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MATHEMATICS
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Non-finite Axiomatizability of the Second Order Arithmetic

by

K. R. APT

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Summary. The following theorem is proved: *Assume $V=L$. Let (A) be the second order arithmetic, and let S be the set of all true Π_n^1 formulas of the language of (A). Then $Cn(S)$ does not contain $Cn(A)$.* As a corollary we get that (A) is not finitely axiomatizable.

By a second order arithmetic we mean the system (A) described in [1] with one argument function variables only, without the symbol ι and with the following schema of comprehension axiom: $\exists a\,\forall x\,(a(x)=0 \leftrightarrow \varphi(x))$, where φ does not contain free variables other than x.

By N we denote the standard model of arithmetic, and by $|\Delta_n^1|$ the ω-structure whose functions are all the Δ_n^1 ones. If R is a collection of formulas, then $A \prec_R B$ denotes the fact that A is an elementary substructure of B with respect to formulas in R, i.e. for every formula $\varphi(x_0, \ldots, x_{k-1})$ from R with free variables indicated, and every sequence $(a_0, \ldots, a_{k-1})$ of elements of A we have $A \models \varphi(a_0, \ldots, a_{k-1})$ if and only if $B \models \varphi(a_0, \ldots, a_{k-1})$.

Let, for $n<\omega$, R_n denote the set of all formulas which are Π_{n+2}^1 or Σ_{n+2}^1.

We shall use the following theorem of Addison (see [2]): If $V=L$, then, for each $n<\omega$, every non-empty Σ_{n+2}^1 family of functions contains a Δ_{n+2}^1 function.

LEMMA 1. *If $V=L$, then $|\Delta_{n+2}^1| \prec_{R_n} N$.*

Proof. We use the slightly modified version of Tarski-Vaught criterion of being an elementary substructure (see [3]). Let $g_0, \ldots, g_{k-1}$ be a sequence of Δ_{n+2}^1 functions. It suffices to prove that if φ is a Π_{n+1}^1 formula, such, that

$$N \models \varphi(g_0, \ldots, g_{k-1}, f) \tag{1}$$

for some f, then (1) holds for some f' being a Δ_{n+2}^1 function. It is easy to see that (1) is Δ_{n+2}^1 relational, with one variable f. Hence the existence of f' is a consequence of Addison's theorem.

LEMMA 2. *If $V=L$ then $|\Delta_{n+2}^1|$ is not a model of (A).*

Proof. Let X be a $\Pi_{n+2}^1-\Sigma_{n+2}^1$, set, and let φ be a Π_{n+2}^1 formula which defines X in N i.e.

$$X=\{n : N \models \varphi(n)\}. \tag{2}$$

Assume, on the contrary, that $|\Delta_{n+2}^1|$ is a model of (A). By a comprehension axiom there exists a Δ_{n+2}^1 function g such that

$$|\Delta_{n+2}^1| \models \forall x\,(g(x)=0 \leftrightarrow \varphi(x)), \quad \text{i.e.}$$

(3) $$|\Delta^1_{n+2}| \models \forall x \left(g(x)=0 \to \varphi(x)\right) \quad \text{and}$$

(4) $$|\Delta^1_{n+2}| \models \forall x \left(\varphi(x) \to g(x)=0\right).$$

By Lemma 1 (3) and (4) hold with $|\Delta^1_{n+2}|$ replaced by N. Hence $N \models \forall x \left(g(x)= =0 \leftrightarrow \varphi(x)\right)$ and because g is a Δ^1_{n+2} function we get by (2) a contradiction.

THEOREM. *Assume $V=L$. Let T be an extension of $Cn(A)$ in the language of (A), and let S be a set of Π^1_n formulas $(n<\omega)$ such that N is a model of $T \cup S$. Then*
1) $Cn(S) \neq T$.
2) $Cn_\omega(S) \neq T$.
3) $Cn_\beta(S) \neq T$.

Proof. It is an immediate consequence of the above lemmata. $|\Delta^1_{n+2}|$ is β-model, because the formula Bord (a) if Π^1_1 one (see [4]).

COROLLARY. For every formula φ we have

$$Cn(\varphi) \neq Cn(A).$$

The author observed that the consideration of the structure $|\Delta^1_{n+2}|$ is the key to the proof of the theorem. The original proof that the structure $|\Delta^1_{n+2}|$ has the required properties was longer than the present one. The present version of the proof that $|\Delta^1_{n+2}|$ has the required properties was suggested to the author by dr L. Pacholski, to whom the author expresses his great gratitude.

After the preparation of the paper the author was informed that Lemma 1 could be found in [5].

The first proof of non-finite axiomatizability of the second order arithmetic with the full comprehension schema was given in [6].

INSTITUTE OF MATHEMATICS, UNIVERSITY, WROCŁAW
(INSTYTUT MATEMATYCZNY, UNIWERSYTET, WROCŁAW)

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К. Р. Апт, **Бесконечная аксиоматизируемость арифметики второй ступени**

Содержание. В настоящей работе доказывается следующая теорема: Пусть $V=L$. Пусть (A) будет арифметикой второй ступени. Если S является множеством всех справедливых Π^1_n формул, то $Cn_\beta(S)$ не содержит в себе $Cn(A)$. В следствии мы получаем, что (A) не может быть конечно аксиоматизируема.