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Polyhedral Combinatorics—Some Recent Developments and Results

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Polyhedral combinatorics deals with characterizing convex hulls of vectors obtained from combinatorial structures, and with deriving min-max relations and algorithms for corresponding combinatorial optimization problems. In this paper, after an introduction discussing the matching polytope (§1) and some algorithmic consequences (§2), we give some illustrations of recent developments (viz., applications of lattice and decomposition techniques (§§3 and 4)), we go into the relation to cutting planes (§5), and we describe some other recent results (§6).

1. A basic example: The matching polytope. We first describe a basic result in polyhedral combinatorics, due to Edmonds [**7**]. Let $G = (V, E)$ be an undirected graph (i.e., V is a finite set (of *vertices*) and E is a collection of pairs (*edges*) of vertices). A subset M of E is called a *matching* if $e' \cap e'' = \emptyset$ whenever $e', e'' \in M$, $e' \neq e''$. The *matching polytope* of G is the set conv.hull$\{\chi^M | M$ matching$\}$ in $\mathbf{R}^E$, where χ^M is the incidence vector of M (i.e., $\chi^M \in \mathbf{R}^E$ with $\chi^M(e) = 1$ if $e \in M$, and $= 0$ otherwise). Edmonds now showed:

THEOREM 1 (EDMONDS'S MATCHING POLYTOPE THEOREM). *The matching polytope of $G = (V, E)$ is equal to the set of vectors $x \in \mathbf{R}^E$ satisfying:*

$$\begin{array}{lll} \text{(i)} & x_e \geq 0 & (e \in E), \\ \text{(ii)} & \sum_{e \ni v} x_e \leq 1 & (v \in V), \\ \text{(iii)} & \sum_{e \subseteq U} x_e \leq \left\lfloor \frac{1}{2}|U| \right\rfloor & (U \subseteq V, |U| \text{ odd}). \end{array} \tag{1}$$

For proofs we refer to [**24**, **30**, **33**].

Edmonds's theorem has the following application. If we are given some "weight" function $c \in \mathbf{R}^E$, we can describe the problem of finding a matching M of maximum "weight" $\sum_{e \in M} c_e$ equivalently as the problem of maximizing $c^{\mathrm{T}}x$ over the matching polytope, that is, by Edmonds's theorem, over $x \in \mathbf{R}^E$ satisfying (1). This last is a linear programming problem, and we can apply

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LP-techniques to solve this problem, and hence to solve the combinatorial optimization problem. Among others, with the help of the ellipsoid method, it can be shown that the maximum matching problem is solvable in polynomial time—see §2.

Another, theoretical, application of Edmonds's theorem is obtained with the duality theorem of linear programming. Let $Ax \leq b$ denote the system (1). Then for any $c \in \mathbf{R}^E$

$$\max\left\{\sum_{e\in M} c_e | M \text{ matching}\right\} = \max\{c^{\mathrm{T}}x | Ax \leq b\}$$
$$= \min\{y^{\mathrm{T}}b | y \geq 0; y^{\mathrm{T}}A = c^{\mathrm{T}}\}. \tag{2}$$

So we have a min-max relation for the maximum matching problem. It was shown by Cunningham and Marsh [**6**] that if c is integer-valued, then the minimum in (2) has an integer optimum solution y. The special case $c \equiv \mathbf{1}$ (the all-one function) is equivalent to the following *Tutte-Berge formula* [**35**, **1**]: the maximum cardinality of a matching in a graph $G = (V, E)$ is equal to

$$\min_{U\subseteq V} \frac{|V| + |U| - \mathcal{O}(V\backslash U)}{2}, \tag{3}$$

where $\mathcal{O}(V\backslash U)$ denotes the number of components of $\langle V\backslash U\rangle$ with an odd number of vertices ($\langle V\backslash U\rangle$ denotes the graph $(V\backslash U, \{e \in E | e \subseteq V\backslash U\})$).

Note that the constraint matrix A in (1) generally is not totally unimodular (a matrix is *totally unimodular* if all subdeterminants belong to $\{0, \pm 1\}$). If G is bipartite (i.e., V can be split into classes V' and V'' (the *color classes*) so that $E \subseteq \{\{v', v''\} | v' \in V', v'' \in V''\}$), then the inequalities (1)(iii) can be deleted as they are implied by the constraints (i) and (ii), as one easily checks. In that case, the theorem is due to Egerváry [**9**] and follows more simply from the fact that if M is totally unimodular and d is integer, then each vertex of the polyhedron determined by $Mx \leq d$ is integer.

Similarly, for bipartite G, the Tutte-Berge formula above reduces to the well-known König-Egerváry theorem [**21**, **9**].

2. Polyhedral combinatorics and polynomial solvability. Above we mentioned obtaining polynomial-time algorithms from polyhedral results with the ellipsoid method. In this section we describe this more precisely.

Suppose that for each graph $G = (V, E)$ we have a collection $\mathcal{F}_G$ of subsets of E. For example:

$$\begin{aligned} &\text{(i)} && \mathcal{F}_G = \{M \subseteq E | M \text{ is a matching}\};\\ &\text{(ii)} && \mathcal{F}_G = \{M \subseteq E | M \text{ is a spanning tree}\};\\ &\text{(iii)} && \mathcal{F}_G = \{M \subseteq E | M \text{ is a Hamiltonian circuit}\}. \end{aligned} \tag{4}$$

With any family $(\mathcal{F}_G | G \text{ graph})$ we can associate the following problem:

Optimization problem. Given a graph $G = (V, E)$ and $c \in \mathbf{Q}^E$, find $M \in \mathcal{F}_G$ maximizing $\sum_{e\in M} c_e$. (5)

So if $(\mathcal{F}_G|G \text{ graph})$ is as in (i), (ii), and (iii), respectively, problem (5) amounts to finding a maximum weighted matching, a maximum weighted spanning tree, and a maximum weighted Hamiltonian circuit, respectively. The last problem is the well-known traveling salesman problem (note that by replacing c by $-c$ (5) becomes a minimization problem).

Given a family $(\mathcal{F}_G|G \text{ graph})$, we are interested in finding, for any graph $G = (V, E)$, a system $Ax \leq b$ of linear inequalities in $x \in \mathbf{R}^E$ so that

$$\text{conv.hull}\{\chi^M | M \in \mathcal{F}_G\} = \{x | Ax \leq b\}. \tag{6}$$

If (6) holds, then for any $c \in \mathbf{R}^E$:

$$\begin{aligned} \max\left\{\sum_{e\in M} c_e \middle| M \in \mathcal{F}_G\right\} &= \max\{c^{\mathrm{T}}x | Ax \leq b\} \\ &= \min\{y^{\mathrm{T}}b | y \geq 0; y^{\mathrm{T}}A = c^{\mathrm{T}}\}, \end{aligned} \tag{7}$$

thus formulating the combinatorial optimization problem as a linear programming problem.

The optimization problem (5) is said to be *solvable in polynomial time* or *polynomially solvable* if it is solvable by an algorithm whose running time is bounded above by a polynomial in the *input size* $|V| + |E| + \text{size}(c)$. Here $\text{size}(c) := \sum_{e\in E} \text{size}(c_e)$, where the size of a rational number p/q is equal to $\log_2(|p|+1) + \log_2|q|$. So size$(c)$ is about the space needed to specify c in binary notation.

It has been shown by Karp and Papadimitriou [**20**] and Grötschel, Lovász, and Schrijver [**16**] that (5) is polynomially solvable if and only if the following problem is solvable in polynomial time:

> *Separation problem.* Given a graph $G = (V, E)$ and $x \in \mathbf{Q}^E$, determine if x belongs to $\text{conv.hull}\{\chi^M | M \in \mathcal{F}_G\}$, and if not, find a separating hyperplane. (8)

Again, "polynomial time" means: time bounded by a polynomial in $|V| + |E| + \sum_{e\in E} \text{size}(x_e)$.

THEOREM 2. *For any fixed family $(\mathcal{F}_G|G \text{ graph})$, the optimization problem* (5) *is polynomially solvable if and only if the separation problem* (8) *is polynomially solvable.*

The theorem implies that with respect to the question of polynomial-time solvability, the approach described above (studying the convex hull) is more or less essential: a combinatorial optimization problem is polynomially solvable if and only if the corresponding convex hulls can be decently described—decently, in the sense of the separation problem.

As an application of Theorem 2, it can be shown that the system (1) of linear inequalities can be tested in polynomial time, although there exist exponentially many constraints (Padberg and Rao [**28**]). Hence, the maximum matching problem is polynomially solvable (in fact, this was shown directly by Edmonds [**7**]).

Theorem 2 can also be used in the negative: if a combinatorial optimization problem is not polynomially solvable (maybe the traveling salesman problem), then the corresponding polytopes have no decent description.

Theorem 2 is shown with the ellipsoid method, for which we refer to the books of Grötschel, Lovász, and Schrijver [**17**] and Schrijver [**32**]. The ellipsoid method does not give practical algorithms, but it may give insight in the complexity of a problem.

There are several variations of Theorem 2. For instance, a similar result holds if we consider collections $\mathcal{F}_G$ of subsets of the vertex set V, instead of subsets of the edge set E. Moreover, we may consider families $(\mathcal{F}_G | G \in \mathcal{G})$, where $\mathcal{G}$ is a subclass of the class of all graphs. Similarly, we can consider directed graphs.

3. Lattices and strongly polynomial algorithms. A first recent development in polyhedral combinatorics is the influence of lattice techniques, to a large extent due to the recently developed *basis reduction method* given by Lenstra, Lenstra, and Lovász [**23**]. In this section we give one illustration of this influence, due to Frank and Tardos [**10**].

The basis reduction method solves the following problem:

Given a nonsingular rational $n\times n$-matrix A, find a basis $b_1, \ldots, b_n$ for the lattice generated by the columns of A satisfying (9)

$$\|b_1\| \cdots \|b_n\| \leq 2^{n(n-1)/4} |\det A|,$$

in time bounded by a polynomial in $\text{size}(A) := \sum_{i,j} \text{size}(a_{ij})$. Here the *lattice generated by* $a_1, \ldots, a_n$ is the set of vectors $\lambda_1 a_1 + \cdots + \lambda_n a_n$ with $\lambda_1, \ldots, \lambda_n \in \mathbf{Z}$. Any linearly independent set of vectors generating the lattice is called a *basis* for the lattice.

One of the many consequences is a polynomial-time algorithm for the following *simultaneous diophantine approximation problem*:

Given $n \in \mathbf{N}$, $a \in \mathbf{Q}^n$, and ε with $0 < \varepsilon < 1$, find an integer vector p and an integer q satisfying $\|a - (1/q)p\| < \varepsilon/q$ and $1 \leq q \leq 2^{n(n+1)/4}\varepsilon^{-n}$. (10)

This can be seen by applying the basis reduction method to the $(n+1)\times(n+1)$-matrix

$$A := \begin{pmatrix} I & a \\ 0 & 2^{-n(n+1)/4}\varepsilon^{n+1} \end{pmatrix}, \tag{11}$$

where I is the $n \times n$ identity matrix.

Frank and Tardos showed that this approximation algorithm yields so-called *strongly polynomial* algorithms. The algorithm for the optimization problem (5) derived from the ellipsoid method performs a number of arithmetic operations, which number is bounded by a polynomial in $|V| + |E| + \text{size}(c)$. (*Arithmetic operations* here are: addition, subtraction, multiplication, division, comparison.)

It would be preferable if the size of the weight function c only influences the sizes of the numbers occurring when executing the algorithm, but not the *number* of arithmetic operations. Therefore, one has defined an algorithm for (5) to be *strongly polynomial* if it consists of a number of arithmetic operations, bounded by a polynomial in $|V| + |E|$, on numbers of size bounded by a polynomial in $|V| + |E| + \text{size}(c)$.

Frank and Tardos however showed the equivalence of the two concepts when applied to (5):

THEOREM 3. *For any family* $(\mathcal{F}_G|G$ *graph), there exists a polynomial-time algorithm for the optimization problem* (5) *if and only if there exists a strongly polynomial algorithm for* (5).

Their result was obtained by constructing a strongly polynomial algorithm for the following problem:

$$\text{Given } n \in \mathbf{N} \text{ and } c \in \mathbf{Q}^n, \text{ find } \tilde{c} \in \mathbf{Z}^n \text{ such that } \|\tilde{c}\|_\infty \leq 2^{9n^3} \text{ and such that: } c^{\mathrm{T}}x > c^{\mathrm{T}}y \Leftrightarrow \tilde{c}^{\mathrm{T}}x > \tilde{c}^{\mathrm{T}}y, \text{ for all } x, y \in \{0,1\}^n. \tag{12}$$

With this method the size of c in the optimization problem can be reduced to $O(|E|^3)$, without changing the optimum solution. Hence any polynomial-time algorithm for the optimization problem yields a strongly polynomial algorithm.

As another interesting recent lattice result we mention Lovász's [**25**] characterization of the *perfect matching lattice* (i.e., the lattice generated by the incidence vectors of perfect matchings in a graph), in the same vein as Edmonds's matching polytope theorem.

4. The coclique polytope and decomposition techniques. As another recent development in polyhedral combinatorics we mention the propagation of decomposition techniques. Fundamental decomposition methods are described in Seymour's paper *Decomposition of regular matroids* [**34**]. Also Burlet, Fonlupt, and Uhry [**2**, **3**] obtained deep decomposition results.

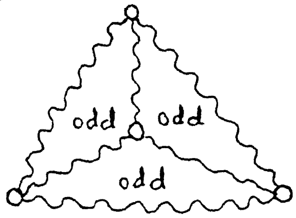
We illustrate the decomposition methods of Seymour by applying them to characterizing the "coclique polytope" of certain graphs. For any undirected graph $G = (V, E)$, a set $C \subseteq V$ is called a *coclique* if it does not contain any edge of G as a subset. The *coclique polytope* of G is the convex hull of the incidence vectors of cocliques in G, i.e., conv.hull$\{\chi^C | C \text{ coclique}\} \subseteq \mathbf{R}^V$.

The problem

$$\text{Given } G = (V, E) \text{ and } c \in \mathbf{R}^V, \text{ find a coclique } C \text{ in } G \text{ maximizing } \textstyle\sum_{v \in C} c_v \tag{13}$$

is NP-complete, and hence probably not polynomially solvable. Therefore, by Theorem 2 (now in the variant with subsets of V instead of E), there is probably no polynomial-time algorithm for the separation problem for coclique polytopes. So we should not expect a decent description for coclique polytopes similar to Edmonds's matching polytope theorem.

For some classes of graphs, however, the coclique polytope has a decent description, e.g., for perfect graphs (including bipartite graphs, line graphs of bipartite graphs, comparability graphs, triangulated graphs, and their complements). Another class of graphs is described in the following theorem of Gerards and Schrijver [**14**]. An undirected graph $G = (V, E)$ is called *odd-K_4-free* if G has no subgraph homeomorphic to



where wriggled lines stand for paths, so that each face in this graph is enclosed by a circuit of odd length.

THEOREM 4. *For any odd-K_4-free graph $G = (V, E)$, the coclique polytope is equal to the set of vectors x in $\mathbf{R}^V$ satisfying:*

$$\begin{array}{lll} \text{(i)} & 0 \le x_v \le 1 & (v \in V), \\ \text{(ii)} & x_v + x_w \le 1 & (\{v, w\} \in E), \\ \text{(iii)} & \sum_{v \in C} x_v \le \left\lfloor \frac{1}{2}|C| \right\rfloor & (C \textit{ circuit with } |C| \textit{ odd}). \end{array} \tag{14}$$

(Here C is a circuit if $C = \{v_1, \dots, v_k\}$ with $\{v_{i-1}, v_i\} \in E$ $(i = 1, 2, \dots, k)$ and $\{v_k, v_1\} \in E$.)

Note that if G is bipartite, then G has no odd circuit, and hence there are no constraints (iii). In that case the theorem reduces to a theorem of Egerváry [**9**].

The theorem implies, with the help of Theorem 2, that problem (13) is polynomially solvable for odd-K_4-free graphs. Indeed, the constraints (14) can be tested for any given $x \in \mathbf{R}^V$ in time bounded by a polynomial in $|V| + |E| + \text{size}(x)$, although there are exponentially many constraints. (The condition (iii) can be tested using a shortest path algorithm.)

We sketch how Theorem 4 can be shown using decomposition techniques (which also yield a direct combinatorial polynomial-time algorithm for the maximum coclique problem for odd-K_4-free graphs). It was shown by Seymour [**34**] that "each regular matroid is obtained by taking 1-,2-, and 3-sums of graphic matroids, cographic matroids, and R_{10}." *Regular* matroids are matroids representable over each field. By a theorem of Tutte [**36**], regular matroids are exactly those binary matroids not containing the Fano-matroid or its dual as a minor.

Seymour's theorem can be equivalently stated as: "Each totally unimodular matrix can be decomposed into network matrices and their transposes and into certain 5×5-matrices." It implies a polynomial-time test for the total unimodularity of matrices, and a polynomial-time algorithm for linear programs over totally unimodular matrices. It also has implications in geometry and graph theory. One of them described by Gerards, Lovász, Schrijver, Seymour, and Truemper [**13**] is as follows.

Consider the following four compositions of graphs $G' = (V', E')$ and $G'' = (V'', E'')$ into a new graph H. *Composition* 1. If $|V' \cap V''| \leq 1$, then $H := (V' \cup V'', E' \cup E'')$. *Composition* 2. If $V' \cap V'' = \{v_1, v_2\} \in E' \cap E''$ and G'' is bipartite, then $H := (V' \cup V'', (E' \cup E'') \backslash \{\{v_1, v_2\}\})$. *Composition* 3. If $V' \cap V'' = \{v_0, v_1, v_2\}$, $E' \cap E'' = \{\{v_0, v_1\}, \{v_0, v_2\}, \{v_1, v_2\}\}$, and v_0 has degree 2 both in G' and in G'', then $H := ((V' \cup V'') \backslash \{v_0\}, (E' \cup E'') \backslash (E' \cap E''))$. *Composition* 4. If $V' \cap V'' = \{v_0, v_1, v_2, v_3\}$, $E' \cap E'' = \{\{v_0, v_1\}, \{v_0, v_2\}, \{v_0, v_3\}\}$, v_0 has degree 3 both in G' and in G'', and G'' is bipartite, then

$$H := ((V' \cup V'') \backslash \{v_0\}, (E' \cup E'') \backslash (E' \cap E'')).$$

Moreover, consider the following operations on a graph $G = (V, E)$. *Operation* 1. If $\{v_0, v_1\}, \{v_1, v_2\}, \{v_2, v_3\} \in E$, where both v_1 and v_2 have degree 2, then

$$H := (V \backslash \{v_1, v_2\}, (E \backslash \{\{v_0, v_1\}, \{v_1, v_2\}, \{v_2, v_3\}\}) \cup \{v_0, v_3\}).$$

Operation 2. If $v_0 \in V$, where $\{v_0, v_1\}, \ldots, \{v_0, v_k\}$ are the edges of G containing v_0, then let $w_1, \ldots, w_k$ be "new" vertices and

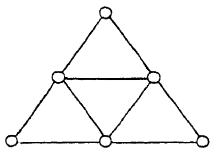
$$\begin{aligned} H := (V \cup \{w_1, \ldots, w_k\}, (E' \backslash \{\{v_0, v_1\}, \ldots, \{v_0, v_k\}\}) \\ \cup \{\{v_0, w_1\}, \ldots, \{v_0, w_k\}, \{w_1, v_1\}, \ldots, \{w_k, v_k\}\}). \end{aligned}$$

THEOREM 5. *An undirected graph is odd-K_4-free if and only if it can be constructed by a series of compositions and operations above starting with the following graphs*:

(i) *graphs $G = (V, E)$ having a vertex v_0 so that the graph $(V \backslash \{v_0\}, E \backslash \{e | e \ni v_0\})$ is bipartite*;

(ii) *planar graphs having exactly two odd facets (an odd facet is a facet enclosed by an odd number of edges)*; (15)

(iii) *the following graph*:



Sufficiency in this theorem is easy to see: each of the graphs in (i), (ii), and (iii) is odd-K_4-free. Moreover, each of the compositions and operations maintains the property of being odd-K_4-free. The content of the theorem is that in this way all odd-K_4-free graphs can be constructed.

In order to derive now Theorem 4, it suffices to prove that each of the graphs (15) has the property described in Theorem 4, and moreover that this property is maintained under each of the compositions and operations above. Showing this is not as hard as the original direct proof of Theorem 4.

If we let $Ax \leq b$ denote the system (14), then by Theorem 4 for odd-K_4-free graphs $G = (V, E)$ and $c \in \mathbf{R}^V$:

$$\max\left\{\sum_{v\in C} c_v | C \text{ coclique}\right\} = \max\{c^{\mathrm{T}}x | Ax \leq b\}$$
$$= \min\{y^{\mathrm{T}}b | y \geq 0; y^{\mathrm{T}}A = c^{\mathrm{T}}\}. \tag{16}$$

Using the above decomposition techniques, Gerards [**12**] showed that if c is integer-valued, the minimum has an integer optimum solution y. In particular, if $c \equiv \mathbf{1}$ (the all-one function) then the maximum size of a coclique is equal to

$$\min\left(|F| + \sum_{i=1}^{t} \left\lfloor \frac{1}{2}|C_i| \right\rfloor\right), \tag{17}$$

where the minimum ranges over all subsets F of E and circuits $C_1, \ldots, C_t$ such that $V = \bigcup F \cup \bigcup_{i=1}^{t} C_i$. This forms an extension of a theorem of König [**22**] for bipartite graphs.

5. Cutting planes. Quite often the problem of characterizing the convex hull of certain $\{0,1\}$-vectors amounts to characterizing, for some polyhedron P, the polyhedron

$$P_{\mathrm{I}} := \text{conv.hull}\{x \in P | x \text{ integral}\}. \tag{18}$$

P_{I} is called the *integer hull* of P. E.g., if $G = (V, E)$ is a graph, and

$$P := \left\{x \in \mathbf{R}^E | x_e \geq 0 \ (e \in E); \sum_{e \ni v} x_e \leq 1 \ (v \in V)\right\}, \tag{19}$$

the integral vectors in P are exactly the incidence vectors of matchings, and hence P_{I} is equal to the matching polytope of G. Similarly, for

$$P := \left\{x \in \mathbf{R}^V | x_v \geq 0 \ (v \in V); \sum_{v \in e} x_v \leq 1 \ (e \in E)\right\}, \tag{20}$$

P_{I} is the coclique polytope of G.

For any rational polyhedron P, there is a procedure of deriving the inequalities determining P_{I} from those determining P—the *cutting plane method*, due to Gomory [**15**]. The following description is due to Chvátal [**4**] and Schrijver [**29**].

Clearly, if H is a *rational half-space*, i.e., H is of form

$$H = \{x \in \mathbf{R}^n | a^{\mathrm{T}}x \leq \beta\}, \tag{21}$$

where $a \in \mathbf{Q}^n$, $a \neq \mathbf{0}$, $\beta \in \mathbf{Q}$, we may assume without loss of generality that a is integral, and that the components of a are relatively prime integers. In that case:

$$H_{\mathrm{I}} = \{x \in \mathbf{R}^n | a^{\mathrm{T}}x \leq \lfloor \beta \rfloor\}. \tag{22}$$

H_{I} arises from H by shifting its bounding hyperplane until it contains integral vectors.

Now define for any set P in $\mathbf{R}^n$:

$$P' := \bigcup_{H \supseteq P} H_{\mathrm{I}}, \tag{23}$$

where H ranges over all rational half-spaces containing P. Since $H \supseteq P$ implies $H_{\mathrm{I}} \supseteq P_{\mathrm{I}}$, it follows that $P' \supseteq P_{\mathrm{I}}$. It can be shown that if P is a rational polyhedron (i.e., a polyhedron determined by linear inequalities with rational coefficients), then P' is a rational polyhedron again.

To P' we can apply this operation again, yielding P''. It is not difficult to find rational polyhedra with $P'' \neq P'$. Each rational polyhedron P thus gives a sequence of polyhedra containing P_{I}:

$$P \supseteq P' \supseteq P'' \supseteq P''' \supseteq \cdots \supseteq P_{\mathrm{I}}. \tag{24}$$

Denoting the $(t+1)$th set in this sequence by $P^{(t)}$, the following can be shown.

THEOREM 6. *For each rational polyhedron P there exists a number t such that $P^{(t)} = P_{\mathrm{I}}$.*

The theorem is the theoretical essence of the termination of the cutting plane method of Gomory. The equation $a^{\mathrm{T}}x = \lfloor \beta \rfloor$ defining H_{I}, or more strictly the hyperplane $\{x | a^{\mathrm{T}}x = \lfloor \beta \rfloor\}$, is called a *cutting plane.*

The smallest t for which $P^{(t)} = P_{\mathrm{I}}$ can be considered as a measure for the complexity of P_{I} relative to that of P. In a sense, P' is near to P, P'' to P', and so on.

Let us study some specific polyhedra related to graphs. Let $G = (V, E)$ be an undirected graph, and let P be the polytope (19), implying that P_{I} is the matching polytope of G. It is not hard to see that for each graph G, the polytope P' is the set of all vectors x in P satisfying

$$\sum_{e \subseteq U} x_e \leq \left\lfloor \frac{1}{2}|U| \right\rfloor \qquad (U \subseteq V, |U| \text{ odd}). \tag{25}$$

(Of course, there are infinitely many half-spaces H containing P, but the corresponding inequalities $a^{\mathrm{T}}x \leq \lfloor \beta \rfloor$ all are implied by the inequalities defining P and by (25).) So Edmonds's matching polytope theorem in fact tells us that $P' = P_{\mathrm{I}}$ for each graph G. ($P = P_{\mathrm{I}}$ for bipartite G, since in that case (25) is implied by the inequalities determining P.)

Next let, for any undirected graph $G = (V, E)$, P be the polytope (20), implying that P_{I} is the coclique polytope of G. It is not difficult to check that the polytope P' is the set of vectors x in P satisfying

$$\sum_{v \in C} x_v \leq \left\lfloor \frac{1}{2}|C| \right\rfloor \qquad (C \text{ odd circuit}). \tag{26}$$

So Theorem 4 states that $P' = P_{\mathrm{I}}$ if G is odd-K_4-free. By Egerváry's theorem $P = P_{\mathrm{I}}$ if and only if G is bipartite. Chvátal [**5**] has shown that there exists no fixed t so that $P^{(t)} = P_{\mathrm{I}}$ for each graph G.

An important computational application of cutting planes is to the traveling salesman problem, which we mention in the following section.

6. The traveling salesman problem and cuts. The well-known *traveling salesman problem* (in its directed, asymmetric form) can be formulated as an integer linear programming problem as follows, for given $n \in \mathbf{N}$ and $c = (c_{ij}) \in \mathbf{R}^{n \times n}$:

$$\text{minimize} \sum_{i,j=1}^{n} x_{ij},$$

such that

$$(*)\begin{cases} x_{ij} \geq 0 & (i,j = 1,\dots,n); \\ \sum_{i \notin U, j \in U} x_{ij} \geq 1 & (\varnothing \neq U \subsetneqq \{1,\dots,n\}); \\ \sum_{j=1}^{n} x_{ij} = 1 & (i = 1,\dots,n); \end{cases} \tag{27}$$
$$x_{ij} \text{ integer} \quad (i,j = 1,\dots,n).$$

Let P be the polytope in $\mathbf{R}^{n \times n}$ determined by $(*)$. It is clear that P_{I} is the convex hull of the incidence vectors of traveling salesman routes. Since the traveling salesman problem is NP-complete, we may not expect a "decent" description of P_{I} in the sense of Theorem 2. In fact, if NP $\neq$ co-NP there is no fixed t such that $P^{(t)} = P_{\mathrm{I}}$ for each n.

On the other hand, cutting planes can be helpful in solving instances of the traveling salesman problem. The traveling salesman problem is equivalent to solving $\min\{c^{\mathrm{T}}x | x \in P_{\mathrm{I}}\}$, while solving $\min\{c^{\mathrm{T}}x | x \in P\}$ is not so difficult (it is polynomially solvable), and it yields a good lower bound for the traveling salesman optimum value (since $P \supseteq P_{\mathrm{I}}$). Good bounds are essential in branch-and-bound procedures for the traveling salesman problem.

Adding *all* cutting planes to $(*)$ to obtain P_{I} seems infeasible, but instead we could add *some* cutting planes in order to obtain a better lower bound. This is a basic ingredient in the recent successes of Crowder, Grötschel, and Padberg in solving large-scale traveling salesman problems (see [**18**, **27**]). Recently, Padberg was able to solve a symmetric 2392-"city" problem using cutting planes.

We shall not go into the details of solving the traveling salesman problem. We describe some theoretical results related to the above, which exhibit some of the connections of polyhedral results with combinatorial min-max relations.

Let $\mathcal{C}$ be a collection of subsets of $V := \{1,\dots,n\}$ satisfying:

$$\begin{aligned} &\text{(i)} \quad \varnothing \notin \mathcal{C},\ V \notin \mathcal{C}; \\ &\text{(ii)} \quad \text{if } T, U \in \mathcal{C},\ T \cap U \neq \varnothing,\ T \cup U \neq V, \text{ then } T \cap U \in \mathcal{C} \\ &\qquad \text{and } T \cup U \in \mathcal{C}. \end{aligned} \tag{28}$$

Such a collection is called a *crossing family.* Consider the polytope P consisting of all $x = (x_{ij}) \in \mathbf{R}^{n \times n}$ satisfying:

$$x_{ij} \geq 0 \quad (i,j = 1,\dots,n), \qquad \sum_{i \notin U,\, j \in U} x_{ij} \geq 1 \quad (U \in \mathcal{C}). \tag{29}$$

Note that (*) in (27) defines a facet of P, for $\mathcal{C} = \mathcal{P}(V)\backslash\{\emptyset, V\}$.

The following theorem was shown in [**31**].

THEOREM 7. *P has integral vertices if and only if*

$$\text{there are no sets } V_1, V_2, V_3, V_4, V_5 \text{ in } \mathcal{C} \text{ such that } V_1 \subseteq V_2 \cap V_3, \quad V_2 \cup V_3 = V,\ V_3 \cup V_4 \subseteq V_5,\ V_3 \cap V_4 = \emptyset. \tag{30}$$

Note that if x is an integral vertex of P, then x is a $\{0,1\}$-vector.

Theorem 7 can be put in a more combinatorial setting. Let $\mathcal{C} \subseteq \mathcal{P}(V)$ be a crossing family and let $D = (V, A)$ be a *directed* graph (i.e., V is a finite set and $A \subseteq V \times V$). Call a subset A' of A a *covering* (for $\mathcal{C}$) if each $U \in \mathcal{C}$ is entered by at least one arc in A' ($a = (v, w)$ *enters* U if $v \notin U$, $w \in U$). Call a subset A' of A a *cut* (*induced by* $\mathcal{C}$) if $A' = \delta_A^-(U) := \{a \in A | a \text{ enters } U\}$ for some $U \in \mathcal{C}$. So each covering intersects each cut.

Consider the polyhedron in $\mathbf{R}^A$ determined by:

$$x_a \geq 0 \quad (a \in A), \qquad \sum_{a \in \delta^-(U)} x_a \geq 1 \quad (U \in \mathcal{C}). \tag{31}$$

Then Theorem 7 is equivalent to:

THEOREM 8. *Each vertex of the polyhedron determined by* (31) *is the incidence vector of a covering, for each directed graph $D = (V, A)$, if and only if* (30) *holds.*

Now we have the following: (30) holds $\Leftrightarrow$ the polyhedron determined by (31) has vertices coming from coverings and facets coming from cuts $\Leftrightarrow$ (by polarity) the polyhedron determined by

$$x_a \geq 0 \quad (a \in A), \qquad \sum_{a \in C} x_a \geq 1 \quad (C \text{ covering}) \tag{32}$$

has *vertices* coming from cuts and *facets* coming from coverings. So Theorem 8 is equivalent to:

THEOREM 9. *Each vertex of the polyhedron determined by* (32) *is the incidence vector of a cut, for each directed graph $D = (V, A)$, if and only if* (30) *holds.*

It follows that if (30) holds, and $c \in \mathbf{Z}_+^A$, then the linear programs of minimizing $c^{\mathrm{T}}x$ over (31) and over (32), respectively, have integral optimum solutions, corresponding to a minimum-weighted covering and a minimum-weighted cut, respectively. In fact, it is shown in [**31**] that if (30) holds, then also the linear programs dual to these programs have integer optimum solutions. By LP-duality this means:

THEOREM 10. *Let $\mathcal{C}$ be a crossing family satisfying* (30), *let $c \in \mathbf{Z}^A$, and let $D = (V, A)$ be a directed graph. Then* (i) *the minimum weight of a covering is equal to the maximum number t of cuts $C_1, \ldots, C_t$ (repetition allowed) so that each arc a of D is in at most c_a of the cuts C_i;* (ii) *the minimum weight of a cut* *is equal to the maximum number t of coverings $C_1, \ldots, C_t$ (repetition allowed) so that each arc a of D is in at most c_a of the coverings C_i.*

We mention the following applications.

1. Let V be partitioned into classes V' and V'', let $\mathcal{C} := \{\{v\} | v \in V'\} \cup \{V \backslash \{v\} | v \in V''\}$, $A \subseteq V'' \times V'$, $c \equiv \mathbf{1}$. Then (i) in Theorem 10 is equivalent to a theorem of König [**22**]: the minimum number of edges covering all vertices in a bipartite graph is equal to the maximum size of a coclique. Similarly, (ii) is equivalent to a theorem of Gupta [**19**]: the minimum degree in a bipartite graph is equal to the maximum number of pairwise disjoint edge sets $E_1, \ldots, E_t$ each covering all vertices.

2. Let $r, s \in V$ be fixed, let $\mathcal{C} := \{U \subseteq V | r \notin U, s \in U\}$, $D = (V, A)$ arbitrary, and $c \equiv \mathbf{1}$. Then (i) in Theorem 10 is equivalent to the (easy) result that the minimum number of edges in a path from r to s in D is equal to the maximum number of pairwise disjoint cuts separating r from s. Assertion (ii) is Menger's theorem [**26**]: the minimum number of edges in a cut separating r from s is equal to the maximum number of pairwise edge-disjoint paths from r to s.

3. Let $r \in V$ be fixed, let $\mathcal{C} := \{U \subseteq V | r \notin U \neq \emptyset\}$, and let $D = (V, A)$ and c be arbitrary. Then (i) in Theorem 10 is equivalent to a theorem of Fulkerson [**11**]: the minimum weight of an r-branching (= a subset of A forming a rooted directed tree with root r) is equal to the maximum number t of r-cuts (= sets of form $\delta_A^-(U)$ with $U \in \mathcal{C}$) (repetition allowed) such that any arc a of D is in at most c_a of these r-cuts. If $c \equiv \mathbf{1}$, assertion (ii) is equivalent to a theorem of Edmonds [**8**]: the minimum size of any r-cut is equal to the maximum number of pairwise disjoint r-branchings.

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