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THE "MIXING CHARACTER ORDER" CAN NOT BE MEASURED BY A FINITE SET OF DIFFERENTIABLE FUNCTIONS

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The "mixing character order" can not be measured by a finite set of differentiable functions.

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Abstract.
Let Γ be the space of all sequences $\gamma = (p_1, p_2, \dots)$, $\Sigma\, p_i = 1$, $p_1 \geq p_2 \geq \dots \geq 0$.
The "mixing character order" (cf. [1, 2]) on Γ is defined by $\gamma \supset \gamma' \Leftrightarrow \sum_{i=1}^{r} p_i \leq \sum_i p_i'$ (γ is more mixed than γ'). A function $f\colon \Gamma \to \mathbb{R}$ is called order preserving if $\gamma \supset \gamma' \Rightarrow f(\gamma) \geq f(\gamma')$. One particular such function is entropy $E(\gamma) = -\sum_i p_i \ln(p_i)$. In this small note we show that no finite set of differentiable functions can suffice to characterize this "mixing order".

1. Introduction and statement of the theorem.

Let Γ be the space of all sequences $\gamma = (p_1, p_2, p_3, \dots)$, $p_i \in \mathbb{R}$, $p_i \geq 0$, such that $\sum_{i=1}^{\infty} p_i = 1$ and $p_i \geq p_{i+1}$ for all i. We define a partial order on Γ by

$$(1.1)\qquad \gamma \supset \gamma' \Leftrightarrow \sum_{i=1}^{r} p_i \leq \sum_{i=1}^{r} p_i' \qquad \text{for all } r = 1, 2, \dots$$

This order has been investigated by Ruch and Mead in [1, 2] in connection with thermo-dynamical considerations. They called it the mixing character order. It appears that for a considerable class of systems evolving in time the mixing character order must always increase in time, a principle which is considerably stronger (in fact infinitely stronger cf. Theorem 1.2 below) than the principle that entropy must always increase. In various other parts of mathematics this order (under varying names) also plays a considerable role [3, 4, 5, 6, 7].
A function $f\colon \Gamma \to \mathbb{R}$ is called order preserving if $\gamma \supset \gamma' \Rightarrow f(\gamma) \supset f(\gamma')$. Entropy, defined by $E(\gamma) = -\Sigma\, p_i \ln(p_i)$ is one such function. The question arises whether perhabs some finite set of functions suffices to determine the mixing order*): in other words we are curious to know how much stronger e.g. the statement $\gamma \supset \gamma'$ is than $E(\gamma) \supset E(\gamma')$.

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*) Some preliminary remarks on this question can be found in [2], however, only for separable functions f, i.e. functions which can be written as $f(\gamma) = \sum_i g(p_i)$ for some $g\colon [0, 1] \to \mathbb{R}$.

One has

1.2. Theorem.

There is no finite set F of differentiable order preserving functions $\Gamma \to \mathbb{R}$ such that $f(\gamma) \geq f(\gamma')$ for all $f \in F$ and γ incomparable to γ' does not happen. (Here differentiable means that first partial derivatives exist for all γ in the interior $\Gamma^0 = \{\gamma = (p_1, p_2, \ldots) \in \Gamma\colon p_i > 0,$ $p_i > p_{i+1}$ for all $i\}$ of Γ.)

This result is an immediate corollary of the following proposition

1.3. Proposition.

Let Γ_n be the subset of Γ consisting of all γ for which $p_{n+1} = p_{n+2} \cdots = 0$. Let F be a determining set of order preserving differentiable functions on Γ_n. Then F has at least n-1 elements.

Note that for Γ_n there does indeed exist a set of n-1 determining functions, viz. the "defining functions" $-\sum_{i=1}^{r} p_i$, $r = 1, \ldots, n$. For Γ a countable set of determining functions does of course exist (the defining functions again). Another, rather pleasing, infinite set of determining functions can be found in [2].

2. Proof of proposition 1.3.

Let F be a determining set of order preserving differentiable functions for Γ_n of m elements with $m < n-1$, $F = \{f_1, \ldots, f_m\}$. Then F defines a differentiable map

$$(2.1) \qquad F\colon \Gamma_n \to \mathbb{R}^m, \quad F(\gamma) = (f_1(\gamma), \ldots, f_m(\gamma))^T$$

where T denotes tranposes.

We can assume that there is an $\gamma \in \Gamma_n^0$, $\gamma = (p_1, \ldots, p_n)$ with $p_1 > p_2 > \ldots > p_n > 0$ such that $J(F)(\gamma) \neq 0$, where $J(F)(\gamma)$ is the Jacobian matrix of F at γ. For otherwise F would be constant on Γ_n^0 and hence by continuity constant on all of Γ_n contradicting that F is determining. Let

$$(2.2) \qquad \gamma' = \gamma + t\delta, \quad \delta = (q_1, \ldots, q_n), \quad \Sigma q_i = 0, \quad q_i \in \mathbb{R}.$$

Then for a given δ for small enough t, $\gamma' \in \Gamma_n$. We also have for $t > 0$ and small enough

$$(2.3) \qquad \gamma \supset \gamma' \Leftrightarrow q_1 \geq 0, \quad q_1 + q_2 \geq 0, \ldots, q_1 + \ldots + q_{n-1} \geq 0.$$

Now let v_i be the i-th column vector of $J(F)(\gamma)$. Then

(2.4) $$F(\gamma') = F(\gamma) + t \, \Sigma \, q_i v_i + o(t) \qquad \text{as } t > 0$$

where o is Landau's small o symbol. Let $V \subset \mathbb{R}^m$ be the vector space spanned by the v_i and $C \subset V$ the negative cone of all $v \in V$ all of whose coordinates are < 0. The mapping $\varphi\colon \mathbb{R}^{n-1} \to V$ defined by $(q_1, \ldots, q_{n-1}) \mapsto \sum_{i=1}^{n} q_i v_i$, $q_n = -\sum_{i=1}^{n-1} q_i$ is linear and hence surjective. Choose $y \in C$. Let $N = \mathrm{Ker}(\varphi)$. Then because $n-1 > m$, $\dim N \geq 1$. So there is a line of points $Z + SW$, $0 \neq W \in \mathbb{N}$ fixed, $S \in \mathbb{R}$ such that $\varphi(Z + SW) = y$ for all $S \in \mathbb{R}$. It easily follows (because $W \neq 0$ and $Z \notin N$) that there is an S such that if $(q_1, \ldots q_{n-1}) = Z + SN$ then $q_1 + \ldots + q_i < 0$ for some i. Set $q_n = -\sum_{i=1}^{n} q_i$. Then using (2.3) and (2.4) we have for small enough $t > 0$ that

$$f_i(\gamma') < f_i(\gamma), \; i = 1, \ldots, m \text{ and } \gamma \neq \gamma'$$

proving that $\{f_1, \ldots, f_m\}$ is not determining. This proves proposition 1.3.

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