ON INVARIANTS AND CANONICAL FORMS FOR LINEAR DYNAMICAL SYSTEMS

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The following text presents no more (nor less) than an outline and possibly a guide to the principal results of [2-5] and some related material [6,7].

A constant, linear, dynamical system is a set of equations

$$\begin{array}{lll} & \dot{x} = Fx + Gu & x_{t+1} = Fx_t + Gu_t \\ (1) & y = Hx & y_t = Hx_t \\ & \text{(continuous time)} & \text{(discrete time)} \end{array}$$

with $u \in \mathbb{R}^m \doteq$ input space or control space, $x \in \mathbb{R}^n$ = state space, $y \in \mathbb{R}^p$ = output space. Here F, G, H are real matrices of the appropriate sizes with constant coefficients. The system is completely given by the triple of matrices (F,G,H). We use $L_{m,n,p}(\mathbb{R})$ to denote the space of all triples of matrices of sizes nxn, nxm, pxn respectively.

Of course the discrete time systems (1) also make sense for matrices (F,G,H) with coefficients in any field.

From the "black box" or "input-output" point of view the system Σ = (F,G,H) assigns the output function

$$(2) \qquad y = f_\Sigma u, \; y(t) = \int_0^t He^{F(t-\tau)}Gu(\tau)d\tau$$

to the input function u(t) if we start in x(0) at time t = 0. From this point of view there is a redundancy about the description of the system by means of a triple of matrices (F,G,H). Indeed let $GL_n(\mathbb{R})$ be the group of invertible real nxn matrices and let $GL_n(\mathbb{R})$ act on $L_{m,n,p}(\mathbb{R})$ according to

$$(3) \qquad (F,G,H)^S = (SFS^{-1}, SG, HS^{-1})$$

Then the input-output maps of Σ = (F,G,H) and of $\Sigma^S = (F,G,H)^S$ (both with starting state x(0) = 0 at time t=0) are exactly the same for all $S \in GL_n(\mathbb{R})$. We thus have an (internal) group of symmetrics $GL_n(\mathbb{R})$ of "basis transformations in state space". (The action just described corresponds to the state space transformation x' = Sx).

Several related questions now rise:

(i) What are the invariants for the action (3)? (Here an <u>invariant</u> is any continuous function f: $L_{m,n,p}(\mathbb{R}) \to \mathbb{R}$ such that $f((F,G,H)^S) = f((F,G,H))$ for all $S \in GL_n(\mathbb{R})$).

(ii) Does (3) describe all the redundancy in the description (F,G,H) of the input-output map (2); can recover "(F,G,H)-up-to-$GL_n(\mathbb{R})$-action" from the input-output data (2). How does one recognize that an input-output map comes from a (finite dimensional) system (F,G,H)?

(iii) Do there exist continuous canonical forms on suitable subspaces of $L_{m,n,p}(\mathbb{R})$? Here a continuous canonical form on a subspace $L' \subset L_{m,n,p}(\mathbb{R})$ is a continuous map $c: L' \to L'$ such that: (a) if $c(F,G,H) = (\bar{F},\bar{G},\bar{H})$ then there is an $S \in GL_n(\mathbb{R})$ such that $(\bar{F},\bar{G},\bar{H}) = (F,G,H)^S$ and (b) $c(F,G,H) = c(\bar{F},\bar{G},\bar{H})$ if and only if there is an $S \in GL_n(\mathbb{R})$ such that $(\bar{F},\bar{G},\bar{H}) = (F,G,H)^S$.

To answer these questions it is necessary to define two more concepts. The system (F,G,H) is said to be completely reachable (cr) if the matrix $R(F,G) = (G\ FG\ \ldots\ F^nG)$ consisting of all the columns of the matrices F^iG, $i = 0, \ldots, n$, has rank n; the system (F,G,H) is said to be completely observable if the matrix $Q(F,H)$ defined by $Q(F,H)^T = (H^T, F^TH^T, \ldots, (F^T)^nH^T)$ has rank n. Here an upper "T" denotes "transposes". These two notions have the meanings suggested by their names, cf. [6]. Let $L^{cr,co}_{m,n,p}(\mathbb{R})$ be the open subspace of $L_{m,n,p}(\mathbb{R})$ consisting of all completely observable and completely reachable triples.

Theorem 1. Every invariant of $GL_n(\mathbb{R})$ acting on $L_{m,n,p}(\mathbb{R})$ can be written as a continuous function in the entries of the 2n-matrices $HG, HFG, \ldots, HF^{2n-1}G$.

Let $\mathcal{A} = (A_0, A_1, A_2, \ldots)$ be a sequence of real pxm matrices. We say that $\mathcal{A}$ is realizable if there exists a triple $(F,G,H) \in L_{m,n,p}(\mathbb{R})$ (for some n) such that $A_i = HF^iG$ for all $i = 1,2,\ldots$. For each $r,s \in \mathbb{N}$ let $\mathcal{H}_{r,s}(\mathcal{A})$ be the block Hankel matrix

$$\mathcal{H}_{r,s}(\mathcal{A}) = \begin{pmatrix} A_0 A_1 & \cdots & A_r \\ A_1 & & \vdots \\ \vdots & & \\ A_s & \cdots & A_{r+s} \end{pmatrix}$$

The answer to question (ii) is now given by

Theorem 2. (Ho, Kalman, Meadowes, Silverman, Tissi, Youla). The sequence $\mathcal{A}$ is realizable by a triple $(F,G,H) \in L_{m,n,p}(\mathbb{R})$ iff there is an n_o such that $n \geq n_o = \operatorname{rank} \mathcal{H}_{n_o-1,n_o-1}(\mathcal{A}) = \operatorname{rank} \mathcal{H}_{r,s}(\mathcal{A})$ for all $r,s \geq n_o-1$. Moreover all realizations of dimension n_o are co and cr and they all are in the same $GL_n(\mathbb{R})$ orbit.

It is now clear from theorem 2, that question (iii) is especially important for the subspace $L^{cr,co}_{m,n,p}(\mathbb{R})$. Before answering it let us take time out to explain why the word continuous in question (iii) is (sometimes) important. First, using delta functions as inputs we see from (2) that knowing the input-output data of a system amounts to knowing the sequence of matrices $HG, HFG, HF^2G, \ldots$. Now suppose we have an unknown black box to be modelled by a linear dynamical system (1). The algorithmic proof of theorem 2 gives us a way of calculating (F,G,H) from $HG, \ldots, HF^{2n-1}G$. Because of measurement errors it would be highly desirable to have a continuous algorithm calculating (F,G,H) from $(HG,\ldots,HF^{2n-1}G)$. Now the existence of such a continuous algorithm is easily seen to be equivalent to the existence of a continuous canonical form. Cf. also [1] for some remarks in a related case.

Theorem 3. There is a continuous canonical form on $L^{cr,co}_{m,n,p}(\mathbb{R})$ if and only if $m = 1$ or $p = 1$.

The proof of this theorem goes via a detailed study of the orbit space $L^{cr,co}_{m,n,p}(\mathbb{R})/GL_n(\mathbb{R})$.

Theorem 4. $L^{co,cr}_{m,n,p}(\mathbb{R})/GL_n(\mathbb{R}) = M^{co,cr}_{m,n,p}(\mathbb{R})$ is a smooth noncompact differentiable manifold (without boundary) of dimension $mn + np$. The natural projection $\pi: L^{cr,co}_{m,n,p}(\mathbb{R}) \to M^{co,cr}_{m,n,p}(\mathbb{R})$ is a locally-trivial principal $GL_n(\mathbb{R})$ bundle which is (globally) trivial iff $p = 1$ or $m = 1$.

From the identification of systems point of view (cf. also just above theorem 3) it is interesting to see if $M^{co,cr}_{m,n,p}(\mathbb{R})$ can be compactified in a system theoretically meaningful way.

Theorem 5. Let $D = B_o + B_1 \frac{d}{dt} + \ldots + B_{n-1} \frac{d^{n-1}}{dt^{n-1}}$ be the linear operator $u(t) \mapsto y(t) = B_o u(t) + \ldots + B_{n-1} \frac{d^{n-1}}{dt^{n-1}} u(t)$, where $B_o, \ldots, B_{n-1}$ are constant real pxm matrices. Then every such operator D arises as a converging limit of input-output maps of systems in $L^{cr,co}_{m,n,p}(\mathbb{R})$. Inversely if $\Sigma_s, s = 1,2, \ldots$ is a sequence of systems in $L^{cr,co}_{m,n,p}(\mathbb{R})$ such that $\lim_{s\to\infty} f_{\Sigma_s} u(t) = fu(t)$ uniformly on each bounded t interval, then f is the (direct) sum of an integral operator of (size pxm and) order $\leq i-1$ and the input-output function of a co and cr system of state space dimension $n-i$.

This provides a partial, but apparently system theoretically maximal, compactification of $M^{cr,co}_{m,n,p}(\mathbb{R})$.

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