AN ACCURATE METHOD WITHOUT DIRECTIONAL BIAS FOR THE NUMERICAL SOLUTION OF A 2-D ELLIPTIC SINGULAR PERTURBATION PROBLEM

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1. INTRODUCTION

We study problems related with the numerical solution of the singular perturbation equation

(1.1) $$L_\varepsilon u \equiv -\varepsilon\, \Delta u + \vec{a}.\nabla u = f,$$

in a two-dimensional region Ω. This equation can be considered as a model equation for more complex real-live problems such as flows described by the Navier-Stokes equation. We refer to equation (1.1) as the convection-diffusion equation; $\vec{a}$ is the convection vector and $\varepsilon > 0$ is the diffusion parameter, which may be small compared to $|\vec{a}|$.

Allthough we study equation (1.1) with constant coefficients, we want to find numerical methods that are applicable for variable $\vec{a}$; i.e. $\vec{a} = \vec{a}(x,y)$ or $\vec{a} = \vec{a}(x,y,u)$. In particular, we are interested in methods that are independent of the direction of $\vec{a}$ and independent of whether the grid is properly refined in possible boundary layers, when ε is small. Therefore, we study methods that do not make use of a-priori knowledge about the solution, the convection direction or proper mesh refinements.

As a simplification of the 2-D equation we also study the 1-D case. For this 1-D problem,

(1.2) $$L_\varepsilon u \equiv +\varepsilon u_{xx} + 2u_x = f,$$

many numerical methods have already been investigated (see e.g. contributions in Hemker-Miller [1979] or Axelsson-Frank-Van Der Sluis [1981]). However, almost none of these methods are suitable for generalization in more dimensions.

An essential difficulty in the numerical solution of (1.1) with $0 < \varepsilon < h$, h the mesh-width, is the different type of approximation that is required in the smooth part of the solution and in the boundary or interior layers. In the smooth part an accurate approximation - possibly of high order - is desirable, whereas for the boundary layer the proper location is of prime importance, with the additional requirement that the effect of the (almost) discontinuity does not disturb the solution in the smooth parts. Large derivatives of the solution inside the layers make that in these layers high order approximation is of no use. Therefore, we are interested in methods that are of low order for unsmooth and of higher order for smooth components in the solution. When these methods are applied, local error estimates may be used for generating adaptive mesh-refinements afterwards.

The problems induced by the small parameter

For large values of ε the numerical solution of (1.1) gives no particular problems. Discretizations

(1.3) $$L_{h,\varepsilon}\, u_{h,\varepsilon} = f_h$$

are known for which $\|u_{h,\varepsilon} - u_\varepsilon\| = O(h^2)$ as $h \to 0$, e.g. the usual central difference or finite element discretizations. The errorbound remains valid for small values of ε:

$$\|u_{h,\varepsilon} - u_\varepsilon\| \le C_\varepsilon h^2 \quad \text{as } h \le h_\varepsilon,$$

but $C_\varepsilon \to \infty$ and $h_\varepsilon \to 0$ as $\varepsilon \to 0$. This means that the error estimate is of no use if we apply these discretizations with finite h and $\varepsilon \to 0$. In fact, for small ε, the usual discretizations may yield quite useless approximations.

EXAMPLE

Discretizing the 1-D model problem

(1.4) $$\varepsilon u_{xx} + 2u_x = 0, \quad x \in [0,\infty), \qquad u(0) = 1\ ,\ u(\infty) = 0,$$

by central differencing:

(1.5) $$\varepsilon\, \Delta_+\Delta_- u_h + (\Delta_+ + \Delta_-)u_h = 0,$$

we find

$$u_{h,\varepsilon}(jh) = \left(\frac{\varepsilon-h}{\varepsilon+h}\right)^j.$$

This is a second order approximation indeed:
for jh fixed and $\left(\frac{h}{\varepsilon}\right) \to 0$

$$|u_{h,\varepsilon}(jh) - u_\varepsilon(jh)| = \left|\left(\frac{\varepsilon-h}{\varepsilon+h}\right)^j - (e^{-2h/\varepsilon})^j\right| \le C\left(\frac{h}{\varepsilon}\right)^2,$$

C independent of j, h and ε.
However, the solution of the reduced difference equation is

(1.6) $$u_{h,0}(jh) = \lim_{\varepsilon\to 0} u_{h,\varepsilon}(jh) = (-1)^j.$$

The influence of the boundary condition at x = 0 is significant over the whole domain of definition, whereas for the reduced differential equation the influence of this boundary condition vanishes in the interior of the domain.

A well-known cure against this spurious influence of the boundary condition is "upwinding" or "artificial diffusion" . In upwinding one-sided differences are used for the discretization of the first order term. In artificial diffusion, the diffusion constant ε is replaced by a larger value $\alpha = \varepsilon + O(h)$. In both cases the spurious influence disappears at the expense of the fact that these discretizations are only accurate of order $O(h)$. In the 1-D case "upwinding" is equivalent with "artificial diffusion" with $\alpha = \varepsilon + h|a|/2$.

EXAMPLE

The solution of the upwind discretization of (1.4)

(1.7) $$\varepsilon\, \Delta_+\Delta_-\, u_h + 2\Delta_+\, u_h = 0$$

is

$$u_{h,\varepsilon}(jh) = \left(\frac{\varepsilon}{\varepsilon+2h}\right)^j .$$

In contrast with the central difference solution, we see that here the influence of the boundary condition vanishes in the interior of the domain as $\varepsilon \to 0$; but this discretization is only first order:
for jh fixed and $(\frac{h}{\varepsilon}) \to 0$ we find

$$|u_{h,\varepsilon}(jh) - u_\varepsilon(jh)| \le C(\frac{h}{\varepsilon}).$$

2. LOCAL MODE ANALYSIS

We want to analyze separately the behaviour of the discretizations (i) in the smooth parts of the solution, and (ii) in the boundary layers. For this we use the local mode analysis (LMA), cf. Brandt [1980] and Brandt and Dinar [1979].We consider equation (1.1) in two particular model problems:
(i) the inhomogeneous problem

(2.1) $$L_{h,\varepsilon}\, u_h = f_h$$

on a regular rectangular discretization of $\mathbb{R}^2$; u_h and f_h are bounded at infinity, and
(ii) the homogeneous problem

(2.2) $$L_{h,\varepsilon}\, u_h = 0$$

in a discretization of the half-space, of which the boundary is a grid-line; boundary conditions are given on this grid-line and u_h is bounded at infinity.

In both cases we consider the discretization of the constant coefficient problem on a regular rectangular grid and we decompose the solution in its Fourier modes (see e.g. Hemker [1980])

(2.3) $$u_h(jh) = \left(\frac{1}{\sqrt{2\pi}}\right)^2 \int \hat{u}_h(\omega)\, e^{+i\omega hj}\, d\omega, \quad j \in \mathbb{Z}^2 ,$$

where $u_{h,\omega} = \hat{u}_h(\omega)\, e^{i\omega hj}$ is *the mode of frequency* ω in u_h; the amplitude of this mode with

$$\omega \in T_h^2 = \{\omega \mid \omega \in \mathbb{C}^2,\ \mathrm{Re}\ \omega_k \in [-\pi/h,\pi/h),\ k = 1,2\}$$

is given by

(2.4) $$\hat{u}_h(\omega) = \left(\frac{h}{\sqrt{2\pi}}\right)^2 \sum_j e^{-i\omega hj}\, u_h(jh).$$

If we consider the problem (2.1), the boundary condition imposes $\omega \in \mathbb{R}^2$; for (2.2) with Ω being the right half-space, with boundary conditions at $x = 0$, we have

Im $\omega_1 > 0$, $I_m\ \omega_2 = 0$. In this paper we restrict the analysis to the model problem (2.1).

The modes being the eigenfunction of the discrete operator L_h, we can define the *characteristic form* $\hat{L}_h(\omega)$ corresponding with the discrete operator L_h, by

$$(2.5) \qquad \widehat{L_h\ u_{h,\omega}} = \widehat{L_h}(\omega)\ \hat{u}_{h,\omega}.$$

This characteristic form $\widehat{L_h}(\omega)$ is the analogue of the *characteristic polynomial* or the *symbol* $\widehat{L}(\omega)$ of the continuous operators L.
We now define consistency and stability of the operator L_h for each mode ω separately.

DEFINITION. The operator L_h is *consistent* with L *of order* p *for mode* $\omega \in T_h^2$ if

$$(2.6) \qquad |\widehat{L_h}(\omega) - \hat{L}(\omega)| \le C\ h^p \text{ for } h \to 0.$$

DEFINITION. The *stability* of L_h *for mode* $\omega \in T_h^2$ is the quantity $|\hat{L}_h(\omega)|$.

DEFINITION. The *numerical (interior) stability* of L_h, a discretization of L, for $\omega \in T_h^2 \cap \mathbb{R}^2$ is

$$(2.7) \qquad |\hat{L}_h(\omega)|\ /\ |\hat{L}(\omega)|.$$

DEFINITION. The operator L_h is *numerical (interior) stable* if

$$(2.8) \qquad \forall\rho > 0\ \ \exists\eta > 0\ \ \forall\omega \in T_h^2 \cap \mathbb{R}^2 \quad |\hat{L}(\omega)| > \rho \to |\hat{L}_h(\omega)|\ /\ |\hat{L}(\omega)| > \eta,$$

where $\eta = \eta(\rho)$ is independent of h.

DEFINITION. The operator $L_{h,\varepsilon}$, a discretization of L_ε is *asymptotically stable* if

$$\forall\rho > 0\ \ \exists\eta > 0\ \ \forall\omega \in T_h^2 \cap \mathbb{R}^2\ \ \lim_{\varepsilon\to 0} |\hat{L}_\varepsilon(\omega)| > \rho \to \lim_{\varepsilon\to 0} \frac{|\hat{L}_{h,\varepsilon}(\omega)|}{|\hat{L}_\varepsilon(\omega)|} > \eta.$$

DEFINITION. The operator $L_{h,\varepsilon}$ is *ε-uniformly stable* if (2.8) holds with $\eta = \eta(\rho)$ independent of h and ε.

EXAMPLE

To study the local behaviour of the discretization (1.5) of our 1-D model problem we find its characteristic form

$$(2.9) \qquad \hat{L}_{h,\varepsilon}(\omega) = \frac{\sin(\omega h/2)}{h/2}\left(-\varepsilon\,\frac{\sin(\omega h/2)}{h/2} + 2i\cos(\omega h/2)\right).$$

Comparing this with the symbol $\hat{L}_\varepsilon(\omega) = -\varepsilon\omega^2 + 2i\omega$ of L we find

(1) the discretization (1.5) is consistent of order 2:

$$|\hat{L}_{h,\varepsilon}(\omega) - \hat{L}_\varepsilon(\omega)| \le C\ h^2\ |\varepsilon\omega^4 + i\omega^3| + O(h^3);$$

(2) the discretization (1.5) is asymptotically unstable:

$$\lim_{\varepsilon\to 0} \hat{L}_{h,\varepsilon}\ (\pi/h) = 0, \quad \text{whereas}$$
$$\lim_{\varepsilon\to 0} \hat{L}_\varepsilon\ (\pi/h) = 2\pi i/h.$$

We find that $u_{h,\pi/h}$ is an unstable mode. This mode corresponds to

$$u_h(jh) = e^{i\pi j} = (-1)^j,$$

cf. eq. (1.6).

If we consider the discretization with artificial diffusion α, we find its characteristic form (2.9) with ε replaced by $\alpha > 0$. This discretization is

(1) consistent of order 1 if $|\alpha - \varepsilon| \le C_1 h$; viz.

$$(2.10)\qquad |\hat{L}_{h,\alpha}(\omega) - \hat{L}_\varepsilon(\omega)| \le C_1 |\alpha-\varepsilon|\ |\omega|^2 + |\hat{L}_{h,\varepsilon}(\omega) - \hat{L}_\varepsilon(\omega)|$$

$$\le \mathcal{O}(|\alpha-\varepsilon|) + \mathcal{O}(h^2) = \mathcal{O}(h),$$

(2) numerical (interior) stable, uniform in ε, if $|\alpha-\varepsilon| \ge C_2 h$; viz.

$$\left|\frac{\hat{L}_{h,\alpha}(\omega)}{\hat{L}_\varepsilon(\omega)}\right| = \left|\frac{\sin(\omega h/2)}{\omega h/2}\right| \left|\frac{\frac{2\alpha}{h}\sin(\omega h/2) - 2i\cos(\omega h/2)}{\varepsilon\omega - 2i}\right|$$

$$\ge \frac{2}{\pi}\cdot\frac{2}{\pi}\left|\frac{\frac{\alpha}{h}\sin(\omega h/2) - i\cos(\omega h/2)}{\frac{\varepsilon}{h}\sin(\omega h/2) - i}\right|$$

$$\ge \frac{4}{\pi^2}\cdot\frac{1}{\sqrt{2}}\ \min(\alpha/h, 1) \ge \frac{2\sqrt{2}}{\pi^2}\min(C_2, 1).$$

There are no spurious unstable modes.

3. THE DEFECT CORRECTION PRINCIPLE

For the solution of linear problems, the defect correction principle is a general technique to approximately solve a problem

$$(3.1)\qquad Lu = f$$

by means of an iteration process

$$(3.2)\qquad \tilde{L}\,u^{(i+1)} = \tilde{L}\,u^{(i)} - L\,u^{(i)} + f, \quad i = 1,2,\ldots .$$

The operator $\tilde{L}$, an approximation to L, is selected such that problems

$$\tilde{L}\,u^{(i+1)} = \tilde{f},$$

with $\tilde{f}$ in a neighbourhood of f, are easy to solve. If $\tilde{L}$ is injective and the iteration process (3.2) converges to a fixed point $\tilde{u}$, then $\tilde{u}$ is clearly a solution of (3.1). The convergence of the iterands to the solution of (3.1) is described by the error amplification operator

$$I - \tilde{L}^{-1}L;$$

the reduction of the residual $r^{(i)} = f - L\,u^{(i)}$ in each step is described by the residual amplification operator

$$I - L\tilde{L}^{-1}.$$

If two equations $\tilde{L}_h u_h = f_h$ and $L_h u_h = f_h$ are both discretizations of a problem $Lu = f$ (respectively consistent of order p and q, $p \le q$) and if $\tilde{L}_h$ satisfies the stability condition

(3.3) $\quad \|\tilde{L}_h^{-1}\| < C, \quad$ uniform in h,

then it is well known (cf. e.g. Hackbusch [1979], Hemker [1981a]) that in the iterative process

$$\begin{cases} \tilde{L}_h u_h^{(1)} = f_h, & (3.4.a) \\ \tilde{L}_h u_h^{(i+1)} = \tilde{L}_h u_h^{(i)} - L_h u_h^{(i)} + f_h, & (3.4.b) \end{cases}$$

$u_h^{(i)}$ satisfies

$$\|u_h^{(i)} - u\| = \mathcal{O}(h^{\min(q,ip)}).$$

This error bound holds without a stability condition (3.3) for the accurate operator L_h.

Direct application of the defect correction principle to the solution of our singular perturbation problem suggest the application of (3.4) with $L_h = L_{h,\varepsilon}$ with the 2nd order central difference (or FEM) discretization and with $\tilde{L}_h = L_{h,\alpha}$, the artificial diffusion discretization. Then, the correction equation (3.4.b) has the simple form

(3.5) $\quad L_{h,\alpha} u_h^{(i+1)} = f_h + (\alpha - \varepsilon)\Delta_h u_h^{(i)}.$

Since $L_{h,\alpha}$ is stable and consistent of order 1 and $L_{h,\varepsilon}$ is consistent of order 2, we obtain

(3.6) $\quad \|u_h^{(1)} - u\| = \mathcal{O}(h) \quad$ and

$$\|u_h^{(i)} - u\| = \mathcal{O}(h^2) \quad \text{for } i > 1.$$

Where $\Delta_h u_h^{(i)}$ is a good approximation to Δu, (i.e. outside the boundary layer) $u_h^{(i+1)}$ is a better approximation to u than $u_h^{(1)}$. The error bounds (3.6), however, hold in the classical sense: for fixed ε and $h \to 0$. For a general $i > 1$, the solution $u_h^{(i)}$ is not better than the central difference approximation, but in the first few iterands the instability of $L_{h,\varepsilon}$ has only a limited influence.

EXAMPLE

For (1.4) we can compute the solutions in the defect correction process explicitly. Application of (3.4) with the operators L_h and $\tilde{L}_h$ as given in (3.5) yields the solutions

$$u_h^{(1)}(jh) = \left(\frac{\varepsilon}{\varepsilon+2h}\right)^j,$$

$$u_h^{(2)}(jh) = \left(\frac{\varepsilon}{\varepsilon+2h}\right)^j \left[1 - \frac{jh}{2} \cdot \frac{2h}{(\varepsilon+2h)}\right],$$

$$\vdots$$

$$u_h^{(m+1)}(jh) = \left(\frac{\varepsilon}{\varepsilon+2h}\right)^j P_m(j,h/\varepsilon),$$

where $P_m(j,h/\varepsilon)$ is an m-th degree polynomial in j depending on the parameter h/ε. It is easily verified that, for ε fixed and $h \to 0$, the solutions are 2nd order accurate for $m = 1,2,\ldots$. For small values of ε/h, $P_m(j,h/\varepsilon)$ changes sign m times for $j = 0,1,2,\ldots, m+1$; i.e. in each iteration step of (3.4) one more oscillation appears in the numerical solution. The influence of the boundary condition at $x = 0$ vanishes in the interior after the first $m+1$ nodal points. By each step of (3.4) the effect of the instability of $L_{h,\varepsilon}$ creeps over one meshpoint further into the numerical solution. Similar effects are found for the process in two dimensions

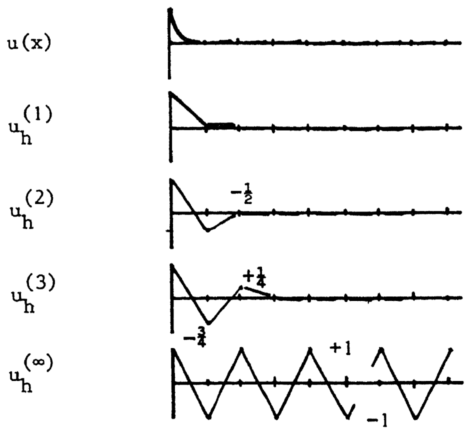

Figure 1

The numerical solutions $u_h^{(i)}$ of equation (1.4) for small values of ε/h

4. A MIXED DEFECT CORRECTION PROCESS

In this section we develop an iterative method of which the stationary solution is asymptotically stable and 2nd order accurate in the smooth parts of the solution. We consider the "mixed defect correction process" (MDCP):

$$\begin{cases} \tilde{L}_h^1 u_h^{(i+\frac{1}{2})} = \tilde{L}_h^1 u_h^{(i)} - L_h^1 u_h^{(i)} + f_h, & (4.1.a) \\ \tilde{L}_h^2 u_h^{(i+1)} = \tilde{L}_h^2 u_h^{(i+\frac{1}{2})} - L_h^2 u_h^{(i+\frac{1}{2})} + f_h. & (4.1.b) \end{cases}$$

For this process the following theorem holds.

THEOREM. *Let both* $\tilde{L}_h^1$ *and* $\tilde{L}_h^2$ *satisfy the stability condition* (3.3) *and let* $L_h^k u_h = f_h$ *and* $\tilde{L}_h^k u_h = f_h$ *be discretizations of order* p_k *and* $q_k \le p_k$ *respectively,* $k = 1,2$. *If for* (4.1) *a stationary solution*

$$u_h^A = \lim_{i\to\infty} u_h^{(i)}$$

exists, then

$$\| u - u_h^A \| \le C\, h^{\min(p_1+q_2,p_2)}. \qquad (4.2)$$

PROOF. See Hemker [1981b] p. 79-81.

For the singular perturbation problem (1.1) we take

(4.3)
a) $L_h^1 = L_{h,\varepsilon}$ the central difference (or FEM) discrete operator,
b) $L_h^2 = \tilde{L}_h^1 = L_{h,\alpha}$ the artificial diffusion discrete operator, and
c) $\tilde{L}_h^2 = 2.\ \mathrm{diag}(L_{h,\alpha})$.

Thus, a pair of iteration steps consists of
1) a defect correction step as in section 3, and
2) a damped Jacobi relaxation step for the solution of the stable discretized system.

If the iteration (4.1) converges, it has not a single fixed point, but it has two stationary solutions $u_h^A = \lim_{i\to\infty} u_h^{(i)}$ and $u_h^B = \lim_{i\to\infty} u_h^{(i+\frac{1}{2})}$. For our choice of operators, the above theorem yields, for a fixed ε, the error bounds

$$\|u_\varepsilon - u_{h,\varepsilon}^A\| \le C_\varepsilon\, h \quad \text{and} \qquad (4.4)$$
$$\|u_\varepsilon - u_{h,\varepsilon}^B\| \le C_\varepsilon\, h^2 ,$$

where u_ε is the exact solution. The defect correction step (4.1.a) generates a 2nd order accurate solution and may introduce high-frequency unstable components. The damped Jacobi relaxation step (4.1.b) is able to reduce the high-frequency errors. Hence we expect that the combined process is not only accurate but also stable. First we demonstrate this for our 1-D problem. In the next section we give the analysis for the 2-D problem.

The stationary solutions u_h^A and u_h^B in (4.1) - (4.3) can be characterized as solutions of linear systems

$$[L_h^1 + L_h^2(\tilde{L}_h^2)^{-1}(L_h^2 - L_h^1)]\, u_h^A = \qquad f_h, \qquad (4.5)$$

and

$$[L_h + (L_h^2 - L_h^1)(\tilde{L}_h^2)^{-1}L_h^2]\, u_h^B = [I + (L_h^2 - L_h^1)(\tilde{L}_h^2)^{-1}]\, f_h, \qquad (4.6)$$

with L_h^1, L_h^2 and $\tilde{L}_h^2$ as in (4.3).
For a brief notation we denote eq. (4.5) as

$$M_{h,\varepsilon}\, u_h^A = f_h.$$

<u>Local mode analysis of the MDCP applied to the 1-D model problem</u>

The characteristic forms of the different discretizations of the 1-D model problem

$$L_\varepsilon u \equiv \varepsilon u'' + 2u' = f \qquad (4.7)$$

are, for central differencing ($L_{h,\varepsilon}$), upwinding ($L_{h,\alpha}$ with $\alpha = \varepsilon + h$) and the MDCP discretization $M_{h,\varepsilon}$ espectively

$$\hat{L}_{h,\varepsilon}(\omega) = -\frac{4\varepsilon}{h^2}\, S^2 \qquad + \frac{4i}{h}\, SC, \qquad (4,8)$$

(4.9) $$\hat{L}_{h,\alpha}(\omega) = -\frac{4\varepsilon}{h^2} S^2[1+\frac{h}{\varepsilon}] + \frac{4i}{h} SC,$$

(4.10) $$\hat{M}_{h,\varepsilon}(\omega) = -\frac{4\varepsilon}{h^2} S^2[1+\frac{h}{\varepsilon} S^2] + \frac{4i}{h} SC\,[1+\frac{h}{\varepsilon+h} S^2],$$

where $S = \sin(\omega h/2)$ and $C = \cos(\omega h/2)$.

THEOREM. *The operator* $M_{h,\varepsilon}$ *defined by the* MDCP *process* (4.1)-(4.3) *applied to the model equation* (4.7) *is consistent of 2nd order and* ε*-uniformly stable.*

PROOF. Comparing $\hat{M}_{h,\varepsilon}$ with $\hat{L}_\varepsilon(\omega)$ we find for all $\omega \in T_h^2 \cap \mathbb{R}^2$

$$|\hat{M}_{h,\varepsilon}(\omega) - \hat{L}_\varepsilon(\omega)| \le |\hat{M}_{h,\varepsilon}(\omega) - \hat{L}_{h,\varepsilon}(\omega)| + |\hat{L}_{h,\varepsilon}(\omega) - \hat{L}_\varepsilon(\omega)|$$

$$= \mathcal{O}(h^2) \qquad \text{for } h \to 0$$

i.e. $M_{h,\varepsilon}$ is consistent of the 2nd order.
For the stability we find

$$\left|\frac{\hat{M}_{h,\varepsilon}(\omega)}{\hat{L}_\varepsilon(\omega)}\right| = \frac{4|S|}{|h\omega|} \cdot \frac{|(\frac{\varepsilon}{h}S + S^3) + iC(1 + \frac{h}{\varepsilon+h} S^2)|}{|\varepsilon\omega + 2i|} .$$

For $0 < h \le \varepsilon$ we find for all $\omega \in T_h^2 \cap \mathbb{R}^2$

$$\frac{|\hat{M}|}{|\hat{L}|} \ge \frac{4}{\pi} \cdot \frac{|\frac{\varepsilon}{h}S + iC|}{|\varepsilon S \frac{\pi}{h} + 2i|} \ge \frac{4}{\pi^2} \; \frac{1}{\sqrt{2}} .$$

For $0 \le \varepsilon \le h$

$$\frac{|\hat{M}|}{|\hat{L}|} \ge \frac{4}{\pi} \; \frac{|S^3 + iC(1 + \frac{1}{2}S^2)|}{|\varepsilon S \frac{\pi}{h} + 2i|} \ge \frac{2}{\pi\sqrt{\pi^2 + 4}} .$$

Thus we find, uniform in ε and h,

$$\inf_{\omega \in T_h^2 \cap \mathbb{R}^2} \frac{|\hat{M}_{h,\varepsilon}(\omega)|}{|\hat{L}_\varepsilon(\omega)|} \ge \frac{\sqrt{2}}{\pi^2} .$$

This inequality implies ε-uniform stability. □

5. LOCAL MODE ANALYSIS APPLIED TO THE 2-D MODEL PROBLEM

An analysis, analogous to the 1-D case, can be made for the 2-D model equation

(5.1) $$L_\varepsilon u \equiv \varepsilon \Delta u + (4 + 2p)\, \vec{a} \nabla u = f.$$

The corresponding difference operator is given by

(5.2) $$L_{h,\varepsilon} \equiv \frac{\varepsilon}{h^2} \begin{bmatrix} & 1 & \\ 1 & -4 & 1 \\ & 1 & \end{bmatrix} + \frac{a_1}{h} \begin{bmatrix} & -p & +p \\ -2 & 0 & 2 \\ -p & +p & \end{bmatrix} + \frac{a_2}{h} \begin{bmatrix} & 2 & p \\ p & 0 & -p \\ -p & -2 & \end{bmatrix} .$$

With $p = 0$ it corresponds to the central difference discretization; with $p = 1$ it describes the FEM discretization on a regular triangulation with piecewise linear trial- and test-functions. Also for the 2-D equation we define the MDCP by (4.1) - (4.3). The 2nd order consistency of the corresponding $M_{h,\varepsilon}$ and its asymptotic stability are proved similarly to the 1-D case.

THEOREM. *The operator* $M_{h,\varepsilon}$, *defined by the process* (4.1) - (4.3), *applied to the model equation* (1.1) *with central difference or finite element discretization for* $L_{h,\varepsilon}$ *and with artificial diffusion,* $\alpha = \varepsilon + C_1 h$, *is consistent of 2nd order and asymptotically stable.*

PROOF. Similar to the 1-D case we find

$$\hat{L}_{h,\varepsilon}(\omega) = -\frac{4\varepsilon}{h^2} S^2 + \frac{4i}{h} T \qquad \text{and}$$

$$\hat{M}_{h,\varepsilon}(\omega) = -\frac{4\varepsilon}{h^2} S^2[1 + \frac{\alpha-\varepsilon}{2\varepsilon} S^2] + \frac{4i}{h} T [1 + \frac{\alpha-\varepsilon}{2\alpha} S^2],$$

where

$$T = a_1 S_\phi(2C_\phi + pC_{\phi+2\theta}) + a_2 S_\theta(2C_\theta + pC_{\theta+2\phi}),$$

$$S = S_\phi^2 + S_\theta^2, \quad S_\phi = \sin(\phi), \quad C_\phi = \cos(\phi)$$

$$\phi = \omega_1 h/2, \quad \theta = \omega_2 h/2.$$

Further $\hat{L}_\varepsilon(\omega) = \frac{-4\varepsilon}{h^2}(\phi^2 + \theta^2) + \frac{2i}{h}(2+p)(a_1\phi + a_2\theta)$. Now it is easy to show that

$$|\hat{M}_{h,\varepsilon}(\omega) - \hat{L}_\varepsilon(\omega)| \le |\hat{M}_{h,\varepsilon}(\omega) - \hat{L}_{h,\varepsilon}(\omega)| + |\hat{L}_{h\varepsilon}(\omega) - \hat{L}_\varepsilon(\omega)| = O(h^2)$$

which proves the consistency.
To prove the asymptotic stability we find

$$\lim_{\varepsilon\to 0} \frac{|\hat{M}_{h,\varepsilon}(\omega)|}{|\hat{L}_\varepsilon(\omega)|} = \left| \frac{i\frac{\alpha}{h} S^4 + [2 + S^2]T}{2(2+p)(a_1\phi + a_2\theta)} \right|.$$

Because of the term $i\frac{\alpha}{h} S^4 \approx i\, C_1 S^4$, $M_{h,0}$ has no unstable modes.
We choose a fixed $\rho > 0$ and consider (ϕ,θ) such that $|\hat{L}_\varepsilon(\omega)| \ge \rho$.
We can write $T = T(\phi,\theta) = (2+p)(\alpha_1\phi + a_2\theta) - R(\phi,\theta)$ with

$$|R(\phi,\theta)| \le C_2\, h^3|\omega|^3, \quad C_2 = C(a_1,a_2,p).$$

Now

$$\lim_{\varepsilon\to 0} \frac{|\hat{M}_{h,\varepsilon}(\omega)|}{|\hat{L}_\varepsilon(\omega)|} = \left| \frac{iC_1 S^4 + (2+S^2)T(\phi,\theta)}{2T(\phi,\theta) + 2R(\phi,\theta)} \right|.$$

For an arbitrary $C_3 > 0$ we consider subregions of $T_h^2 \cap \mathbb{R}^2$:

$$A = \{(\phi,\theta) \mid T(\phi,\theta) \ge C_3\, h^3|\omega|^3 \text{ and } |a_1\phi + a_2\theta| \ge \rho\};$$

$$B = \{(\phi,\theta) \mid T(\phi,\theta) \le C_3\, h^3|\omega|^3 \text{ and } |a_1\phi + a_2\theta| \ge \rho\}.$$

Because $(2+p)(a_1\phi + a_2\theta) = R(\phi,\theta) + T(\phi,\theta)$, we know for all $(\phi,\theta) \in B$ that $h|\omega| \geq C\rho^{1/3}$
For $(\phi,\theta) \in A$ we have

$$\lim_{\varepsilon\to 0} \frac{|\hat{M}_{h,\varepsilon}(\omega)|}{|\hat{L}_\varepsilon(\omega)|} \geq \frac{|(2+S^2)T(\phi,\theta)|}{|2\,T(\phi,\theta)| + |2C_2\,T(\phi,\theta)/C_3|} \geq \frac{C_3}{C_2 + C_3}$$

and for $(\phi,\theta) \in B$

$$\lim_{\varepsilon\to 0} \frac{|\hat{M}_{h,\varepsilon}(\omega)|}{|\hat{L}_\varepsilon(\omega)|} \geq \left| \frac{C_1 S^4}{2(C_2 + C_3)h^3|\omega|^3} \right| = C\,h|\omega| > C\rho^{1/3}.$$

Thus, for a given $\rho > 0$, and for all $\omega \in T_h^2 \cap \mathbb{R}^2$ such that $\lim_{\varepsilon\to 0} L_\varepsilon(\omega) \geq \rho$, we have

$$\lim_{\varepsilon\to 0} \frac{|\hat{M}_{h,\varepsilon}(\omega)|}{|\hat{L}_\varepsilon(\omega)|} \geq \eta(\rho)$$

i.e. $M_{h,\varepsilon}$ is asymptotically stable. □

REMARK.

The MDCP method as described above can conveniently be imbedded into an iterative process for the solution of the discrete system. Only the first step in (4.1) requires the solution of a linear system, the 2nd step is the application of a single relaxation sweep. If an iterative method for the solution of (4.1.a) is used, a sufficient number of iteration steps for its solution should be interchanged with a single step (4.1.b). If an efficient iterative method is used, such as a multiple grid method, possibly a few iteration steps for (4.1.a) are sufficient to obtain the derived effects. It is likely that also only a few iteration steps of the MDCP process are sufficient to obtain approximations to u_h^A and u_h^B that have essentially the properties of u_h^A and u_h^B. Here further research is required.

REMARK.

The MDCP-method makes use of the fact that the solution of equation $L_\alpha u = f$, with $\alpha = \varepsilon + \mathcal{O}(h)$, is an approximate solution of the equation $L_\varepsilon u = f$.
The method does not make use of any particular knowledge about the convection direction or about the location or the shape of boundary or interior layers.

6. NUMERICAL EXAMPLES

For a number of problems (1.1) we have computed the numerical solution. In all problems we took for $L_{h,\varepsilon}$ the finite element discretization on a regular triangulation and for $L_{h,\alpha}$ the artificial diffusion discretization with $\alpha = \varepsilon + h/2$.
By 3 different methods the solution was computed:

1) by the method of artificial diffusion (AD), i.e. $u_h^{(1)}$, the solution of $L_{h,\alpha}\, u_h^{(1)} = f_h$.

2) by a single defect correction step (DCP), i.e. $u_h^{(2)}$ in eq. (3.5)
3) by the iterative process (4.1) - (4.3). The stationary solution after the 2nd order correction step (u_h^B) is denoted by (J;DCP) and the solution after Jacobi-relaxation (u_h^A) by (DCP;J).

For four typical problems we compare the results of the computations. The 4 problems are:

1. *A problem with a smooth solution*

$$(6.1) \qquad \varepsilon \Delta u + u_x = f(x,y) \qquad \text{on } [0,1]^2,$$

with Dirichlet boundary conditions. The boundary conditions and f(x,y) are chosen such that

$$(6.2) \qquad u(x,y) = \sin(\pi x)\sin(\pi y) + \cos(\pi x)\cos(3\pi y)$$

is the solution.

2. *A problem with an exponential boundary layer*

The same problem (6.1), with the Dirichlet boundary conditions and f(x,y) such that

$$(6.3) \qquad u(x,y) = \sin(\pi x)\sin(\pi y) + \cos(\pi x)\cos(3\pi y) + (\exp(-x/\varepsilon) - \exp(-1/\varepsilon))/(1-\exp(-1/\varepsilon))$$

is the solution.

3. *A problem with a parabolic boundary layer*

$$(6.4) \qquad \varepsilon \Delta u - u_x = f(x,y) \qquad \text{on } [0,1]^2,$$

with Dirichlet boundary conditions and f(x,y) chosen such that

$$(6.5) \qquad u(x,y) = \sin(\pi x)\sin(\pi y) + \cos(\pi x)\cos(3\pi y) + - \sqrt{\frac{-x_0}{x-x_0}}\; e^{\frac{-(y-y_0)^2}{4\varepsilon(x-x_0)}},$$

with $x_0 = -1$ and $y_0 = 0$, is the solution.

4. *A problem with a parabolic interior layer*

The problem (6.4) with the boundary conditions and f(x,y) chosen such that (6.5) is a solution with $x_0 = -0.1$ and $y_0 = 0.5$.

In the tables 6.1 - 6.4 we show for $\varepsilon = 10^{-6}$ the maximal error at the meshpoints in the whole unit square and (in italics) on a properly selected subregion, away from the boundaries, where the solution of the problem is smooth. We give the error on a regular square mesh with h = 1/8, 1/16, 1/32. Further we give the ratio of the error when the mesh-size is halved.

	h = 1/8 error	ratio	h = 1/16 error	ratio	h = 1/32 error
AD	0.973	1.52	0.640	1.60	0.399
	0.790	*1.37*	*0.578*	*1.50*	*0.380*
DCP	0.635	1.74	0.365	1.97	0.185
	0.635	*1.76*	*0.360*	*2.08*	*0.173*
(J;DCP)	0.507	2.39	0.212	3.64	0.0583
	0.507	*3.40*	*0.149*	*4.45*	*0.0335*
(DCP;J)	0.429	3.09	0.139	3.22	0.0432
	0.429	*3.35*	*0.128*	*4.40*	*0.0291*

TABLE 6.1. Problem 1: smooth solution, $\varepsilon = 10^{-6}$.

	h = 1/8 error	ratio	h = 1/16 error	ratio	h = 1/32 error
AD	0.973	1.52	0.640	1.60	0.399
	0.790	*1.37*	*0.578*	*1.52*	*0.380*
DCP	1.08	1.28	0.845	1.28	0.662
	0.635	*1.76*	*0.360*	*2.08*	*0.173*
(J;DCP)	1.11	1.18	0.944	1.19	0.792
	0.608	*3.82*	*0.159*	*4.75*	*0.0335*
(DCP;J)	0.727	1.21	0.603	1.19	0.506
	0.459	*3.48*	*0.132*	*4.54*	*0.0291*

TABLE 6.2. Problem 2: exponential boundary layer, $\varepsilon = 10^{-6}$

	h = 1/8 error	ratio	h = 1/16 error	ratio	h = 1/32 error
AD	1.21	1.56	0.777	1.00	0.776
	0.799	*1.38*	*0.578*	*1.52*	*0.380*
DCP	0.813	1.19	0.684	0.99	0.694
	0.660	*1.61*	*0.409*	*2.09*	*0.196*
(J;DCP)	0.552	1.08	0.511	0.91	0.560
	0.552	*3.76*	*0.147*	*4.50*	*0.0327*
(DCP;J)	0.441	0.92	0.478	0.98	0.489
	0.441	*3.45*	*0.128*	*4.40*	*0.0291*

TABLE 6.3. Problem 3: parabolic boundary layer, $\varepsilon = 10^{-6}$.

	h = 1/8 error	ratio	h = 1/16 error	ratio	h = 1/32 error
AD	1.11	1.52	0.730	1.61	0.453
	0.573	*2.08*	*0.275*	*1.44*	*0.191*
DCP	0.835	1.74	0.481	1.32	0.364
	0.399	*1.86*	*0.214*	*1.95*	*0.110*
(J;DCP)	0.735	1.71	0.427	1.43	0.298
	0.286	*1.95*	*0.147*	*5.53*	*0.0266*
(DCP;J)	0.677	2.00	0.339	1.13	0.300
	0.247	*2.01*	*0.123*	*5.67*	*0.0217*

TABLE 6.4. Problem 4: parabolic interior layer, $\varepsilon = 10^{-6}$..

We notice that for $\varepsilon = 10^{-6}$ and for the given mesh-sizes, the (J;DCP) and the (DCP;J) solutions show 2nd order convergence in the smooth parts of the solutions. Thus, they show the local interior behaviour as it was predicted by the local mode analysis. The DCP solution only shows 1st order convergence for these h/ε ratios, whereas the AD solutions even show less convergence.

	h = 1/8 error	ratio	h = 1/16 error	ratio	h = 1/32 error
AD	0.630	2.47	0.0255	1.71	0.0149
DCP	0.0740	3.65	0.0203	4.02	0.00505
(J; DCP)	0.0780	3.65	0.0214	4.01	0.00533
(DCP;J)	0.0693	3.46	0.0201	3.89	0.00516

TABLE 6.5. Problem 2: $\varepsilon = 1.0$.

In table 6.5 we show the results of problem 2, now with $\varepsilon = 1.0$. Here, of course, we recognize the classical convergence rates already for h = 1/8, 1/16, 1/32; viz. the AD solution shows 1st order convergence, the DCP and (J;DCP) solutions are 2nd order and (DCP;J) is slightly less than 2nd order accurate.

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