale-Weil representation), cf. [Hazewinkel, 1981], which is more likely to be useful.

(A result like proposition 6.6 holds generally also for higher dimensional Heisenberg groups (and hence for all 2-step nilpotent Lie groups) and I would like to pose the question whether it holds for every fibre linear system with linear base (and suitable output maps "linear" in the fibres).

Things change dramatically if instead of using observation like (6.5) one uses an observation equation

$$(6.7) \qquad dy = x_3 dt + dv$$

E.g. the system

$$(6.8) \qquad dx_1 = dw, \; dx_2 = x_1 dt, \; dx_3 = \frac{1}{2} x_1^2 dt, \; dy = x_3 dt + dv$$

has the Weyl algebra $W_1 = \mathbb{R}\langle x_1, \frac{\partial}{\partial x_1}\rangle$ as a subalgebra. This is perhaps not surprising because the map $(x_1, x_2, x_3) \to x_3$ is not "homogeneous" with respect to H. Indeed there is no action of H on $\mathbb{R}$ which makes this map H-equivariant. There is an action of H on $\underline{\underline{R}}^2$ which makes $(x_1,x_2,x_3) \to (x_2,x_3)$ H-equivariant. This, at first sight, would make an observation equation like

$$(6.9) \qquad \begin{pmatrix} dy_1 \\ dy_2 \end{pmatrix} = \begin{pmatrix} x_2 \\ x_3 \end{pmatrix} dt + \begin{pmatrix} dv_1 \\ dv_2 \end{pmatrix}$$

permissible, and this would also give a subalgebra W_1 in the estimation Lie algebra. However, in (6.9) the noises do not enter in a group-equivariant way. To achieve that one needs observation equations like

$$(6.10) \qquad \begin{pmatrix} dy_1 \\ dy_2 \end{pmatrix} = \begin{pmatrix} x_2 \\ x_3 \end{pmatrix} dt + \begin{pmatrix} dv_1 \\ x_1 dv_1 \end{pmatrix}$$

And this raises the general question of obtaining a D-M-Z type equation for an (unnormalized) conditional density for more general systems

$$(6.11) \qquad dx = f(x)dt + G(x)dw, \quad dy = h(x)dt + J(x)dv$$

With this open question I would like to conclude this paper.

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